Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: S.Y.B.SC SEM.-IV

SUBJECT: MTH-404: VECTOR CALCULUS

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MTH 404: VECTOR CALCULUS

Unit -1: Product of Vectors **Marks-15**

1.1 Scalar Product

1.2 Vector Product

1.3 Scalar Triple Product

1.4 Vector Product of Three Vectors

1.5 Reciprocal Vector

Unit-2: Vector functions **Marks-15**

2.1 Vector functions of a single variable.

2.2 Limits and continuity.

2.3 Differentiability, Algebra of differentiation.

2.4 Curves in space, Velocity and acceleration.

2.5 Vector function of two or three variables.

2.6 Limits, Continuity, Partial Differentiation

Unit-3: The Vector Operator Del **Marks-15**

3.1 The vector differentiation operator del.

3.2 Gradient.

3.3 Divergence and curl.

3.4 Formulae involving del. Invariance.

Unit-4: Vector Integration **Marks-15**

4.1 Ordinary integrals of vectors.

4.2 Line integrals.

4.3 Surface integrals.

Recommended Book:

1. Vector Analysis by Murray R Spiegel, Schaum's Series, McGraw Hill Book Company.

Reference Book:

1. Vector Calculus by Shanti Narayan and P.K. Mittal, S. Chand & Co., New Delhi

Learning Outcomes:

a) understand scalar and vector products

b) understand vector valued functions and their limits and continuity and use them to estimate velocity and acceleration of partials.

c) Calculate the curl and divergence of a vector field.

d) Set up and evaluate line integrals of functions along curves.

UNIT -1: PRODUCT OF VECTORS

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Scalar Product or Dot Product: The scalar product or dot product of two vectors $\bar{A}$ and $\bar{B}$ is denoted by $\bar{A}.\bar{B}$ and defined as $\bar{A}.\bar{B} = AB\cos\theta$,

Where $|\bar{A}| = A$, $|\bar{B}| = B$ and θ is angle between vectors $\bar{A}$ and $\bar{B}$.

Remark: 1) $\bar{A}.\bar{B} = \bar{B}.\bar{A}$ i.e. scalar product is commutative.

2) $\bar{A}.(\bar{B}+\bar{C}) = \bar{A}.\bar{B} + \bar{A}.\bar{C}$ (Distributive law)

3) $m(\bar{A}.\bar{B}) = (m\bar{A}).\bar{B} = \bar{A}.(m\bar{B}) = (\bar{A}.\bar{B})m$ for any scalar m.

4) $\bar{\imath}.\bar{\imath} = \bar{\jmath}.\bar{\jmath} = \bar{k}.\bar{k} = 1$ and $\bar{\imath}.\bar{\jmath} = \bar{\jmath}.\bar{k} = \bar{k}.\bar{\imath} = 0$, where $\bar{\imath}, \bar{\jmath}, \bar{k}$ are unit vectors along x, y, z axis respectively.

5) If $\bar{A} = A_1\bar{\imath} + A_2\bar{\jmath} + A_3\bar{k}$ and $\bar{B} = B_1\bar{\imath} + B_2\bar{\jmath} + B_3\bar{k}$ then

$\bar{A}.\bar{B} = A_1B_1 + A_2B_2 + A_3B_3$

6) If $\bar{A} = A_1\bar{\imath} + A_2\bar{\jmath} + A_3\bar{k}$, then $\bar{A}.\bar{A} = (A_1)^2 + (A_2)^2 + (A_3)^2 = |\bar{A}|^2$

i.e. $|\bar{A}| = \sqrt{(A_1)^2 + (A_2)^2 + (A_3)^2}$

7) Non-zero vectors $\bar{A}$ and $\bar{B}$ are perpendicular iff $\bar{A}.\bar{B} = 0$

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Ex. Find $\bar{a}.\bar{b}$ for $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 4\bar{\imath} - 4\bar{\jmath} + 7\bar{k}$

Solution: Let $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 4\bar{\imath} - 4\bar{\jmath} + 7\bar{k}$

$\therefore\ \bar{a}.\bar{b} = (\bar{\imath} - 2\bar{\jmath} + \bar{k}).(4\bar{\imath} - 4\bar{\jmath} + 7\bar{k}) = (1)(4)+(-2)(-4)+(1)(7) = 4 + 8 + 7 = 19$

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Ex. Find $\bar{a}.\bar{b}$ for $\bar{a} = \bar{\jmath} + 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + \bar{k}$

Solution: Let $\bar{a} = \bar{\jmath} + 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + \bar{k}$

$\therefore\ \bar{a}.\bar{b} = (\bar{\jmath} + 2\bar{k}).(2\bar{\imath} + \bar{k}) = (0)(2)+(1)(0)+(2)(1) = 0+0+2 = 2$

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Ex. Find $\bar{a}.\bar{b}$ for $\bar{a} = \bar{\jmath} - 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + 3\bar{\jmath} - 2\bar{k}$

Solution: Let $\bar{a} = \bar{\jmath} - 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + 3\bar{\jmath} - 2\bar{k}$

$\therefore\ \bar{a}.\bar{b} = (\bar{\jmath} - 2\bar{k}).(2\bar{\imath} + 3\bar{\jmath} - 2\bar{k}) = (0)(2)+(1)(3)+(-2)(-2) = 0 + 3 + 4 = 7$

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Ex. For what value of m the vectors $\bar{a}$ and $\bar{b}$ are perpendicular to each other

i) $\bar{a} = m\bar{\imath} + 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 4\bar{\imath} - 9\bar{\jmath} + 2\bar{k}$, ii) $\bar{a} = 5\bar{\imath} - 9\bar{\jmath} + 2\bar{k}$ and $\bar{b} = m\bar{\imath} + 2\bar{\jmath} + \bar{k}$

Solution: i) Let $\bar{a} = m\bar{\imath} + 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 4\bar{\imath} - 9\bar{\jmath} + 2\bar{k}$ are perpendicular to each other

$\therefore\ \bar{a}.\bar{b} = 0$

$\Rightarrow (m\bar{\imath} + 2\bar{\jmath} + \bar{k}).(4\bar{\imath} - 9\bar{\jmath} + 2\bar{k}) = 0$

$\Rightarrow (m)(4) + (2)(-9) + (1)(2) = 0$

$\Rightarrow 4m - 18 + 2 = 0$

$\Rightarrow$ 4m= 16

$\Rightarrow$ m = 4

ii) Let $\bar{a} = 5\bar{\imath} - 9\bar{\jmath} + \overline{2k}$ and $\bar{b} = m\bar{\imath} + 2\bar{\jmath} + \bar{k}$ are perpendicular to each other

$\therefore\ \bar{a}\,.\,\bar{b} = 0$

$\Rightarrow (5\bar{\imath} - 9\bar{\jmath} + 2\bar{k}).(m\bar{\imath} + 2\bar{\jmath} + \bar{k}) = 0$

$\Rightarrow (5)(m) + (-9)(2)+(2)(1)= 0$

$\Rightarrow 5m - 18 + 2= 0$

$\Rightarrow 5m= 16$

$\Rightarrow m = \frac{16}{5}$

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Ex. Find the angle between the vectors $\bar{a}$ and $\bar{b}$ where $\bar{a} = \bar{\imath} - \bar{\jmath}$ and $\bar{b} = \bar{\jmath} - \bar{k}$

Solution: Let θ be the angle between the vectors $\bar{a} = \bar{\imath} - \bar{\jmath}$ and $\bar{b} = \bar{\jmath} - \bar{k}$

$$\therefore \cos\theta = \frac{\bar{a}\,.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{(1)(0)+(-1)(1)+(0)(-1)}{\sqrt{1^2+(-1)^2+0^2}\sqrt{0^2+1^2+(-1)^2}} = \frac{0-1-0}{\sqrt{2}\sqrt{2}} = \frac{-1}{2}$$

$$\therefore \theta = \frac{2\pi}{3}$$

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Ex. Find the angle between the vectors $3\bar{\imath} - 2\bar{\jmath} - 6\bar{k}$ and $4\bar{\imath} - \bar{\jmath} + 8\bar{k}$

Solution: Let θ be the angle between the vectors $\bar{a} = 3\bar{\imath} - 2\bar{\jmath} - 6\bar{k}$ and $\bar{b} = 4\bar{\imath} - \bar{\jmath} + 8\bar{k}$

$$\therefore \cos\theta = \frac{\bar{a}\,.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{(3)(4)+(-2)(-1)+(-6)(8)}{\sqrt{3^2+(-2)^2+(-6)^2}\sqrt{4^2+(-1)^2+(8)^2}} = \frac{12+2-48}{\sqrt{49}\sqrt{81}} = \frac{-34}{63}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-34}{63}\right)$$

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Ex. If $\bar{a}$ and $\bar{b}$ are two vectors such that $|\bar{a}| = 4, |\bar{b}| = 3$ and $\bar{a}\,.\,\bar{b} = 6$.

Find the angle between the vectors $\bar{a}$ and $\bar{b}$

Solution: Let θ be the angle between the vectors $\bar{a}$ and $\bar{b}$

such that $|\bar{a}| = 4, |\bar{b}| = 3$ and $\bar{a}\,.\,\bar{b} = 6$

$$\therefore \cos\theta = \frac{\bar{a}\,.\bar{b}}{|\bar{a}||\bar{b}|} = \frac{6}{(4)(3)} = \frac{1}{2} \qquad \therefore \theta = \frac{\pi}{3}$$

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Ex. For any vector $\bar{r}$, prove that $\bar{r} = (\bar{r}.\bar{\imath})\bar{\imath} + (\bar{r}.\bar{\jmath})\bar{\jmath} + (\bar{r}.\bar{k})\bar{k}$

Proof: Let $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ be any vector, then

$\bar{r}.\bar{\imath} = (x\bar{\imath} + y\bar{\jmath} + z\bar{k}).\bar{\imath} = x$

$\bar{r}.\bar{\jmath} = (x\bar{\imath} + y\bar{\jmath} + z\bar{k}).\bar{\jmath} = y$

$\bar{r}.\bar{k} = (x\bar{\imath} + y\bar{\jmath} + z\bar{k}).\bar{k} = z$

$\therefore\ (\bar{r}.\bar{\imath})\bar{\imath} + (\bar{r}.\bar{\jmath})\bar{\jmath} + (\bar{r}.\bar{k})\bar{k} = x\bar{\imath} + y\bar{\jmath} + z\bar{k} = \bar{r}$ Hence proved.

Ex. For any two vectors $\bar{a}$ and $\bar{b}$ prove that $|\bar{a}+\bar{b}|^2+|\bar{a}-\bar{b}|^2 = 2(|\bar{a}|^2+|\bar{b}|^2)$

Proof: Consider

$$\text{LHS} = |\bar{a}+\bar{b}|^2+|\bar{a}-\bar{b}|^2$$
$$= (\bar{a}+\bar{b}).(\bar{a}+\bar{b})+(\bar{a}-\bar{b}).(\bar{a}-\bar{b})$$
$$= \bar{a}.\bar{a}+\bar{a}.\bar{b}+\bar{b}.\bar{a}+\bar{b}.\bar{b}+\bar{a}.\bar{a}-\bar{a}.\bar{b}-\bar{b}.\bar{a}+\bar{b}.\bar{b}$$
$$= 2\bar{a}.\bar{a}+2\bar{b}.\bar{b}$$
$$= 2(|\bar{a}|^2+|\bar{b}|^2)$$
$$= \text{RHS}$$

Hence proved.

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Ex. If $\bar{a}+\bar{b}+\bar{c} = \bar{0}$, $|\bar{a}| = 3$, $|\bar{b}| = 5$ and $|\bar{c}| = 7$, Find the angle between $\bar{a}$ and $\bar{b}$

Solution: Let $\bar{a}+\bar{b}+\bar{c} = \bar{0}$

$\therefore \bar{a}+\bar{b} = -\bar{c}$

$\therefore (\bar{a}+\bar{b}).(\bar{a}+\bar{b}) = (-\bar{c}).(-\bar{c})$

$\therefore \bar{a}.\bar{a}+\bar{a}.\bar{b}+\bar{b}.\bar{a}+\bar{b}.\bar{b} = \bar{c}.\bar{c}$

$\therefore |\bar{a}|^2+2\bar{a}.\bar{b}+|\bar{b}|^2 = |\bar{c}|^2$

$\therefore 9+2\bar{a}.\bar{b}+25 = 49$ $\quad \because |\bar{a}| = 3, |\bar{b}| = 5$ and $|\bar{c}| = 7$

$\therefore 2\bar{a}.\bar{b} = 15$

$\therefore 2|\bar{a}||\bar{b}|\cos\theta = 15$ where θ is angle between vectors $\bar{a}$ and $\bar{b}$

$\therefore 2(3)(5)\cos\theta = 15$

$\therefore \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ be the angle between vectors $\bar{a}$ and $\bar{b}$.

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Vector Product or Cross Product: The vector product or cross product of two vectors $\bar{A}$ and $\bar{B}$ is denoted by $\bar{A}\times\bar{B}$ and defined as $\bar{A}\times\bar{B} = AB\sin\theta\hat{u}$

Where $|\bar{A}| = A$, $|\bar{B}| = B$, θ is angle between vectors $\bar{A}$ and $\bar{B}$ and $\hat{u}$ is unit vector indicating the direction of $\bar{A}\times\bar{B}$.

Remark: 1) $\bar{A}\times\bar{B} = -\bar{B}\times\bar{A}$ i.e. vector product is not commutative.

2) $\bar{A}\times(\bar{B}+\bar{C}) = \bar{A}\times\bar{B}+\bar{A}\times\bar{C}$ (Distributive law)

3) $m(\bar{A}\times\bar{B}) = (m\bar{A})\times\bar{B} = \bar{A}\times(m\bar{B}) = (\bar{A}\times\bar{B})m$ for any scalar m.

4) $\bar{i}\times\bar{i} = \bar{j}\times\bar{j} = \bar{k}\times\bar{k} = \bar{0}$ and $\bar{i}\times\bar{j} = \bar{k}$, $\bar{j}\times\bar{k} = \bar{i}$, $\bar{k}\times\bar{i} = \bar{j}$,

where $\bar{i}$, $\bar{j}$, $\bar{k}$ are unit vectors along x, y, z axis resp.

5) If $\bar{A} = A_1\bar{i}+A_2\bar{j}+A_3\bar{k}$ and $\bar{B} = B_1\bar{i}+B_2\bar{j}+B_3\bar{k}$ then

$$\bar{A}\times\bar{B}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix} \text{ and } \bar{A}\times\bar{A}=\bar{B}\times\bar{B}=\bar{0}$$

6) Non-zero vectors $\bar{A}$ and $\bar{B}$ are parallel iff $\bar{A}\times\bar{B}=\bar{0}$

7) Vectors $\bar{A}$ and $\bar{B}$ both are perpendicular to vector $\bar{A}\times\bar{B}$ because $\bar{A}.(\bar{A}\times\bar{B})=0$ and $\bar{B}.(\bar{A}\times\bar{B})=0$

8) Area of parallelogram with sides $\bar{A}$ and $\bar{B}=|\bar{A}\times\bar{B}|$

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Ex. Find $\bar{a}\times\bar{b}$ for $\bar{a}=\bar{\jmath}-2\bar{k}$ and $\bar{b}=2\bar{\imath}+3\bar{\jmath}-2\bar{k}$

Solution: Let $\bar{a}=\bar{\jmath}-2\bar{k}$ and $\bar{b}=2\bar{\imath}+3\bar{\jmath}-2\bar{k}$

$$\therefore\ \bar{a}\times\bar{b}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ 0 & 1 & -2\\ 2 & 3 & -2\end{vmatrix}=(-2+6)\bar{\imath}-(0+4)\bar{\jmath}+(0-2)\bar{k}=4\bar{\imath}-4\bar{\jmath}-2\bar{k}$$

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Ex. If $\bar{p}=-3\bar{\imath}+4\bar{\jmath}-7\bar{k}$ and $\bar{q}=6\bar{\imath}+2\bar{\jmath}-3\bar{k}$, then find $\bar{p}\times\bar{q}$. Verify that $\bar{p}$ and $\bar{p}\times\bar{q}$ are perpendicular to each other and also verify that $\bar{q}$ and $\bar{p}\times\bar{q}$ are perpendicular to each other.

Proof: Let $\bar{p}=-3\bar{\imath}+4\bar{\jmath}-7\bar{k}$ and $\bar{q}=6\bar{\imath}+2\bar{\jmath}-3\bar{k}$

$$\therefore\ \bar{p}\times\bar{q}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ -3 & 4 & -7\\ 6 & 2 & -3\end{vmatrix}=(-12+14)\bar{\imath}-(9+42)\bar{\jmath}+(-6-24)\bar{k}=2\bar{\imath}-51\bar{\jmath}-30\bar{k}$$

Now $\bar{p}.(\bar{p}\times\bar{q})=(-3\bar{\imath}+4\bar{\jmath}-7\bar{k}).(2\bar{\imath}-51\bar{\jmath}-30\bar{k})=-6-204+210=0$

Hence $\bar{p}$ and $\bar{p}\times\bar{q}$ are perpendicular to each other

Again $\bar{q}.(\bar{p}\times\bar{q})=(6\bar{\imath}+2\bar{\jmath}-3\bar{k}).(2\bar{\imath}-51\bar{\jmath}-30\bar{k})=12-102+90=0$

Hence $\bar{q}$ and $\bar{p}\times\bar{q}$ are perpendicular to each other is proved.

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Ex. If $\bar{a}$ and $\bar{b}$ are two vectors, then prove that $|\bar{a}\times\bar{b}|^2+(\bar{a}.\bar{b})^2=|\bar{a}|^2|\bar{b}|^2$

Proof: Let θ is angle between any two vectors $\bar{a}$ and $\bar{b}$.

$\therefore \bar{a}\times\bar{b}=|\bar{a}||\bar{b}|\sin\theta\hat{u}$ and $\bar{a}.\bar{b}=|\bar{a}||\bar{b}|\cos\theta$

$\therefore |\bar{a}\times\bar{b}|=|\bar{a}||\bar{b}|\sin\theta$ and $\bar{a}.\bar{b}=|\bar{a}||\bar{b}|\cos\theta$

$\therefore |\bar{a}\times\bar{b}|^2+(\bar{a}.\bar{b})^2=|\bar{a}|^2|\bar{b}|^2\sin^2\theta+|\bar{a}|^2|\bar{b}|^2\cos^2\theta$

$\therefore |\bar{a}\times\bar{b}|^2+(\bar{a}.\bar{b})^2=|\bar{a}|^2|\bar{b}|^2$ Hence proved.

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Ex. If $|\bar{a}|=13$, $|\bar{b}|=5$ and $\bar{a}.\bar{b}=60$ then find $|\bar{a}\times\bar{b}|$.

Solution: Let $|\bar{a}|=13$, $|\bar{b}|=5$ and $\bar{a}.\bar{b}=60$

As $|\bar{a}\times\bar{b}|^2+(\bar{a}.\bar{b})^2=|\bar{a}|^2|\bar{b}|^2$

$\therefore |\bar{a} \times \bar{b}|^2 + (60)^2 = (13)^2(5)^2$

$\therefore |\bar{a} \times \bar{b}|^2 = 4225 - 3600 = 625$

$\therefore |\bar{a} \times \bar{b}| = 25$

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Ex. If the position vectors of three points A, B and C are $\bar{\imath} + 2\bar{\jmath} + 3\bar{k}$, $4\bar{\imath} + \bar{\jmath} + 5\bar{k}$ and $7(\bar{\imath} + \bar{k})$ respectively, then find $\overline{AB} \times \overline{AC}$

Solution: Let $\bar{\imath} + 2\bar{\jmath} + 3\bar{k}$, $4\bar{\imath} + \bar{\jmath} + 5\bar{k}$ and $7(\bar{\imath} + \bar{k})$ are the position vectors of three points A, B and C respectively.

$\therefore \overline{AB} = (4\bar{\imath} + \bar{\jmath} + 5\bar{k}) - (\bar{\imath} + 2\bar{\jmath} + 3\bar{k}) = 3\bar{\imath} - \bar{\jmath} + 2\bar{k}$

& $\overline{AC} = (7\bar{\imath} + 7\bar{k}) - (\bar{\imath} + 2\bar{\jmath} + 3\bar{k}) = 6\bar{\imath} - 2\bar{\jmath} + 4\bar{k}$

$$\therefore \overline{AB} \times \overline{AC} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 3 & -1 & 2 \\ 6 & -2 & 4 \end{vmatrix} = 0\bar{\imath} - 0\bar{\jmath} + 0\bar{k} = \bar{0}$$

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Scalar Triple Product or Box Product: The scalar triple product or box product of three vectors $\bar{A} = A_1\bar{\imath} + A_2\bar{\jmath} + A_3\bar{k}$, $\bar{B} = B_1\bar{\imath} + B_2\bar{\jmath} + B_3\bar{k}$ and $\bar{C} = C_1\bar{\imath} + C_2\bar{\jmath} + C_3\bar{k}$ is denoted by $[\bar{A}\ \bar{B}\ \bar{C}]$ and defined as $[\bar{A}\ \bar{B}\ \bar{C}] = \bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

Properties of Scalar Triple Product:

1) $\bar{A} \cdot (\bar{B} \times \bar{C}) = \bar{B} \cdot (\bar{C} \times \bar{A}) = \bar{C} \cdot (\bar{A} \times \bar{B})$
2) $\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C}$
3) $\bar{A} \cdot (\bar{A} \times \bar{C}) = 0$
4) $\bar{A}$, $\bar{B}$ and $\bar{C}$ are coplanar iff $\bar{A} \cdot (\bar{B} \times \bar{C}) = 0$
5) Volume of parallelepiped with sides $\bar{A}$, $\bar{B}$ and $\bar{C} = |\bar{A} \cdot (\bar{B} \times \bar{C})|$

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Ex. Find the scalar triple product of $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$, $\bar{b} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$ and $\bar{c} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$

Solution: Let $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$, $\bar{b} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$ and $\bar{c} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$

$$\therefore \bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = (-1-2) + 2(-2-1) + (4-1) = -3 - 6 + 3 = -6$$

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Ex. If the edges $\bar{a} = -3\bar{\imath} + 7\bar{\jmath} + 5\bar{k}$, $\bar{b} = -5\bar{\imath} + 7\bar{\jmath} - 3\bar{k}$ and $\bar{c} = 7\bar{\imath} - 5\bar{\jmath} - 3\bar{k}$ meet at a vertex point, find the volume of the parallelepiped.

Solution: Let $\bar{a} = -3\bar{\imath} + 7\bar{\jmath} + 5\bar{k}$, $\bar{b} = -5\bar{\imath} + 7\bar{\jmath} - 3\bar{k}$ and $\bar{c} = 7\bar{\imath} - 5\bar{\jmath} - 3\bar{k}$ meet at a vertex point.

∴ The volume of the parallelepiped = $\left|\bar{a}.(\bar{b}\times\bar{c})\right|$

Now $\bar{a}.(\bar{b}\times\bar{c}) = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$

= -3(-21- 15) - 7(15+21) + 5(25-49)

= 108 – 252 - 120

= - 264

∴ The volume of the parallelepiped = $|-264|$ = 264 cu. units.

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Vector Triple Product: Let $\bar{A}$, $\bar{B}$ and $\bar{C}$ be any three vectors, then $\bar{A}\times(\bar{B}\times\bar{C})$ is called the vector triple product.

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Ex. Show that $\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}.\bar{C})\bar{B} - (\bar{A}.\bar{B})\bar{C}$

Proof: Let $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$, $\bar{B} = B_1\bar{i} + B_2\bar{j} + B_3\bar{k}$ and $\bar{C} = C_1\bar{i} + C_2\bar{j} + C_3\bar{k}$, then

$$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{A}\times\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= (A_1\bar{i} + A_2\bar{j} + A_3\bar{k})\times[(B_2C_3 - B_3C_2)\bar{i} - (B_1C_3 - B_3C_1)\bar{j} + (B_1C_2 - B_2C_1)\bar{k}]$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ B_2C_3 - B_3C_2 & B_3C_1 - B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

$$= (A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3)\bar{i}$$
$$-(A_1B_1C_2 - A_1B_2C_1 - A_3B_2C_3 + A_3B_3C_2)\bar{j}$$
$$+(A_1B_3C_1 - A_1B_1C_3 - A_2B_2C_3 + A_2B_3C_2)\bar{k}$$

$$=(A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3)\bar{i}$$
$$+(A_3B_2C_3 - A_3B_3C_2 - A_1B_1C_2 + A_1B_2C_1)\bar{j}$$
$$+(A_1B_3C_1 - A_1B_1C_3 - A_2B_2C_3 + A_2B_3C_2)\bar{k} \quad \ldots\ldots (1)$$

& $(\bar{A}.\bar{C})\bar{B} - (\bar{A}.\bar{B})\bar{C} = (A_1C_1 + A_2C_2 + A_3C_3)(B_1\bar{i} + B_2\bar{j} + B_3\bar{k})$

$$- (A_1B_1 + A_2B_2 + A_3B_3)(C_1\bar{i} + C_2\bar{j} + C_3\bar{k})$$

$$= (A_1B_1C_1 + A_2B_1C_2 + A_3B_1C_3 - A_1B_1C_1 - A_2B_2C_1 - A_3B_3C_1)\bar{i}$$
$$+(A_1B_2C_1 + A_2B_2C_2 + A_3B_2C_3 - A_1B_1C_2 - A_2B_2C_2 - A_3B_3C_2)\bar{j}$$
$$+(A_1B_3C_1 + A_2B_3C_2 + A_3B_3C_3 - A_1B_1C_3 - A_2B_2C_3 - A_3B_3C_3)\bar{k}$$

$$= (A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3)\bar{i}$$
$$+(A_3B_2C_3 - A_3B_3C_2 - A_1B_1C_2 + A_1B_2C_1)\bar{j}$$
$$+(A_1B_3C_1 - A_1B_1C_3 - A_2B_2C_3 + A_2B_3C_2)\bar{k} \quad \ldots\ldots (2)$$

∴ From equation (1) and (2), we have

$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} . \bar{C})\bar{B} - (\bar{A} . \bar{B})\bar{C}$ Hence proved.

==

Ex. Show that $(\bar{A} \times \bar{B}) \times \bar{C} = (\bar{A} . \bar{C})\bar{B} - (\bar{B} . \bar{C})\bar{A}$

Proof: Consider $(\bar{A} \times \bar{B}) \times \bar{C} = - \bar{C} \times (\bar{A} \times \bar{B})$

$= - [(\bar{C} . \bar{B})\bar{A} - (\bar{C} . \bar{A})\bar{B}]$

$= (\bar{A} . \bar{C})\bar{B} - \bar{A} (\bar{B} . \bar{C})\bar{A}$

Hence proved.

==

Ex. Prove that $\bar{A} \times (\bar{B} \times \bar{C}) + \bar{B} \times (\bar{C} \times \bar{A}) + \bar{C} \times (\bar{A} \times \bar{B}) = \bar{0}$

Proof: Consider

LHS = $\bar{A} \times (\bar{B} \times \bar{C}) + \bar{B} \times (\bar{C} \times \bar{A}) + \bar{C} \times (\bar{A} \times \bar{B})$

$= (\bar{A} . \bar{C})\bar{B} - (\bar{A} . \bar{B})\bar{C} + (\bar{A} . \bar{B})\bar{C} - (\bar{B} . \bar{C})\bar{A} + (\bar{B} . \bar{C})\bar{A} - (\bar{A} . \bar{C})\bar{B}$

$= \bar{0}$

Hence proved.

==

Ex. Show that $\bar{\imath} \times (\bar{a} \times \bar{\imath}) + \bar{\jmath} \times (\bar{a} \times \bar{\jmath}) + \bar{k} \times (\bar{a} \times \bar{k}) = 2\bar{a}$

Proof: Consider

LHS = $\bar{\imath} \times (\bar{a} \times \bar{\imath}) + \bar{\jmath} \times (\bar{a} \times \bar{\jmath}) + \bar{k} \times (\bar{a} \times \bar{k})$

$= (\bar{\imath} . \bar{\imath})\bar{a} - (\bar{\imath} . \bar{a})\bar{\imath} + (\bar{\jmath} . \bar{\jmath})\bar{a} - (\bar{\jmath} . \bar{a})\bar{\jmath} + (\bar{k} . \bar{k})\bar{a} - (\bar{k} . \bar{a})\bar{k}$

$= \bar{a} - (\bar{\imath} . \bar{a})\bar{\imath} + \bar{a} - (\bar{\jmath} . \bar{a})\bar{\jmath} + \bar{a} - (\bar{k} . \bar{a})\bar{k}$

$= 3\bar{a} - [(\bar{\imath} . \bar{a})\bar{\imath} + (\bar{\jmath} . \bar{a})\bar{\jmath} + (\bar{k} . \bar{a})\bar{k}]$

$= 3\bar{a} - \bar{a}$

$= 2\bar{a}$

= RHS.

Hence proved.

==

Ex. Find the value of $\bar{a} \times (\bar{b} \times \bar{c})$ if $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}, \bar{b} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$ and $\bar{c} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$

Solution: Let $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}, \bar{b} = 2\bar{\imath} + \bar{\jmath} + \bar{k}$ and $\bar{c} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\bar{\imath} + 3\bar{\jmath} + 3\bar{k}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix} = -9\bar{\imath} - 6\bar{\jmath} - 3\bar{k} = -3(3\bar{\imath} + 2\bar{\jmath} + \bar{k})$$

Ex. Find the value of $\bar{a} \times (\bar{b} \times \bar{c})$ if

$\bar{a} = 2\bar{\imath} - 10\bar{\jmath} + 2\bar{k}, \bar{b} = 3\bar{\imath} + \bar{\jmath} + 2\bar{k}$ and $\bar{c} = 2\bar{\imath} + \bar{\jmath} + 3\bar{k}$

Solution: Let $\bar{a} = 2\bar{\imath} - 10\bar{\jmath} + 2\bar{k}, \bar{b} = 3\bar{\imath} + \bar{\jmath} + 2\bar{k}$ and $\bar{c} = 2\bar{\imath} + \bar{\jmath} + 3\bar{k}$

$$\therefore\ \bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} = \bar{\imath} - 5\bar{\jmath} + \bar{k}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & -10 & 2 \\ 1 & -5 & 1 \end{vmatrix} = 0\bar{\imath} - 0\bar{\jmath} + 0\bar{k} = \bar{0}$$

==

Ex. If $\bar{a} = 3\bar{\imath} + 2\bar{\jmath} - 4\bar{k}, \bar{b} = 5\bar{\imath} - 3\bar{\jmath} + 6\bar{k}$ and $\bar{c} = 5\bar{\imath} - \bar{\jmath} + 2\bar{k}$, find

i) $\bar{a} \times (\bar{b} \times \bar{c})$ ii) $(\bar{a} \times \bar{b}) \times \bar{c}$ and show that they are not equal.

Solution: Let $\bar{a} = 3\bar{\imath} + 2\bar{\jmath} - 4\bar{k}, \bar{b} = 5\bar{\imath} - 3\bar{\jmath} + 6\bar{k}$ and $\bar{c} = 5\bar{\imath} - \bar{\jmath} + 2\bar{k}$

$$\text{i) } \bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 5 & -3 & 6 \\ 5 & -1 & 2 \end{vmatrix} = 0\bar{\imath} + 20\bar{\jmath} + 10\bar{k}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 3 & 2 & -4 \\ 0 & 20 & 10 \end{vmatrix} = 100\bar{\imath} - 30\bar{\jmath} + 60\bar{k} = 10(10\bar{\imath} - 3\bar{\jmath} + 6\bar{k})$$

$$\text{ii) } \bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 3 & 2 & -4 \\ 5 & -3 & 6 \end{vmatrix} = 0\bar{\imath} - 38\bar{\jmath} - 19\bar{k}$$

$$\therefore (\bar{a} \times \bar{b}) \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 0 & -38 & -19 \\ 5 & -1 & 2 \end{vmatrix} = -95\bar{\imath} - 95\bar{\jmath} + 190\bar{k} = -95(\bar{\imath} + \bar{\jmath} - 2\bar{k})$$

From (i) and (ii) $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$ is proved.

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Ex. Verify that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\,.\bar{c})\bar{b} - (\bar{a}\,.\bar{b})\bar{c}$ for

$\bar{a} = \bar{\imath} + 2\bar{\jmath} + 3\bar{k}$, $\bar{b} = 2\bar{\imath} - \bar{\jmath} + \bar{k}$ and $\bar{c} = 3\bar{\imath} + 2\bar{\jmath} - 5\bar{k}$

Proof: Let $\bar{a} = \bar{\imath} + 2\bar{\jmath} + 3\bar{k}$, $\bar{b} = 2\bar{\imath} - \bar{\jmath} + \bar{k}$ and $\bar{c} = 3\bar{\imath} + 2\bar{\jmath} - 5\bar{k}$.

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & -1 & 1 \\ 3 & 2 & -5 \end{vmatrix} = (5-2)\bar{\imath} - (-10-3)\bar{\jmath} + (4+3)\bar{k} = 3\bar{\imath} + 13\bar{\jmath} + 7\bar{k}$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & 2 & 3 \\ 3 & 13 & 7 \end{vmatrix} = (14-39)\bar{\imath} - (7-9)\bar{\jmath} + (13-6)\bar{k} = -25\bar{\imath} + 2\bar{\jmath} + 7\bar{k} \ \ldots(1)$$

Now $\bar{a}\,.\bar{c} = (\bar{\imath} + 2\bar{\jmath} + 3\bar{k}).(3\bar{\imath} + 2\bar{\jmath} - 5\bar{k}) = 3 + 4 - 15 = -8$

& $\bar{a}\,.\bar{b} = (\bar{\imath} + 2\bar{\jmath} + 3\bar{k}).(2\bar{\imath} - \bar{\jmath} + \bar{k}) = 2 - 2 + 3 = 3$

$\therefore (\bar{a}\,.\bar{c})\bar{b} - (\bar{a}\,.\bar{b})\bar{c} = (-8)(2\bar{\imath} - \bar{\jmath} + \bar{k}) - 3(3\bar{\imath} + 2\bar{\jmath} - 5\bar{k})$

$= -16\bar{\imath} + 8\bar{\jmath} - 8\bar{k} - 9\bar{\imath} - 6\bar{\jmath} + 15\bar{k}$

$= -25\bar{\imath} + 2\bar{\jmath} + 7\bar{k} \ \ldots \ldots (2)$

∴ from (1) and (2) $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \,.\, \bar{c})\bar{b}$ - $(\bar{a} \,.\, \bar{b})\bar{c}$ is verified.

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Scalar Product of Four Vectors: Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are any four vectors, then $(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D})$ is called scalar product of four vectors.

Vector Product of Four Vectors: Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are any four vectors, then $(\bar{A} \times \bar{B})\times(\bar{C} \times \bar{D})$ is called vector product of four vectors.

Lagrange's Identity: Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are any four vectors, then

$(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D}) = \begin{vmatrix} \bar{A}.\bar{C} & \bar{B}.\bar{C} \\ \bar{A}.\bar{D} & \bar{B}.\bar{D} \end{vmatrix}$ is called Lagrange's identity.

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Ex. Prove that $(\bar{B} \times \bar{C}).(\bar{A} \times \bar{D})+(\bar{C} \times \bar{A}).(\bar{B} \times \bar{D})+(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D}) = 0$

Proof: Consider

LHS $=(\bar{B} \times \bar{C}).(\bar{A} \times \bar{D})+(\bar{C} \times \bar{A}).(\bar{B} \times \bar{D})+(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D})$

$= \begin{vmatrix} \bar{B}.\bar{A} & \bar{C}.\bar{A} \\ \bar{B}.\bar{D} & \bar{C}.\bar{D} \end{vmatrix} + \begin{vmatrix} \bar{C}.\bar{B} & \bar{A}.\bar{B} \\ \bar{C}.\bar{D} & \bar{A}.\bar{D} \end{vmatrix} + \begin{vmatrix} \bar{A}.\bar{C} & \bar{B}.\bar{C} \\ \bar{A}.\bar{D} & \bar{B}.\bar{D} \end{vmatrix}$ by Lagrange's identity

$= (\bar{A}.\bar{B})(\bar{C}.\bar{D})-(\bar{A}.\bar{C})(\bar{B}.\bar{D})+(\bar{B}.\bar{C})(\bar{A}.\bar{D})-(\bar{A}.\bar{B})(\bar{C}.\bar{D})+(\bar{A}.\bar{C})(\bar{B}.\bar{D})-(\bar{B}.\bar{C})(\bar{A}.\bar{D})$

$= 0$

Hence proved.

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Ex. If $\bar{A} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$, $\bar{B} = 2\bar{\imath} + \bar{\jmath} + 3\bar{k}$, $\bar{C} = \bar{\imath} - \bar{\jmath} + \bar{k}$ and $\bar{D} = 3\bar{\imath} + \bar{\jmath} + 2\bar{k}$, evaluate

i) $(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D})$ and ii)$(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D})$

Solution: Let $\bar{A} = \bar{\imath} + 2\bar{\jmath} - \bar{k}$, $\bar{B} = 2\bar{\imath} + \bar{\jmath} + 3\bar{k}$, $\bar{C} = \bar{\imath} - \bar{\jmath} + \bar{k}$ and $\bar{D} = 3\bar{\imath} + \bar{\jmath} + 2\bar{k}$

$\therefore \bar{A} \times \bar{B} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = 7\bar{\imath} - 5\bar{\jmath} - 3\bar{k}$

$\& \; \bar{C} \times \bar{D} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -3\bar{\imath} + \bar{\jmath} + 4\bar{k}$

i) $(\bar{A} \times \bar{B}).(\bar{C} \times \bar{D}) =(7\bar{\imath} - 5\bar{\jmath} - 3\bar{k}).(-3\bar{\imath} + \bar{\jmath} + 4\bar{k}) = -21 - 5 -12 = -38$

ii) $(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D})= \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 7 & -5 & -3 \\ -3 & 1 & 4 \end{vmatrix} = -17\bar{\imath} - 19\bar{\jmath} - 8\bar{k}$

===

Ex. If $\bar{a} = 2\bar{\imath} + \bar{\jmath} - \bar{k}$, $\bar{b} = -\bar{\imath} + 2\bar{\jmath} - 4\bar{k}$ and $\bar{c} = \bar{\imath} + \bar{\jmath} + \bar{k}$, find $(\bar{a} \times \bar{b}).(\bar{a} \times \bar{c})$

Solution: Let $\bar{a} = 2\bar{\imath} + \bar{\jmath} - \bar{k}$, $\bar{b} = -\bar{\imath} + 2\bar{\jmath} - 4\bar{k}$ and $\bar{C} = \bar{\imath} + \bar{\jmath} + \bar{k}$

$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -2\bar{\imath} + 9\bar{\jmath} + 5\bar{k}$

$$\& \ \bar{a} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2\bar{\imath} - 3\bar{\jmath} + \bar{k}$$

$\therefore (\bar{a} \times \bar{b}).(\bar{a} \times \bar{c}) = (-2\bar{\imath} + 9\bar{\jmath} + 5\bar{k}).(2\bar{\imath} - 3\bar{\jmath} + \bar{k}) = -4 -27+5 = -26$

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Reciprocal System of Vector: If $\bar{a}, \bar{b}$ and $\bar{c}$ are any three non-coplanar vectors so that $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0$, then the three vectors $\overline{a'}, \overline{b'}$ and $\overline{c'}$ defined by

$\overline{a'} = \frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}$, $\overline{b'} = \frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ and $\overline{c'} = \frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ are called reciprocal system of vectors.

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Properties of Reciprocal System of Vector:

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i) If $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors, then $\bar{a}.\overline{a'} = \bar{b}.\overline{b'} = \bar{c}.\overline{c'} = 1$

Proof : Consider $\bar{a}.\overline{a'} = \bar{a}.\frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]} = \frac{\bar{a}.(\bar{b}\times\bar{c})}{\bar{a}.(\bar{b}\times\bar{c})} = 1$

Similarly $\bar{b}.\overline{b'} = 1$ and $\bar{c}.\overline{c'} = 1$

$\therefore \bar{a}.\overline{a'} = \bar{b}.\overline{b'} = \bar{c}.\overline{c'} = 1$ is proved.

==

ii) If $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors, then

$\bar{a} \times \overline{a'} + \bar{b} \times \overline{b'} + \bar{c} \times \overline{c'} = \bar{0}$

Proof : Let $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors.

$$\therefore \bar{a} \times \overline{a'} + \bar{b} \times \overline{b'} + \bar{c} \times \overline{c'} = \bar{a} \times \frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]} + \bar{b} \times \frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]} + \bar{c} \times \frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$= \frac{\bar{a}\times(\bar{b}\times\bar{c}) + \bar{b}\times(\bar{c}\times\bar{a}) + \bar{c}\times(\bar{a}\times\bar{b})}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$= \frac{\bar{0}}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$= \bar{0}$$

==

iii) If $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors, then

$\bar{a}.\overline{a'} + \bar{b}.\overline{b'} + \bar{c}.\overline{c'} = 3$

Proof : Let $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors.

$$\therefore \bar{a}.\overline{a'} + \bar{b}.\overline{b'} + \bar{c}.\overline{c'} = \bar{a}.\frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]} + \bar{b}.\frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]} + \bar{c}.\frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$= \frac{\bar{a}.(\bar{b}\times\bar{c}) + \bar{b}.(\bar{c}\times\bar{a}) + \bar{c}.(\bar{a}\times\bar{b})}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$= \frac{[\bar{a}\ \bar{b}\ \bar{c}] + [\bar{b}\ \bar{c}\ \bar{a}] + [\bar{c}\ \bar{a}\ \bar{b}]}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$= \frac{3[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]}$$
$$= 3$$

Hence proved.

==

iv) The product of any vector of one system with a vector of reciprocal system which does not correspond to it is zero i.e. $\bar{a}.\bar{b}' = \bar{a}.\bar{c}' = \bar{b}.\bar{a}' = \bar{b}.\bar{c}' = \bar{c}.\bar{a}' = \bar{c}.\bar{b}' = 0$

Proof : Consider $\bar{a}.\bar{b}' = \bar{a}.\frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]} = \frac{\bar{a}.(\bar{c}\times\bar{a})}{\bar{a}.(\bar{b}\times\bar{c})} = \frac{0}{\bar{a}.(\bar{b}\times\bar{c})} = 0$

Similarly $\bar{a}.\bar{c}' = 0$, $\bar{b}.\bar{a}' = 0$, $\bar{b}.\bar{c}' = 0$, $\bar{c}.\bar{a}' = 0$, $\bar{c}.\bar{b}' = 0$

$\therefore \bar{a}.\bar{b}' = \bar{a}.\bar{c}' = \bar{b}.\bar{a}' = \bar{b}.\bar{c}' = \bar{c}.\bar{a}' = \bar{c}.\bar{b}' = 0$ is proved.

==

v) The orthogonal triad of vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ is self reciprocal. i.e. $\bar{i}' = \bar{i}, \bar{j}' = \bar{j}, \bar{k}' = \bar{k}$.

Proof : Let $\bar{i}', \bar{j}', \bar{k}'$ be the reciprocal system to $\bar{i}, \bar{j}, \bar{k}$ then

$$\bar{i}' = \frac{\bar{j}\times\bar{k}}{[\bar{i}\ \bar{j}\ \bar{k}]} = \frac{\bar{i}}{1} = \bar{i}$$

Similarly $\bar{j}' = \bar{j}$ and $\bar{k}' = \bar{k}$

$\therefore$ The orthogonal triad of vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ is self reciprocal is proved.

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Ex. Find the set of vectors reciprocal to the set $-\bar{i} + \bar{j} + \bar{k}$, $\bar{i} + \bar{j} + \bar{k}$, $\bar{i} + \bar{j} - \bar{k}$

Solution : Let $\bar{a}', \bar{b}', \bar{c}'$ be the reciprocal system to

$\bar{a} = -\bar{i} + \bar{j} + \bar{k}, \bar{b} = \bar{i} + \bar{j} + \bar{k}, \bar{c} = \bar{i} + \bar{j} - \bar{k}$.

$\therefore \bar{a}' = \frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}, \bar{b}' = \frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ and $\bar{c}' = \frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ …… (1)

Now $[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 2 + 2 + 0 = 4$

$\bar{b}\times\bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -2\bar{i} + 2\bar{j} + 0\bar{k} = -2\bar{i} + 2\bar{j}$

$\bar{c}\times\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix} = 2\bar{i} + 0\bar{j} + 2\bar{k} = 2\bar{i} + 2\bar{k}$

$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0\bar{i} + 2\bar{j} - 2\bar{k} = 2\bar{j} - 2\bar{k}$

From (1), we get set of vectors reciprocal as

$\bar{a}' = \frac{-2\bar{i} + 2\bar{j}}{4} = \frac{1}{2}(-\bar{i} + \bar{j})$,

$\overline{b'} = \frac{2\bar{\imath} + 2\bar{k}}{4} = \frac{1}{2}(\bar{\imath} + \bar{k})$

and $\overline{c'} = \frac{2\bar{\jmath} - 2\bar{k}}{4} = \frac{1}{2}(\bar{\jmath} - \bar{k})$

==

Ex. Find the set of vectors reciprocal to the set $2\bar{\imath} + 3\bar{\jmath} - \bar{k}$, $\bar{\imath} - \bar{\jmath} - 2\bar{k}$, $-\bar{\imath} + 2\bar{\jmath} + 2\bar{k}$

Solution : Let $\overline{a'}, \overline{b'}, \overline{c'}$ be the reciprocal system to

$\bar{a} = 2\bar{\imath} + 3\bar{\jmath} - \bar{k}, \bar{b} = \bar{\imath} - \bar{\jmath} - 2\bar{k}, \bar{c} = -\bar{\imath} + 2\bar{\jmath} + 2\bar{k}.$

$\therefore \overline{a'} = \frac{\bar{b}\times\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]}, \overline{b'} = \frac{\bar{c}\times\bar{a}}{[\bar{a}\,\bar{b}\,\bar{c}]}$ and $\overline{c'} = \frac{\bar{a}\times\bar{b}}{[\bar{a}\,\bar{b}\,\bar{c}]}$ …… (1)

Now $[\bar{a}\,\bar{b}\,\bar{c}] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 4 - 0 - 1 = 3$

$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2\bar{\imath} + 0\bar{\jmath} + \bar{k} = 2\bar{\imath} + \bar{k}$

$\bar{c} \times \bar{a} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -8\bar{\imath} + 3\bar{\jmath} - 7\bar{k}$

$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = -7\bar{\imath} + 3\bar{\jmath} - 5\bar{k}$

From (1), we get set of vectors reciprocal as

$\overline{a'} = \frac{2\bar{\imath} + \bar{k}}{3} = \frac{2}{3}\bar{\imath} + \frac{1}{3}\bar{k},$

$\overline{b'} = \frac{-8\bar{\imath} + 3\bar{\jmath} - 7\bar{k}}{3} = -\frac{8}{3}\bar{\imath} + \bar{\jmath} - \frac{7}{3}\bar{k}$

and $\overline{c'} = \frac{-7\bar{\imath} + 3\bar{\jmath} - 5\bar{k}}{3} = -\frac{7}{3}\bar{\imath} + \bar{\jmath} - \frac{5}{3}\bar{k}$

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MULTIPLE CHOICE QUESTIONS [MCQ'S]

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11) The scalar product is also called …….

A) dot product B) vector product C) box product D) None of these

2) If θ is angle between the vectors $\bar{A}$ and $\bar{B}$ with $|\bar{A}| = A$, $|\bar{B}| = B$, then scalar product of two vectors $\bar{A}$ and $\bar{B}$ is denoted by $\bar{A}.\bar{B}$ and defined as $\bar{A}.\bar{B}$ = ……

A) AB cotθ B) AB cosθ C) AB sinθ D) None of these

3) The scalar product of two vectors is a ……

A) scalar B) vector C) both scalar and vector D) None of these

4) If $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$ are unit vectors along x, y, z axis respectively, then $\bar{\imath}.\bar{\imath} = \bar{\jmath}.\bar{\jmath} = \bar{k}.\bar{k}$ = ……

A) 0 B) 1 C) -1 D) None of these

5) If $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$ are unit vectors along x, y, z axis respectively, then $\bar{\imath}.\bar{\jmath} = \bar{\jmath}.\bar{k} = \bar{k}.\bar{\imath} =$ ……

A) 0 B) 1 C) -1 D) None of these

6) If $\bar{A} = A_1\bar{\imath} + A_2\bar{\jmath} + A_3\bar{k}$ and $\bar{B} = B_1\bar{\imath} + B_2\bar{\jmath} + B_3\bar{k}$ then $\bar{A}.\bar{B} =$ ……

A) 0 B) $\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$ C) $A_1B_1+A_2B_2+A_3B_3$ D) None of these

7) Non-zero vectors $\bar{A}$ and $\bar{B}$ are perpendicular if and only if $\bar{A}.\bar{B} =$ ……

A) 0 B) 1 C) -1 D) None of these

8) The scalar product of two vectors is commutative is…

A) true B) false

9) If $\bar{a} = \bar{\imath} - 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 4\bar{\imath} - 4\bar{\jmath} + 7\bar{k}$, then $\bar{a}.\bar{b} =$ ……

A) 2 B) 7 C) 19 D) 0

10) If $\bar{a} = \bar{\jmath} + 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + \bar{k}$, then $\bar{a}.\bar{b} =$ ……

A) 2 B) 7 C) 19 D) 0

11) If $\bar{a} = \bar{\jmath} - 2\bar{k}$ and $\bar{b} = 2\bar{\imath} + 3\bar{\jmath} - 2\bar{k}$, then $\bar{a}.\bar{b} =$ ……

A) 2 B) 7 C) 19 D) 0

12) The vectors $\bar{a} = m\bar{\imath} + 2\bar{\jmath} + \bar{k}$ and $\bar{b} = 4\bar{\imath} - 9\bar{\jmath} + 2\bar{k}$ are perpendicular to each other if m = ……

A) 2 B) 0 C) 4 D) 3

13) The angle between the vectors $\bar{a} = \bar{\imath} - \bar{\jmath}$ and $\bar{b} = \bar{\jmath} - \bar{k}$ is ……

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) π

14) If $\bar{a}$ and $\bar{b}$ are two vectors such that $|\bar{a}| = 4$, $|\bar{b}| = 3$ and $\bar{a}.\bar{b} = 6$, then the angle between the vectors $\bar{a}$ and $\bar{b}$ is ……

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) π

15) For any two vectors $\bar{a}$ and $\bar{b}$, $|\bar{a}+\bar{b}|^2+|\bar{a}-\bar{b}|^2 =$ ……

A) $2(|\bar{a}|^2-|\bar{b}|^2)$ B) $(|\bar{a}|^2+|\bar{b}|^2)$ C) $2(|\bar{a}|^2+|\bar{b}|^2)$ D) $|\bar{a}|^2-|\bar{b}|^2$

16)The vector product is also called …….

A) dot product B) cross product C) box product D) None of these

17) If θ is angle between the vectors $\bar{A}$ and $\bar{B}$ with $|\bar{A}| = A$, $|\bar{B}| = B$ and $\hat{u}$ is unit vector indicating the direction of $\bar{A} \times \bar{B}$, then vector product of two vectors $\bar{A}$ and $\bar{B}$ is denoted by $\bar{A} \times \bar{B}$ and defined as $\bar{A} \times \bar{B} =$ ……

A) AB sinθ B) AB cosθ C) AB sinθ $\hat{u}$ D) None of these

18) The vector product of two vectors is a ……

A) scalar B) vector C) both scalar and vector D) None of these

19) The vector product of two vectors is commutative is…

A) true B) false

20) If $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$ and $\bar{B} = B_1\bar{i} + B_2\bar{j} + B_3\bar{k}$ then $\bar{A} \times \bar{B}$ = ……

A) 0 B) $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$ C) $A_1B_1+A_2B_2+A_3B_3$ D) None of these

21) Non-zero vectors $\bar{A}$ and $\bar{B}$ are parallel to each other if and only if $\bar{A} \times \bar{B}$ = ……

A) $\bar{0}$ B) 1 C) π D) $-\pi$

22) If $\bar{i}$, $\bar{j}$, $\bar{k}$ are unit vectors along x, y, z axis respectively, then $\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k}$ = …

A) π B) $\bar{0}$ C) 1 D) $-\pi$

23) Area of parallelogram with sides $\bar{A}$ and $\bar{B}$ =

A) $\bar{A} . \bar{B}$ B) $\bar{A} \times \bar{B}$ C) $|\bar{A} \times \bar{B}|$ D) None of these

24) If $\bar{a} = \bar{j} - 2\bar{k}$ and $\bar{b} = 2\bar{i} + 3\bar{j} - 2\bar{k}$, then $\bar{a} \times \bar{b}$ = ……

A) $\bar{i} - 4\bar{j} - 2\bar{k}$ B) $4\bar{i} - 4\bar{j} - 2\bar{k}$ C) $4\bar{i} - \bar{j} - 2\bar{k}$ D) None of these

25) If $\bar{p} = -3\bar{i} + 4\bar{j} - 7\bar{k}$ and $\bar{q} = 6\bar{i} + 2\bar{j} - 3\bar{k}$, then $\bar{p} \times \bar{q}$ = ……

A) $2\bar{i} - 51\bar{j} - 30\bar{k}$ B) $2\bar{i} - 5\bar{j} - 30\bar{k}$ C) $2\bar{i} - 51\bar{j} - 3\bar{k}$ D) None of these

26) If $\bar{a}$ and $\bar{b}$ are two vectors, then prove that $|\bar{a} \times \bar{b}|^2 + (\bar{a}.\bar{b})^2$ = ……

A) $|\bar{a}|^2 + |\bar{b}|^2$ B) $2|\bar{a}|^2|\bar{b}|^2$ C) $|\bar{a}|^2|\bar{b}|^2$ D) None of these

27) If $|\bar{a}| = 13$, $|\bar{b}| = 5$ and $\bar{a}.\bar{b} = 60$ then find $|\bar{a} \times \bar{b}|$

A) 10 B) 25 C) 18 D) None of these

28) The scalar triple product is also called …….

A) dot product B) vector product C) box product D) None of these

29) The scalar triple product of three vectors is a ……

A) scalar B) vector C) both scalar and vector D) None of these

30) If $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$, $\bar{B} = B_1\bar{i} + B_2\bar{j} + B_3\bar{k}$ and $\bar{C} = C_1\bar{i} + C_2\bar{j} + C_3\bar{k}$, then $[\bar{A}\ \bar{B}\ \bar{C}] = \bar{A} . (\bar{B} \times \bar{C})$ = ……

A) $A_1B_1+A_2B_2+A_3B_3$ B) $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$ C) $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$ D) None of these

31) If $\bar{a} = \bar{i} - 2\bar{j} + \bar{k}$, $\bar{b} = 2\bar{i} + \bar{j} + \bar{k}$ and $\bar{c} = \bar{i} + 2\bar{j} - \bar{k}$, then $\bar{a} . (\bar{b} \times \bar{c})$ = ……

A) 0 B) 1 C) -6 D) None of these

32) $\bar{A}$, $\bar{B}$ and $\bar{C}$ are coplanar if and only if $\bar{A} . (\bar{B} \times \bar{C})$ = ……

A) 0 B) 1 C) -1 D) None of these

33) Volume of parallelepiped with sides $\bar{A}$, $\bar{B}$ and $\bar{C}$ = ……

A) $\bar{A} \times (\bar{B} \times \bar{C})$ B) $|\bar{A} . (\bar{B} \times \bar{C})|$ C) $\bar{A} . (\bar{B} \times \bar{C})$ D) None of these

34) If the edges $\bar{a} = -3\bar{i} + 7\bar{j} + 5\bar{k}$, $\bar{b} = -5\bar{i} + 7\bar{j} - 3\bar{k}$ and $\bar{c} = 7\bar{i} - 5\bar{j} - 3\bar{k}$ meet at vertex point, then the volume of the parallelopiped is ……

A) 264 B) -264 C) 0 D) None of these

35) $\bar{A} \cdot (\bar{A} \times \bar{C}) =$ ……

A) 0 B) C C) A D) None of these

36) Let $\bar{A}$, $\bar{B}$ and $\bar{C}$ be any three vectors, then $\bar{A} \times (\bar{B} \times \bar{C})$ is called the ……

A) vector product B) scalar triple product
C) vector triple product D) None of these

37) $\bar{A} \times (\bar{B} \times \bar{C}) =$ ……

A) $\bar{A}(\bar{B} \cdot \bar{C})$ B) $(\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$ C) $(\bar{A} \cdot \bar{B})\bar{C} - (\bar{A} \cdot \bar{C})\bar{B}$ D) None of these

38) If $\bar{a} = 2\bar{i} - 10\bar{j} + 2\bar{k}$, $\bar{b} = 3\bar{i} + \bar{j} + 2\bar{k}$ and $\bar{c} = 2\bar{i} + \bar{j} + 3\bar{k}$, then $\bar{a} \times (\bar{b} \times \bar{c}) =$ …

A) $\bar{0}$ B) 0 C) $\bar{i} + \bar{j} + \bar{k}$ D) None of these

39) Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are any four vectors, then $(\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D})$ is called ……of four vectors.

A) vector product B) scalar product C) scalar triple product D) None of these

40) Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are any four vectors, then $(\bar{A} \times \bar{B}) \times (\bar{C} \times \bar{D})$ is called ……of four vectors.

A) vector product B) scalar product C) scalar triple product D) None of these

41) Let $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are any four vectors, then $(\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D}) =$ …… is called Lagrange's identity.

A) $\begin{vmatrix} \bar{A}.\bar{B} & 0 \\ 0 & \bar{C}.\bar{D} \end{vmatrix}$ B) $\begin{vmatrix} 1 & \bar{C}.\bar{D} \\ \bar{A}.\bar{B} & 1 \end{vmatrix}$ C) $\begin{vmatrix} \bar{A}.\bar{C} & \bar{B}.\bar{C} \\ \bar{A}.\bar{D} & \bar{B}.\bar{D} \end{vmatrix}$ D) None of these

42) If $\bar{a} = 2\bar{i} + \bar{j} - \bar{k}$, $\bar{b} = -\bar{i} + 2\bar{j} - 4\bar{k}$ and $\bar{c} = \bar{i} + \bar{j} + \bar{k}$, then $(\bar{a} \times \bar{b}) \cdot (\bar{a} \times \bar{c}) =$ ……

A) 26 B) -26 C) 0 D) None of these

43) $(\bar{B} \times \bar{C}) \cdot (\bar{A} \times \bar{D}) + (\bar{C} \times \bar{A}) \cdot (\bar{B} \times \bar{D}) + (\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D}) =$ ……

A) 0 B) 1 C) -1 D) None of these

44) If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are any three non-coplanar vectors so that $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0$, then the three vectors $\overline{a'}$, $\overline{b'}$ and $\overline{c'}$ defined by $\overline{a'} = \frac{\bar{b} \times \bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}$, $\overline{b'} = \frac{\bar{c} \times \bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ and $\overline{c'} = \frac{\bar{a} \times \bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ are called …… system of vectors.

A) homogeneous B) non-homogeneous C) reciprocal D) None of these

45) If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\overline{a'}$, $\overline{b'}$, $\overline{c'}$ are reciprocal system of vectors, then $\bar{a} \cdot \overline{a'} = \bar{b} \cdot \overline{b'} = \bar{c} \cdot \overline{c'} =$……

A) 0 B) 1 C) -1 D) None of these

46) If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\overline{a'}$, $\overline{b'}$, $\overline{c'}$ are reciprocal system of vectors, then $\bar{a} \times \overline{a'} + \bar{b} \times \overline{b'} + \bar{c} \times \overline{c'} =$ ……

A) $\bar{0}$ B) $\bar{1}$ C) 3 D) None of these

47) If $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors, then

$\bar{a}.\overline{a'} + \bar{b}.\overline{b'} + \bar{c}.\overline{c'} = 3$

A) 0 B) 1 C) 3 D) None of these

48) The reciprocal system of vectors to the vectors $\bar{\imath}, \bar{\jmath}, \bar{k}$ is ……

A) $\bar{\imath}, \bar{\jmath}, \bar{k}$ B) $\bar{\jmath}, \bar{k}, \bar{\imath}$ C) $\bar{k}, \bar{\imath}, \bar{\jmath}$ D) None of these

49) If $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors, then

$\bar{a}.\overline{b'} = \bar{a}.\overline{c'} = \bar{b}.\overline{a'} = \bar{b}.\overline{c'} = \bar{c}.\overline{a'} = \bar{c}.\overline{b'} = \ldots\ldots$

A) 3 B) 1 C) 0 D) None of these

50) If $\bar{a}, \bar{b}, \bar{c}$ and $\overline{a'}, \overline{b'}, \overline{c'}$ are reciprocal system of vectors, then $\overline{a'} = \ldots\ldots$

A) $\frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ B) $\frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ C) $\frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ D) None of these

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UNIT-2: VECTOR FUNCTIONS

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Vector functions of a single variable: A function $\bar{v}$: $R \to R^3$ defined by

$\bar{v} = v_1(t)\bar{\imath} + v_2(t)\bar{\jmath} + v_3(t)\bar{k}$ is called a vector function of a single variable t.

Limit of Vector Function: Let $\bar{v}(t) = v_1(t)\bar{\imath} + v_2(t)\bar{\jmath} + v_3(t)\bar{k}$ be a vector function of a scalar variable t. If for small ε > 0, there exist $\delta > 0$ depends on ε such that $|\bar{v}(t) - \bar{l}| < \varepsilon$ whenever $0 < |t - a| < \delta$. Then $\bar{l}$ is said to be limit of $\bar{v}(t)$ as $t \to a$. Denoted by $\lim_{t \to a} \bar{v}(t) = \bar{l}$.

Algebra of Limits:

If $\lim_{t \to a} \bar{v}(t) = \bar{l}$ and $\lim_{t \to a} \bar{u}(t) = \bar{m}$ then

i) $\lim_{t \to a} [\bar{v}(t) \pm \bar{u}(t)] = \bar{l} \pm \bar{m}$

ii) $\lim_{t \to a} [\bar{v}(t) . \bar{u}(t)] = \bar{l} . \bar{m}$

iii) $\lim_{t \to a} [\bar{v}(t) \times \bar{u}(t)] = \bar{l} \times \bar{m}$

iv) $\lim_{t \to a} [\frac{\bar{v}(t)}{\bar{u}(t)}] = \frac{\bar{l}}{\bar{m}}$ provided $\bar{m} \neq \bar{0}$

Continuity of Vector Function: A vector function $\bar{v} = \bar{v}(t)$ of a scalar variable t is said to be continuous at $t = t_0$ if $\lim_{t \to t_0} \bar{v}(t) = \bar{v}(t_0)$.

Remark: A vector function $\bar{v} = \bar{v}(t)$ of a scalar variable t is said to be continuous in an interval (a, b) if it is continuous at every point in (a, b).

Differentiability of Vector Function: Let $\bar{v}(t) = v_1(t)\bar{\imath} + v_2(t)\bar{\jmath} + v_3(t)\bar{k}$ be a vector function of a scalar variable t and $\overline{\delta v}$ be change in $\bar{v}$ corresponding to small change δt in t. If $\lim_{\delta t \to 0} \frac{\overline{\delta v}}{\delta t} = \lim_{\delta t \to 0} \frac{\bar{v}(t+\delta t) - \bar{v}(t)}{\delta t}$ exist and finite, then $\bar{v}(t)$ is said to be differentiable w.r.t.t and $\frac{\overline{dv}}{dt} = \lim_{\delta t \to 0} \frac{\overline{\delta v}}{\delta t}$ is called derivative of $\bar{v}$ w.r.t.t.

Remark: i) $\overline{v'}(t_0) = (\frac{\overline{dv}}{dt})_{t = t_0} = \lim_{\delta t \to 0} \frac{\bar{v}(t_0+\delta t) - \bar{v}(t_0)}{\delta t} = \lim_{t \to t_0} \frac{\bar{v}(t) - \bar{v}(t_0)}{t - t_0}$

is called derivative of $\bar{v}(t)$ at point $t = t_0$.

ii) $\frac{d^2\bar{v}}{dt^2} = \frac{d}{dt}(\frac{\overline{dv}}{dt})$ is called second order derivative of $\bar{v}$ w.r.t.t.

iii) $\frac{d^3\bar{v}}{dt^3} = \frac{d}{dt}(\frac{d^2\bar{v}}{dt^2})$ is called third order derivative of $\bar{v}$ w.r.t.t.

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Theorem: If $\bar{v}(t)$ is differentiable at t = t_0, then $\bar{v}(t)$ is continuous at t = t_0.

Proof: Let $\bar{v}(t)$ is differentiable at t = t_0

$$\Rightarrow \bar{v}'(t_0) = \lim_{t\to t_0} \frac{\bar{v}(t)-\bar{v}(t_0)}{t-t_0} \text{ is exists and finite …… (1)}$$

Consider

$$\lim_{t\to t_0}[\bar{v}(t)-\bar{v}(t_0)] = \lim_{t\to t_0} \frac{\bar{v}(t)-\bar{v}(t_0)}{t-t_0} \times (t-t_0)$$

$$= = \lim_{t\to t_0} \frac{\bar{v}(t)-\bar{v}(t_0)}{t-t_0} \times \lim_{t\to t_0}(t-t_0)$$

$$= \bar{v}'(t_0) \times 0$$

$$\therefore \lim_{t\to t_0} \bar{v}(t) - \bar{v}(t_0) = \bar{0}$$

$$\therefore \lim_{t\to t_0} \bar{v}(t) = \bar{v}(t_0)$$

i.e. $\bar{v}(t)$ is continuous at t=t_0.

==

Ex.: Show that $\bar{v}(t) = t\bar{i} + |t|\,\bar{j}$ is continuous but not differentiable at point t = 0.

Proof : Let $\bar{v}(t) = t\bar{i} + |t|\,\bar{j}$

$\therefore \bar{v}(0) = 0\bar{i} + |0|\,\bar{j} = \bar{0}$

and $\lim_{t\to 0}\bar{v}(t) = \lim_{t\to 0}(t\bar{i} + |t|\,\bar{j}) = 0\bar{i} + |0|\,\bar{j} = \bar{0} = \bar{v}(0)$

$\therefore \bar{v}(t) = t\bar{i} + |t|\,\bar{j}$ is continuous at point t = 0.

Now $\lim_{t\to 0} \frac{\bar{v}(t)-\bar{v}(0)}{t-0} = \lim_{t\to 0} \frac{t\bar{i} + |t|\,\bar{j} - 0}{t}$

$$= \lim_{t\to 0} \left(\bar{i} + \frac{|t|}{t}\,\bar{j}\right)$$

$$= \bar{i} + \lim_{t\to 0} \frac{|t|}{t}\,\bar{j}$$

But $\lim_{t\to 0^+} \frac{|t|}{t} = \lim_{t\to 0^+} \frac{t}{t} = 1$ and $\lim_{t\to 0^-} \frac{|t|}{t} = \lim_{t\to 0^-} \frac{-t}{t} = -1$

$\therefore \lim_{t\to 0} \frac{\bar{v}(t)-\bar{v}(0)}{t-0}$ does not exist.

Hence $\bar{v}(t) = t\bar{i} + |t|\,\bar{j}$ is continuous but not differentiable at point t = 0 is proved.

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Theorem: If $\bar{u}$ and $\bar{v}$ are differentiable vector functions of scalar variable t then

$$\frac{d}{dt}(\bar{u} + \bar{v}) = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}.$$

Proof: Let $\bar{w} = \bar{u} + \bar{v}$

Let $\overline{\delta u}$, $\overline{\delta v}$ and $\overline{\delta w}$ are the changes in $\bar{u}$, $\bar{v}$ and $\bar{w}$ corresponding to small change δt in t respectively.

$\therefore \bar{w} + \delta\bar{w} = (\bar{u} + \delta\bar{u}) + (\bar{v} + \delta\bar{v})$

$\therefore \delta\bar{w} = \delta\bar{u} + \delta\bar{v}$ …… (i)

Dividing (i) by δt and taking limit as $\delta t \to 0$, we get,

$$\lim_{\delta t \to 0} \frac{\delta\bar{w}}{\delta t} = \lim_{\delta t \to 0} \left(\frac{\delta\bar{u}}{\delta t} + \frac{\delta\bar{v}}{\delta t}\right)$$

$$= \lim_{\delta t \to 0} \frac{\delta\bar{u}}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\bar{v}}{\delta t}$$

$\therefore \frac{d\bar{w}}{dt} = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$ $\because \bar{u}$ and $\bar{v}$ are differentiable vector functions.

i.e. $\frac{d}{dt}(\bar{u} + \bar{v}) = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$ Hence proved.

==

Theorem: If $\bar{u}$ and $\bar{v}$ are differentiable vector functions of scalar variable t then

$$\frac{d}{dt}(\bar{u} - \bar{v}) = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$$

Proof: Let $\bar{w} = \bar{u} - \bar{v}$

Let $\overline{\delta u}$, $\overline{\delta v}$ and $\overline{\delta w}$ are the changes in $\bar{u}$, $\bar{v}$ and $\bar{w}$ corresponding to small change δt in t respectively.

$\therefore \bar{w} + \delta\bar{w} = (\bar{u} + \delta\bar{u}) - (\bar{v} + \delta\bar{v})$

$\therefore \delta\bar{w} = \delta\bar{u} - \delta\bar{v}$ …… (i)

Dividing (i) by δt and taking limit as $\delta t \to 0$, we get,

$$\lim_{\delta t \to 0} \frac{\delta\bar{w}}{\delta t} = \lim_{\delta t \to 0} \left(\frac{\delta\bar{u}}{\delta t} - \frac{\delta\bar{v}}{\delta t}\right)$$

$$= \lim_{\delta t \to 0} \frac{\delta\bar{u}}{\delta t} - \lim_{\delta t \to 0} \frac{\delta\bar{v}}{\delta t}$$

$\therefore \frac{d\bar{w}}{dt} = \frac{d\bar{u}}{dt} - \frac{d\bar{v}}{dt}$ $\because \bar{u}$ and $\bar{v}$ are differentiable vector functions.

i.e. $\frac{d}{dt}(\bar{u} - \bar{v}) = \frac{d\bar{u}}{dt} - \frac{d\bar{v}}{dt}$ Hence proved.

==

Theorem: If $\bar{u}$ and $\bar{v}$ are differentiable vector functions of scalar variable t then

$$\frac{d}{dt}(\bar{u}.\bar{v}) = \bar{u}.\frac{d\bar{v}}{dt} + \bar{v}.\frac{d\bar{u}}{dt}$$

Proof: Let $\phi = \bar{u}.\bar{v}$

Let $\overline{\delta u}$, $\overline{\delta v}$ and $\delta\phi$ are the changes in $\bar{u}$, $\bar{v}$ and ϕ corresponding to small change δt in t respectively.

$\therefore \phi + \delta\phi = (\bar{u} + \delta\bar{u}).(\bar{v} + \delta\bar{v})$

$\therefore \bar{u}.\bar{v} + \delta\phi = \bar{u}.\bar{v} + \bar{u}.\delta\bar{v} + \delta\bar{u}.\bar{v} + \delta\bar{u}.\delta\bar{v}$

$\therefore \delta\phi = \bar{u}.\delta\bar{v} + \bar{v}.\delta\bar{u} + \delta\bar{u}.\delta\bar{v}$ …… (i)

Dividing (i) by δt and taking limit as $\delta t \to 0$, we get,

$$\lim_{\delta t \to 0} \frac{\delta\phi}{\delta t} = \lim_{\delta t \to 0} (\bar{u}.\frac{\delta\bar{v}}{\delta t} + \bar{v}.\frac{\delta\bar{u}}{\delta t} + \delta\bar{u}.\frac{\delta\bar{v}}{\delta t})$$

$$= \bar{u}.\lim_{\delta t \to 0} \frac{\delta\bar{v}}{\delta t} + \bar{v}.\lim_{\delta t \to 0} \frac{\delta\bar{u}}{\delta t} + \lim_{\delta t \to 0} \delta\bar{u}.\frac{\delta\bar{v}}{\delta t}$$

As $\bar{u}$ and $\bar{v}$ are differentiable vector functions and $\delta t \to 0 \Rightarrow \delta\bar{u} \to 0$, we get,

$$\therefore \frac{d\phi}{dt} = \bar{u}.\frac{d\bar{v}}{dt} + \bar{v}.\frac{d\bar{u}}{dt}$$

i.e. $\frac{d}{dt}(\bar{u}.\bar{v}) = \bar{u}.\frac{d\bar{v}}{dt} + \bar{v}.\frac{d\bar{u}}{dt}$ Hence proved.

==

Corollary: If $\bar{u}$ is differentiable vector function of scalar variable t then

$$\frac{d\bar{u}^2}{dt} = 2\bar{u}.\frac{d\bar{u}}{dt} \text{ and } \bar{u}.\frac{d\bar{u}}{dt} = u\frac{du}{dt}, \text{ where } u = |\bar{u}|$$

Proof: As $\bar{u}^2 = \bar{u}.\bar{u} = u^2$ where $u = |\bar{u}|$

$\therefore \frac{d\bar{u}^2}{dt} = \frac{d}{dt}(\bar{u}.\bar{u}) = \bar{u}.\frac{d\bar{u}}{dt} + \bar{u}.\frac{d\bar{u}}{dt} = 2\bar{u}.\frac{d\bar{u}}{dt}$ …… (1)

& $\frac{d\bar{u}^2}{dt} = \frac{du^2}{dt} = 2u\frac{du}{dt}$ …… (2)

From (1) and (2), we get,

$$2\bar{u}.\frac{d\bar{u}}{dt} = 2u\frac{du}{dt}$$

i.e. $\bar{u}.\frac{d\bar{u}}{dt} = u\frac{du}{dt}$ Hence proved.

==

Theorem: If $\bar{u}$ and $\bar{v}$ are differentiable vector functions of scalar variable t then

$$\frac{d}{dt}(\bar{u} \times \bar{v}) = \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$$

Proof: Let $\bar{w} = \bar{u} \times \bar{v}$

Let $\overline{\delta u}$, $\overline{\delta v}$ and $\delta\bar{w}$ are the changes in $\bar{u}$, $\bar{v}$ and $\bar{w}$ corresponding to small change δt in t respectively.

$\therefore \bar{w} + \delta\bar{w} = (\bar{u} + \delta\bar{u}) \times (\bar{v} + \delta\bar{v})$

$\therefore \bar{u} \times \bar{v} + \delta\bar{w} = \bar{u} \times \bar{v} + \bar{u} \times \delta\bar{v} + \delta\bar{u} \times \bar{v} + \delta\bar{u} \times \delta\bar{v}$

$\therefore \delta\bar{w} = \bar{u} \times \delta\bar{v} + \delta\bar{u} \times \bar{v} + \delta\bar{u} \times \delta\bar{v}$ …… (i)

Dividing (i) by δt and taking limit as δt → 0, we get,

$$\lim_{\delta t \to 0} \frac{\delta\bar{w}}{\delta t} = \lim_{\delta t \to 0} \left(\bar{u} \times \frac{\delta\bar{v}}{\delta t} + \frac{\delta\bar{u}}{\delta t} \times \bar{v} + \delta\bar{u} \times \frac{\delta\bar{v}}{\delta t}\right)$$

$$= \bar{u} \times \lim_{\delta t \to 0} \frac{\delta\bar{v}}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\bar{u}}{\delta t} \times \bar{v} + \lim_{\delta t \to 0} \delta\bar{u} \times \frac{\delta\bar{v}}{\delta t}$$

As $\bar{u}$ and $\bar{v}$ are differentiable vector functions and δt → 0 ⇒ $\delta\bar{u} \to 0$, we get,

$$\therefore \frac{d\bar{w}}{dt} = \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$$

i.e. $\frac{d}{dt}(\bar{u} \times \bar{v}) = \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$ Hence proved.

==

Corollary: $\frac{d}{dt}\, \bar{u} \times (\bar{v} \times \bar{w}) = \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times (\frac{d\bar{v}}{dt} \times \bar{w}) + \bar{u} \times (\bar{v} \times \frac{d\bar{w}}{dt})$

Proof: Consider

$$\frac{d}{dt}\, \bar{u} \times (\bar{v} \times \bar{w}) = \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times \frac{d}{dt}(\bar{v} \times \bar{w})$$

$$= \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times [\frac{d\bar{v}}{dt} \times \bar{w} + \bar{v} \times \frac{d\bar{w}}{dt}]$$

$$= \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times \left(\frac{d\bar{v}}{dt} \times \bar{w}\right) + \bar{u} \times \left(\bar{v} \times \frac{d\bar{w}}{dt}\right)$$

Hence proved.

==

Corollary: $\frac{d}{dt}\, [\bar{u}\ \bar{v}\ \bar{w}] = [\frac{d\bar{u}}{dt}\, \bar{v}\ \bar{w}] + [\bar{u}\, \frac{d\bar{v}}{dt}\, \bar{w}] + [\bar{u}\ \bar{v}\ \frac{d\bar{w}}{dt}]$

Proof: Consider

$$\frac{d}{dt}[\bar{u}\ \bar{v}\ \bar{w}] = \frac{d}{dt}\, \bar{u}.(\bar{v} \times \bar{w})$$

$$= \frac{d\bar{u}}{dt}.(\bar{v} \times \bar{w}) + \bar{u}.\frac{d}{dt}(\bar{v} \times \bar{w})$$

$$= \frac{d\bar{u}}{dt}.(\bar{v} \times \bar{w}) + \bar{u}.[\frac{d\bar{v}}{dt} \times \bar{w} + \bar{v} \times \frac{d\bar{w}}{dt}]$$

$$= \frac{d\bar{u}}{dt}.(\bar{v} \times \bar{w}) + \bar{u}.\left(\frac{d\bar{v}}{dt} \times \bar{w}\right) + \bar{u}.\left(\bar{v} \times \frac{d\bar{w}}{dt}\right)$$

$$= [\frac{d\bar{u}}{dt}\, \bar{v}\ \bar{w}] + [\bar{u}\, \frac{d\bar{v}}{dt}\, \bar{w}] + [\bar{u}\ \bar{v}\ \frac{d\bar{w}}{dt}]$$

Hence proved.

==

Theorem: If a vector function $\bar{u}$ and a scalar function ϕ are differentiable functions of scalar variable t then $\frac{d}{dt}(\phi\bar{u}) = \phi\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}\bar{u}$

Proof: Let $\bar{w} = \phi\bar{u}$

Let $\delta\bar{u}$, $\delta\phi$ and $\delta\bar{w}$ are the changes in $\bar{u}$, ϕ and $\bar{w}$ corresponding to small change δt in t respectively.

$\therefore \bar{w} + \delta\bar{w} = (\phi + \delta\phi)(\bar{u} + \delta\bar{u})$

$\therefore \phi\bar{u} + \delta\bar{w} = \phi\bar{u} + \phi\delta\bar{u} + \delta\phi\bar{u} + \delta\phi\delta\bar{u}$

$\therefore \delta\bar{w} = \phi\delta\bar{u} + \delta\phi\bar{u} + \delta\phi\delta\bar{u}$ …… (i)

Dividing (i) by δt and taking limit as δt → 0, we get,

$$\lim_{\delta t\to 0}\frac{\delta\bar{w}}{\delta t} = \lim_{\delta t\to 0}\left(\phi\frac{\delta\bar{u}}{\delta t} + \frac{\delta\phi}{\delta t}\bar{u} + \frac{\delta\phi}{\delta t}\delta\bar{u}\right)$$

$$= \phi\lim_{\delta t\to 0}\frac{\delta\bar{u}}{\delta t} + \lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}\bar{u} + \lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}\delta\bar{u}$$

As a vector function $\bar{u}$ and a scalar function ϕ are differentiable functions of scalar variable t and δt → 0 ⇒ $\delta\bar{u} \to 0$, we get,

$$\therefore \frac{d\bar{w}}{dt} = \phi\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}\bar{u}$$

i.e. $\frac{d}{dt}(\phi\bar{u}) = \phi\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}\bar{u}$ Hence proved.

==

Corollary: If k is constant scalar then $\frac{d}{dt}(k\bar{u}) = k\frac{d\bar{u}}{dt}$

Proof: Consider

$$\frac{d}{dt}(k\bar{u}) = k\frac{d\bar{u}}{dt} + \frac{dk}{dt}\bar{u} = k\frac{d\bar{u}}{dt} + 0\bar{u} = k\frac{d\bar{u}}{dt}$$

Hence proved.

==

Theorem: If $\bar{u}$ a differentiable vector function of a scalar s and s is the differentiable scalar function of scalar variable t then $\frac{d\bar{u}}{dt} = \frac{ds}{dt}\frac{d\bar{u}}{ds}$

Proof: Let $\delta\bar{u}$ and δs are the changes in $\bar{u}$ and s corresponding to change δt in t, then

$$\frac{\delta\bar{u}}{\delta t} = \frac{\delta s}{\delta t}\frac{\delta\bar{u}}{\delta s}$$

By taking limit as δt → 0, we get,

$$\lim_{\delta t\to 0}\frac{\delta\bar{u}}{\delta t} = \lim_{\delta t\to 0}\left(\frac{\delta s}{\delta t}\frac{\delta\bar{u}}{\delta s}\right)$$

$$= \lim_{\delta t\to 0}\frac{\delta s}{\delta t}\lim_{\delta t\to 0}\frac{\delta\bar{u}}{\delta s}$$

As δt → 0 ⇒ $\delta s \to 0$, we get,

$$\therefore \lim_{\delta t\to 0}\frac{\delta\bar{u}}{\delta t} = \lim_{\delta t\to 0}\frac{\delta s}{\delta t}\lim_{\delta s\to 0}\frac{\delta\bar{u}}{\delta s}$$

$\therefore \frac{d\bar{u}}{dt} = \frac{ds}{dt}\frac{d\bar{u}}{ds}$ ∵ $\bar{u}$ and s are differentiable functions.

Hence proved.

==

Theorem: If $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ is a differentiable vector function of a scalar variable t, then $\frac{d}{dt}\bar{f}(t) = \frac{df_1(t)}{dt}\bar{i} + \frac{df_2(t)}{dt}\bar{j} + \frac{df_3(t)}{dt}\bar{k}$

Proof: Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$.

Let $\delta f_1, \delta f_2, \delta f_3$ and $\delta\bar{f}$ are the changes in f_1, f_2, f_3 and $\bar{f}$ corresponding to change δt in t.

$\therefore \bar{f} + \delta\bar{f} = (f_1 + \delta f_1)\bar{i} + (f_2 + \delta f_2)\bar{j} + (f_3 + \delta f_3)\bar{k}$

$\therefore f_1\bar{i} + f_2\bar{j} + f_3\bar{k} + \delta\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k} + \delta f_1\bar{i} + \delta f_2\bar{j} + \delta f_3\bar{k}$

$\therefore \delta\bar{f} = \delta f_1\bar{i} + \delta f_2\bar{j} + \delta f_3\bar{k}$. …… (i)

Dividing equation (i) by δt and taking limit as $\delta t \to 0$, we get,

$$\lim_{\delta t \to 0}\frac{\delta\bar{f}}{\delta t} = \lim_{\delta t \to 0}\left(\frac{\delta f_1}{\delta t}\bar{i} + \frac{\delta f_2}{\delta t}\bar{j} + \frac{\delta f_3}{\delta t}\bar{k}\right)$$

$$= \lim_{\delta t \to 0}\frac{\delta f_1}{\delta t}\bar{i} + \lim_{\delta t \to 0}\frac{\delta f_2}{\delta t}\bar{j} + \lim_{\delta t \to 0}\frac{\delta f_3}{\delta t}\bar{k}$$

As $\bar{f}$ is differentiable ⇒limit of LHS is exists ⇒limit of RHS is also exists

$\therefore \frac{d}{dt}\bar{f}(t) = \frac{df_1(t)}{dt}\bar{i} + \frac{df_2(t)}{dt}\bar{j} + \frac{df_3(t)}{dt}\bar{k}$ Hence proved.

===

Ex.: Show that $\bar{u}(t)$ is constant vector function on [a, b] iff $\frac{d\bar{u}}{dt} = \bar{0}$ on [a, b]

Proof: Suppose $\bar{u}(t) = \bar{c}, \forall t \in [a, b]$

$$\therefore \frac{d\bar{u}}{dt} = \lim_{\delta t \to 0}\frac{\bar{u}(t+\delta t) - \bar{u}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0}\frac{\bar{c} - \bar{c}}{\delta t}$$

$\therefore \frac{d\bar{u}}{dt} = \bar{0} \quad \forall t \in [a, b]$

Conversely: Suppose $\frac{d\bar{u}}{dt} = \bar{0} \quad \forall t \in [a, b]$

Let $\bar{u}(t) = u_1(t)\bar{i} + u_2(t)\bar{j} + u_3(t)\bar{k}$

$\therefore \frac{d\bar{u}}{dt} = \frac{du_1}{dt}\bar{i} + \frac{du_2}{dt}\bar{j} + \frac{du_3}{dt}\bar{k}$

$\therefore \frac{d\bar{u}}{dt} = \bar{0} \Rightarrow \frac{du_1}{dt}\bar{i} + \frac{du_2}{dt}\bar{j} + \frac{du_3}{dt}\bar{k} = \bar{0}$

$\Rightarrow \frac{du_1}{dt} = 0, \frac{du_2}{dt} = 0 \text{ and } \frac{du_3}{dt} = 0$

$\Rightarrow u_1, u_2$ and u_3 are constants.

Let $u_1(t) = c_1$, $u_2(t) = c_2$ and $u_3(t) = c_3$

$\bar{u}(t) = c_1\bar{i} + c_2\bar{j} + c_3\bar{k} = \bar{c}$ a constant vector $\forall t \in [a, b]$

Hence proved.

===

Ex.: Show that a differentiable vector function $\bar{u}(t)$ is of constant magnitude iff $\bar{u}.\frac{d\bar{u}}{dt} = 0\ \forall\ t \in [a, b]$

Proof: Let $\bar{u}$ is of constant magnitude $\forall\ t \in [a, b]$

$\Leftrightarrow |\bar{u}| = u$ is constant $\forall\ t \in [a, b]$

$\Leftrightarrow \bar{u}.\bar{u} = u^2$ is constant $\forall\ t \in [a, b]$

$\Leftrightarrow \frac{d}{dt}(\bar{u}.\bar{u}) = 0\ \forall\ t \in [a, b]$

$\Leftrightarrow 2\bar{u}.\frac{d\bar{u}}{dt} = 0\ \forall\ t \in [a, b]$

$\Leftrightarrow \bar{u}.\frac{d\bar{u}}{dt} = 0\ \forall\ t \in [a, b]$

Hence proved.

==

Ex.: Show that a non-constant vector function $\bar{u}(t)$ is of constant direction iff $\bar{u} \times \frac{d\bar{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$

Proof: Let $\bar{u} = u\hat{u}$, where $\hat{u}$ is unit vector along $\bar{u}$.

$\therefore\ \bar{u} \times \frac{d\bar{u}}{dt} = (u\hat{u}) \times \frac{d}{dt}(u\hat{u})$

$= (u\hat{u}) \times [u\frac{d\hat{u}}{dt} + \hat{u}\frac{du}{dt}]$

$= (u\hat{u}) \times (u\frac{d\hat{u}}{dt}) + (u\hat{u}) \times \hat{u}\frac{du}{dt}$

$= u^2(\hat{u} \times \frac{d\hat{u}}{dt}) + u\frac{du}{dt}(\hat{u} \times \hat{u})$

$\therefore\ \bar{u} \times \frac{d\bar{u}}{dt} = u^2(\hat{u} \times \frac{d\hat{u}}{dt})$ …… (1) $\because \hat{u} \times \hat{u} = \bar{0}$

Suppose $\bar{u}$ is of constant direction $\forall\ t \in [a, b]$

$\therefore\ \hat{u}$ is of constant direction $\forall\ t \in [a, b]$

$\therefore\ \hat{u}$ is constant vector $\forall\ t \in [a, b]$ $\because$ magnitude of $\hat{u}$ is constant

$\therefore \frac{d\hat{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$

$\therefore$ From (1) $\bar{u} \times \frac{d\bar{u}}{dt} = u^2(\hat{u} \times \bar{0}) = \bar{0}\ \forall\ t \in [a, b]$

Conversely: Suppose $\bar{u} \times \frac{d\bar{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$

$\therefore$ From (1) $u^2(\hat{u} \times \frac{d\hat{u}}{dt}) = \bar{0}\ \forall\ t \in [a, b]$

$\therefore\ \hat{u} \times \frac{d\hat{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$ …… (2) $\because u \neq 0$ as $\bar{u}$ is non-constant vector.

Also $\hat{u}.\frac{d\hat{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$ …… (3) $\because$ magnitude of $\hat{u}$ is constant.

$\therefore$ From (2) and (3) $\frac{d\hat{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$

$\therefore\ \hat{u}$ is constant vector $\forall\ t \in [a, b]$

$\therefore\ \hat{u}$ and hence $\bar{u}$ is of constant direction $\forall\ t \in [a, b]$

Hence proved.

==

Ex.: Evaluate $\lim_{t\to 0}[(t^2+1)\bar{\imath} + (\frac{3^{2t}-1}{t})\bar{\jmath} + (1+2t)^{\frac{1}{t}}\bar{k}]$

Sol. Consider $\lim_{t\to 0}[(t^2+1)\bar{\imath} + (\frac{3^{2t}-1}{t})\bar{\jmath} + (1+2t)^{\frac{1}{t}}\bar{k}]$

$= \lim_{t\to 0}(t^2+1)\bar{\imath} + \lim_{t\to 0}(\frac{3^{2t}-1}{t})\bar{\jmath} + \lim_{t\to 0}(1+2t)^{\frac{1}{t}}\bar{k}$

$= (0+1)\bar{\imath} + \log 3^2\,\bar{\jmath} + \lim_{t\to 0}[(1+2t)^{\frac{1}{2t}}]^2\bar{k}$ $\quad \because \lim_{t\to 0}(\frac{a^t-1}{t}) = \log a$

$= \bar{\imath} + 2\log 3\,\bar{\jmath} + e^2\bar{k}$ $\quad \because \lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$

==

Ex.: If $\bar{f}(t) = \frac{\sin 2t}{t}\bar{\imath} + \cos t\,\bar{\jmath}$, $t \neq 0$ and $\bar{f}(0) = x\bar{\imath} + \bar{\jmath}$ is continuous at t = 0, then find the value of x.

Sol. Let $\bar{f}(t) = \frac{\sin 2t}{t}\bar{\imath} + \cos t\,\bar{\jmath}$, $t \neq 0$ and $\bar{f}(0) = x\bar{\imath} + \bar{\jmath}$ is continuous at t = 0

$\therefore \lim_{t\to 0}\bar{f}(t) = \bar{f}(0)$

$\therefore \bar{f}(0) = \lim_{t\to 0}(\frac{\sin 2t}{t}\bar{\imath} + \cos t\,\bar{\jmath})$

$\therefore x\bar{\imath} + \bar{\jmath} = \lim_{t\to 0}(\frac{\sin 2t}{t})\bar{\imath} + \lim_{t\to 0}\cos t\,\bar{\jmath}$

$\therefore x\bar{\imath} + \bar{\jmath} = \lim_{t\to 0}2(\frac{\sin 2t}{2t})\bar{\imath} + \cos 0\,\bar{\jmath}$

$\therefore x\bar{\imath} + \bar{\jmath} = 2(1)\bar{\imath} + \bar{\jmath}$

$\therefore x\bar{\imath} + \bar{\jmath} = 2\bar{\imath} + \bar{\jmath}$

$\therefore x = 2$

==

Ex.: If $\bar{f}(t) = \cos t\,\bar{\imath} + \sin t\,\bar{\jmath} + \tan t\,\bar{k}$, find $\bar{f}'(t)$ and $\left|\bar{f}'(\frac{\pi}{4})\right|$.

Solution: Let $\bar{f}(t) = \cos t\,\bar{\imath} + \sin t\,\bar{\jmath} + \tan t\,\bar{k}$

$\therefore \bar{f}'(t) = -\sin t\,\bar{\imath} + \cos t\,\bar{\jmath} + \sec^2 t\,\bar{k}$

$\therefore \bar{f}'(\frac{\pi}{4}) = -\sin\frac{\pi}{4}\,\bar{\imath} + \cos\frac{\pi}{4}\,\bar{\jmath} + \sec^2\frac{\pi}{4}\,\bar{k} = -\frac{1}{\sqrt{2}}\bar{\imath} + \frac{1}{\sqrt{2}}\bar{\jmath} + 2\bar{k}$

$\therefore \left|\bar{f}'(\frac{\pi}{4})\right| = \sqrt{(-\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 + 2^2} = \sqrt{5}$

==

Ex.: If $\bar{r} = (t^2+1)\bar{\imath} + (4t-3)\bar{\jmath} + (2t^2-6t)\bar{k}$, find i) $\frac{d\bar{r}}{dt}$, ii) $\left|\frac{d\bar{r}}{dt}\right|$, iii) $\frac{d^2\bar{r}}{dt^2}$ iv) $\left|\frac{d^2\bar{r}}{dt^2}\right|$ at t = 2.

Solution: Let $\bar{r} = (t^2+1)\bar{\imath} + (4t-3)\bar{\jmath} + (2t^2-6t)\bar{k}$

$\therefore \frac{d\bar{r}}{dt} = (2t)\bar{\imath} + (4)\bar{\jmath} + (4t-6)\bar{k}$ and

$\frac{d^2\bar{r}}{dt^2} = 2\bar{i} + 0\,\bar{j} + 4\,\bar{k}$

At t = 2, we have,

i) $\frac{d\bar{r}}{dt} = 4\bar{i} + 4\,\bar{j} + 2\bar{k} = 2(2\bar{i} + 2\bar{j} + \bar{k})$

ii) $\left|\frac{d\bar{r}}{dt}\right| = 2\sqrt{2^2 + 2^2 + 1^2} = 6$

iii) $\frac{d^2\bar{r}}{dt^2} = 2\bar{i} + 4\,\bar{k} = 2(\bar{i} + 2\bar{k})$

iv) $\left|\frac{d^2\bar{r}}{dt^2}\right| = 2\sqrt{1^2 + 2^2} = 2\sqrt{5}$

==

Ex.: If $\bar{r} = (t+1)\bar{i} + (t^2+t+1)\,\bar{j} + (t^3+t^2+t+1)\,\bar{k}$, find $\frac{d\bar{r}}{dt}$ and $\frac{d^2\bar{r}}{dt^2}$

Solution: Let $\bar{r} = (t+1)\bar{i} + (t^2+t+1)\,\bar{j} + (t^3+t^2+t+1)\bar{k}$

$\therefore \frac{d\bar{r}}{dt} = \bar{i} + (2t+1)\,\bar{j} + (3t^2+2t+1)\bar{k}$ and

$\frac{d^2\bar{r}}{dt^2} = 0\bar{i} + 2\,\bar{j} + (6t+2)\,\bar{k}$

i.e. $\frac{d^2\bar{r}}{dt^2} = 2[\,\bar{j} + (3t+1)\,\bar{k}]$

==

Ex.: If $\bar{r} = e^{-t}\bar{i} + \log(t^2+1)\,\bar{j} - \tan t\bar{k}$, find i) $\frac{d\bar{r}}{dt}$, ii) $\frac{d^2\bar{r}}{dt^2}$, ii) $\left|\frac{d\bar{r}}{dt}\right|$, iv) $\left|\frac{d^2\bar{r}}{dt^2}\right|$ at t = 0.

Solution: Let $\bar{r} = e^{-t}\bar{i} + \log(t^2+1)\,\bar{j} - \tan t\bar{k}$

$\therefore \frac{d\bar{r}}{dt} = -e^{-t}\bar{i} + \frac{2t}{t^2+1}\bar{j} - sec^2 t\bar{k}$

and $\frac{d^2\bar{r}}{dt^2} = e^{-t}\bar{i} + 2[\frac{t^2+1-t(2t)}{(t^2+1)^2}]\,\bar{j} - 2sec t.\sec t.\tan t\bar{k}$

$= e^{-t}\bar{i} + 2[\frac{1-t^2}{(t^2+1)^2}]\,\bar{j} - 2sec^2 t.\tan t\bar{k}$

At t = 0, we have,

i) $\frac{dr}{dt} = -\bar{i} + 0\bar{j} - \bar{k} = -\bar{i} - \bar{k}$

ii) $\frac{d^2\bar{r}}{dt^2} = \bar{i} + 2\bar{j} - 0\bar{k} = \bar{i} + 2\bar{j}$

iii) $\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

iv) $\left|\frac{d^2\bar{r}}{dt^2}\right| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$

==

Ex.: If $\bar{r} = sint\bar{i} + \cos t\,\bar{j} + t\bar{k}$, find i) $\frac{d\bar{r}}{dt}$, ii) $\frac{d^2\bar{r}}{dt^2}$, ii) $\left|\frac{d\bar{r}}{dt}\right|$, iv) $\left|\frac{d^2\bar{r}}{dt^2}\right|$.

Solution: Let $\bar{r} = sint\bar{i} + \cos t\,\bar{j} + t\bar{k}$

i) $\frac{d\bar{r}}{dt} = cost\bar{i} - \sin t\,\bar{j} + \bar{k}$

ii) $\frac{d^2\bar{r}}{dt^2}$ = $-sint\bar{\imath} - cost\,\bar{\jmath} + 0\bar{k}$

$= -(sint\bar{\imath} + cost\,\bar{\jmath})$

iii) $\left|\frac{d\bar{r}}{dt}\right| = \sqrt{(cost)^2 + (-sint)^2 + 1^2} = \sqrt{2}$

iv) $\left|\frac{d^2\bar{r}}{dt^2}\right| = \sqrt{(-sint)^2 + (-cost)^2} = 1$

==

Ex.: If $\bar{r} = e^{kt}\bar{a} + e^{-kt}\bar{b}$, where $\bar{a}, \bar{b}$ are constant vectors and k is constant scalar, then show that $\ddot{\bar{r}} = k^2\bar{r}$, where $\ddot{\bar{r}} = \frac{d^2\bar{r}}{dt^2}$

Proof: Let $\bar{r} = e^{kt}\bar{a} + e^{-kt}\bar{b}$, where $\bar{a}, \bar{b}$ are constant vectors and k is constant scalar.

$\therefore \frac{d\bar{r}}{dt} = ke^{kt}\bar{a} - ke^{-kt}\bar{b}$

$\therefore \frac{d^2\bar{r}}{dt^2} = k^2e^{kt}\bar{a} + k^2e^{-kt}\bar{b}$

$= k^2(e^{kt}\bar{a} + e^{-kt}\bar{b})$

$\therefore \ddot{\bar{r}} = k^2\bar{r}$

Hence proved.

==

Ex.: If $\bar{r} = (sinht)\bar{a} + (cosht)\bar{b}$, where $\bar{a}, \bar{b}$ are constant vectors, then show that $\frac{d^2\bar{r}}{dt^2} = \bar{r}$

Proof: Let $\bar{r} = (sinht)\bar{a} + (cosht)\bar{b}$, where $\bar{a}, \bar{b}$ are constant vectors.

$\therefore \frac{d\bar{r}}{dt} = (cosht)\bar{a} + (sinht)\bar{b}$

$\therefore \frac{d^2\bar{r}}{dt^2} = (sinht)\bar{a} + (cosht)\bar{b}$

$\therefore \frac{d^2\bar{r}}{dt^2} = \bar{r}$

Hence proved.

==

Ex.: If $\bar{r} = cosnt\bar{\imath} + sinnt\,\bar{\jmath}$, where n is constant, then show that

i) $\bar{r}.\frac{d\bar{r}}{dt} = 0$ ii) $\bar{r} \times \frac{d\bar{r}}{dt} = n\bar{k}$ iii) $\frac{d^2\bar{r}}{dt^2} = -n^2\bar{r}$

Proof: Let $\bar{r} = cosnt\bar{\imath} + sinnt\,\bar{\jmath}$, where n is constant.

$\therefore \frac{d\bar{r}}{dt} = -nsinnt\bar{\imath} + ncosnt\,\bar{\jmath}$

i) $\bar{r}.\frac{d\bar{r}}{dt} = (cosnt\bar{\imath} + sinnt\,\bar{\jmath})(-nsinnt\bar{\imath} + ncosnt\,\bar{\jmath})$

$= -ncosntsinnt + nsinntcosnt$

$\therefore \bar{r}.\frac{d\bar{r}}{dt} = 0$

ii) $\bar{r} \times \frac{d\bar{r}}{dt} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ cosnt & sinnt & 0 \\ -nsinnt & ncosnt & 0 \end{vmatrix}$

$= 0\bar{\imath} + 0\bar{\jmath} + (ncos^2nt + nsin^2nt)\bar{k}$

$= n\bar{k}$

and iii) As $\frac{dr}{dt} = -nsinnt\bar{\imath} + ncosnt\,\bar{\jmath}$

$\therefore \frac{d^2\bar{r}}{dt^2} = -n^2 cosnt\bar{\imath} - n^2 sinnt\,\bar{\jmath}$

$= -n^2(cosnt\bar{\imath} + sinnt\,\bar{\jmath})$

$\therefore \frac{d^2\bar{r}}{dt^2} = -n^2\bar{r}$

Hence proved.

==

Ex.: If $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$, where $\bar{a}, \bar{b}$ are constant vectors and ω is constant scalar, then prove that i) $\bar{r} \times \frac{d\bar{r}}{dt} = \omega(\bar{a} \times \bar{b})$ ii) $\frac{d^2\bar{r}}{dt^2} = -\omega^2\bar{r}$

Proof: Let $\bar{r} = \bar{a}\cos\omega t + \bar{b}\sin\omega t$, where $\bar{a}, \bar{b}$ are constant vectors and ω is constant scalar.

$\therefore \frac{d\bar{r}}{dt} = -\omega\bar{a}\sin\omega t + \omega\bar{b}\cos\omega t$

i) $\bar{r} \times \frac{d\bar{r}}{dt} = (\bar{a}\cos\omega t + \bar{b}\sin\omega t) \times (-\omega\bar{a}\sin\omega t + \omega\bar{b}\cos\omega t)$

$= \omega[-(\bar{a} \times \bar{a})\cos\omega t\sin\omega t + (\bar{a} \times \bar{b})\cos^2\omega t - (\bar{b} \times \bar{a})\sin^2\omega t + (\bar{b} \times \bar{b})\sin\omega t\cos\omega t]$

$= \omega[\bar{0} + (\bar{a} \times \bar{b})\cos^2\omega t + (\bar{a} \times \bar{b})\sin^2\omega t + \bar{0}]$

$\because \bar{a} \times \bar{a} = \bar{b} \times \bar{b} = \bar{0}$ and $\bar{b} \times \bar{a} = -\bar{a} \times \bar{b}$

$= \omega(\bar{a} \times \bar{b})$

ii) As $\frac{d\bar{r}}{dt} = -\omega\bar{a}\sin\omega t + \omega\bar{b}\cos\omega t$

$\therefore \frac{d^2\bar{r}}{dt^2} = -\omega^2\bar{a}\cos\omega t - \omega^2\bar{b}\sin\omega t$

$= -\omega^2(\bar{a}\cos\omega t + \bar{b}\sin\omega t)$

$\therefore \ddot{\bar{r}} = -\omega^2\bar{r}$ Hence proved.

==

Ex.: If $\bar{A} = 5t^2\bar{\imath} + t\bar{\jmath} - t^3\bar{k}$ and $\bar{B} = sint\bar{\imath} - cost\bar{\jmath}$, then find $\frac{d}{dt}(\bar{A}.\bar{B})$ and $\frac{d}{dt}(\bar{A}.\bar{A})$

Solution: Let $\bar{A} = 5t^2\bar{\imath} + t\bar{\jmath} - t^3\bar{k}$ and $\bar{B} = sint\bar{\imath} - cost\bar{\jmath}$.

$\therefore \bar{A}.\bar{B} = 5t^2 sint - tcost$

$\therefore \frac{d}{dt}(\bar{A}.\bar{B}) = 10tsint + 5t^2cost - cost + tsint$

$\therefore \frac{d}{dt}(\bar{A}.\bar{B}) = 11t\sin t + 5t^2\cos t - \cos t.$

Now $\bar{A}.\bar{A} = 25t^4 + t^2 + t^6$

$\therefore \frac{d}{dt}(\bar{A}.\bar{A}) = 100t^3 + 2t + 6t^5$

==

Ex.: If $\bar{a} = t^2\bar{i} + t\bar{j} + (2t+1)\bar{k}$ and $\bar{b} = (2t-3)\bar{i} + \bar{j} - t\bar{k}$, then find i) $\frac{d}{dt}(\bar{a}.\bar{b})$, ii) $\frac{d}{dt}(\bar{a}\times\bar{b})$

Solution: Let $\bar{a} = t^2\bar{i} + t\bar{j} + (2t+1)\bar{k}$ and $\bar{b} = (2t-3)\bar{i} + \bar{j} - t\bar{k}$.

$\therefore \bar{a}.\bar{b} = t^2(2t-3) + t - t(2t+1) = 2t^3 - 3t^2 + t - 2t^2 - t = 2t^3 - 5t^2$

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ t^2 & t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$

$= (-t^2-2t-1)\,\bar{i} - (-t^3-4t^2-2t+6t+3)\,\bar{j} + (t^2-2t^2+3t)\bar{k}$

$= (-t^2-2t-1)\,\bar{i} + (t^3+4t^2-4t-3)\,\bar{j} + (-t^2+3t)\bar{k}$

i) $\frac{d}{dt}(\bar{a}.\bar{b}) = 6t^2 - 10t$

At t = 1, we have

$\therefore \frac{d}{dt}(\bar{a}.\bar{b}) = 6 - 10 = -4$

ii) $\frac{d}{dt}(\bar{a}\times\bar{b}) = \frac{d}{dt}[(-t^2-2t-1)\,\bar{i} + (t^3+4t^2-4t-3)\,\bar{j} + (-t^2+3t)\bar{k}]$

$= (-2t-2)\,\bar{i} + (3t^2+8t-4)\,\bar{j} + (-2t+3)\bar{k}$

At t = 1, we have,

$\frac{d}{dt}(\bar{a}\times\bar{b}) = -4\,\bar{i} + 7\,\bar{j} + \bar{k}$

==

Ex.: Prove that $\frac{d}{dt}\left(\bar{r}.\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right) = \bar{r}.\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}$

Proof: Consider

LHS $= \frac{d}{dt}\left(\bar{r}.\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)$

$= \frac{d\bar{r}}{dt}.\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2} + \bar{r}.\frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2} + \bar{r}.\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}$

$= 0 + 0 + \bar{r}.\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}$

$= \bar{r}.\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}$

= RHS Hence proved.

==

Ex.: Find $\frac{d}{dt}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right]$ and $\frac{d^2}{dt^2}\left[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}\right]$

Proof: Consider

i) $\frac{d}{dt}[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}] = \frac{d}{dt}(\bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2})$

$= \frac{d\bar{r}}{dt}.\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2} + \bar{r}.\frac{d^2\bar{r}}{dt^2} \times \frac{d^2\bar{r}}{dt^2} + \bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^3\bar{r}}{dt^3}$

$= 0 + 0 + \bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^3\bar{r}}{dt^3}$

$= \bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^3\bar{r}}{dt^3}$

$= [\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}]$

ii) $\frac{d^2}{dt^2}[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}] = \frac{d}{dt}\{\frac{d}{dt}[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}]\} = \frac{d}{dt}[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}]$

$= [\frac{d\bar{r}}{dt}\ \frac{d\bar{r}}{dt}\ \frac{d^3\bar{r}}{dt^3}] + [\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}] + [\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dt^4}]$

$= 0 + [\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}] + [\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dt^4}]$

$\therefore \frac{d^2}{dt^2}[\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^2\bar{r}}{dt^2}] = [\bar{r}\ \frac{d^2\bar{r}}{dt^2}\ \frac{d^3\bar{r}}{dt^3}] + [\bar{r}\ \frac{d\bar{r}}{dt}\ \frac{d^4\bar{r}}{dt^4}]$

==

Curves in Space: Let $\overline{r(t)} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be a position vector of a point P(t), then

i) $\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$ is the tangent to the curve in space at P.

i) $\bar{T} = \frac{d\bar{r}}{ds} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}}$ is called unit tangent vector to the curve in space at P.

Where $\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$

ii) $\frac{d\bar{T}}{ds} = \frac{\frac{d\bar{T}}{dt}}{\frac{ds}{dt}}$ is the normal vector to the curve in space at P.

iii) $\bar{N} = \frac{\frac{d\bar{T}}{ds}}{\left|\frac{d\bar{T}}{ds}\right|}$ is an unit normal vector to the curve in space at P.

iv) $k = \left|\frac{d\bar{T}}{ds}\right|$ is the curvature of the curve in space at P.

v) $\rho = \frac{1}{k}$ is the radius of curvature at P.

Velocity: Let $\overline{r(t)} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be a position of a particle moving along a curve at time t, then $\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$ is called the velocity of a particle at time t.

Acceleration: Let $\overline{r(t)} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be a position of a particle moving along a curve at time t, then $\bar{a} = \frac{d\bar{V}}{dt} = \frac{d^2\bar{r}}{dt^2}$ is called an acceleration of a particle at time t.

Speed: Let $\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$ is velocity of a particle at time t, then $v = |\bar{v}|$ is called speed of a particle at time t.

==

Ex.: Find the tangential and normal components of acceleration of a particle.

Solution: Let $\overline{r(t)}$ = x(t)$\bar{i}$ + y(t)$\bar{j}$ + z(t)$\bar{k}$ be a position vector of a particle at time t,

then $\bar{v} = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$ is the velocity of a particle at time t.

Now $\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\bar{T} = v\bar{T}$ where $v = |\bar{v}| = \frac{ds}{dt}$ is speed of particle.

Which shows that velocity is always along the tangent to the curve.

i.e. Tangential component of velocity = v

and normal component of velocity = 0.

Now $\bar{a} = \frac{d\bar{V}}{dt} = \frac{d}{dt}(v\bar{T})$

$= \frac{dv}{dt}\bar{T} + v\frac{d\bar{T}}{dt}$

$= \frac{dv}{dt}\bar{T} + v\frac{d\bar{T}}{ds}\frac{ds}{dt}$

$= \frac{dv}{dt}\bar{T} + v(k\bar{N})v$ $\because \frac{d\bar{T}}{ds} = k\bar{N}$ *and* $\frac{ds}{dt} = v$

$= \frac{dv}{dt}\bar{T} + kv^2\bar{N}$

$\therefore$ Tangential component of acceleration $= \frac{dv}{dt}$

and normal component of acceleration $= kv^2$

==

Remark: i) As $\bar{T}$ is perpendicular to $\bar{N}$ $\therefore |\bar{a}|^2 = (\frac{dv}{dt})^2 + (kv^2)^2$

i.e.(Magnitude of acceleration)^2 = (Tangential component of acceleration)^2 +(Normal component of acceleration)^2

ii) Unit Tangent $\bar{T} = \frac{\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|}$

iii) Tangential component of acceleration $= \ddot{\bar{r}}.\bar{T}$

iv) Normal component of acceleration $= \sqrt{|\bar{a}|^2 - (\ddot{\bar{r}}.\bar{T})^2}$

==

Ex.: Find unit tangent vector to any point on the curve x = acost, y = asint, z = bt

Solution: The position vector of any point P(x, y, z) for the given curve

x = acost, y = asint, z = bt is

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ = acost $\bar{i}$ + asint $\bar{j}$ + bt$\bar{k}$

$\therefore$ The tangent vector to the curve at point P(x, y, z) is

$\frac{d\bar{r}}{dt}$ = -asint $\bar{i}$ + acost $\bar{j}$ + b$\bar{k}$

$\therefore \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{(-asint)^2 + (acost)^2 + b^2} = \sqrt{a^2 + b^2}$

$\therefore$ The unit tangent vector to the curve at point P(x, y, z) is

$\bar{T} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}} = \frac{1}{\sqrt{a^2+b^2}}$(-asint $\bar{i}$ +acost $\bar{j}$ + b$\bar{k}$)

==

Ex.: A curve is given by the equations $x = t^2+1$, $y = 4t-3$, $z = 2t^2+ 6t$. Find the angle between tangents at t = 1 and at t = 2

Solution: The position vector of a point P(x, y, z) for the given curve

$x = t^2+1$, $y = 4t-3$, $z = 2t^2+ 6t$ is

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = (t^2+1)\,\bar{i} + (4t - 3)\,\bar{j} + (2t^2 + 6t)\bar{k}$

∴ The tangent vector to the curve at point P(x, y, z) is

$\frac{d\bar{r}}{dt} = 2t\,\bar{i} + 4\,\bar{j} + (4t + 6)\bar{k}$

∴ Tangents at t = 1 and at t = 2 are

$\bar{T_1} = [\frac{d\bar{r}}{dt}]_{t=1} = 2\bar{i} + 4\bar{j} + 10\bar{k} = 2(\bar{i} + 2\bar{j} + 5\bar{k})$ and

$\bar{T_2} = [\frac{d\bar{r}}{dt}]_{t=2} = 4\bar{i} + 4\bar{j} + 14\bar{k} = 2(2\bar{i} + 2\bar{j} + 7\bar{k})$

∴ $T_1 = |\bar{T_1}| = 2\sqrt{1^2 + 2^2 + 5^2} = 2\sqrt{30}$ and

$T_2 = |\bar{T_2}| = 2\sqrt{2^2 + 2^2 + 7^2} = 2\sqrt{57}$

∴ The angle θ between this tangents $\bar{T_1}$ and $\bar{T_2}$ is given by

$\cos\theta = \frac{\bar{T_1}.\bar{T_2}}{T_1 T_2} = \frac{4[2+4+35]}{4\sqrt{30}\sqrt{57}} = \frac{41}{3\sqrt{190}}$ i.e. $\theta = \cos^{-1}(\frac{41}{3\sqrt{190}})$

==

Ex.: If $\bar{a}, \bar{b}, \bar{c}$ are constant vectors, then $\bar{r} = t^2\bar{a} + t\bar{b} + \bar{c}$ is the path of a particle moving with constant acceleration.

Proof: Let $\bar{r} = t^2\bar{a} + t\bar{b} + \bar{c}$ be the path of a particle, where $\bar{a}, \bar{b}, \bar{c}$ are constant vectors.

∴ Velocity and acceleration of particle are

$\bar{v} = \frac{d\bar{r}}{dt} = 2t\,\bar{a} + \bar{b}$ and $\bar{a} = \frac{d\bar{v}}{dt} = 2\bar{a}$

Here the acceleration of particle is constant.

Thus the particle with path $\bar{r} = t^2\bar{a} + t\bar{b} + \bar{c}$ is moving with constant acceleration is proved.

==

Ex.: For the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$. Find the velocity and acceleration of the particle moving along the curve at t = 0.

Solution: Let a particle moves along the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$

∴ The position vector of a particle is

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = e^t\cos t\,\bar{i} + e^t\sin t\,\bar{j} + e^t\bar{k}$

∴ The velocity and acceleration of a particle at any time t are

$\bar{v} = \frac{d\bar{r}}{dt}$ = e^t(cost-sint) $\bar{\imath}$ + e^t (sint + cost) $\bar{\jmath}$ + $e^t\bar{k}$ and

$\bar{a} = \frac{d\bar{v}}{dt}$ = e^t(cost-sint-sint-cost) $\bar{\imath}$ + e^t(sint + cost + cost − sint) $\bar{\jmath}$ + $e^t\bar{k}$

= -2e^tsint $\bar{\imath}$ + 2e^tcost $\bar{\jmath}$ + $e^t\bar{k}$

∴ The velocity and acceleration of a particle at time t = 0 are

$\bar{v} = \bar{\imath} + \bar{\jmath} + \bar{k}$ and $\bar{a} = 2\bar{\jmath} + \bar{k}$

==

Ex.: A particle moves along the curve x = 4cost, y = 4sint, z = 6t. Find the velocity and acceleration at time t = 0, t = $\frac{\pi}{2}$. Also find the magnitude of the velocity and acceleration at any time t

Solution: Let a particle moves along the curve x = 4cost, y = 4sint, z = 6t

∴ The position vector of a particle is

$\bar{r}$ = x$\bar{\imath}$ + y$\bar{\jmath}$ + z$\bar{k}$ = 4cost $\bar{\imath}$ +4sint $\bar{\jmath}$ + 6t$\bar{k}$

∴ The velocity and acceleration of a particle at any time t are

$\bar{v} = \frac{d\bar{r}}{dt}$ = -4sint $\bar{\imath}$ +4cost $\bar{\jmath}$ + 6$\bar{k}$ and

$\bar{a} = \frac{d\bar{v}}{dt}$ = -4cost $\bar{\imath}$ -4sint $\bar{\jmath}$

∴ The velocity and acceleration at time t = 0 are

$\bar{v}$ = 4 $\bar{\jmath}$ + 6$\bar{k}$ and

$\bar{a} = \frac{d\bar{v}}{dt}$ = -4 $\bar{\imath}$

Again the velocity and acceleration at time t = $\frac{\pi}{2}$ are

$\bar{v}$ = -4 $\bar{\imath}$ + 6$\bar{k}$ and

$\bar{a} = \frac{d\bar{v}}{dt}$ = - 4$\bar{\jmath}$

Now the magnitude of the velocity and acceleration at any time t

∴ $|\bar{v}| = \sqrt{(-4sint)^2 + (4cost)^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$ and

$|\bar{a}| = \sqrt{(-4cost)^2 + (-4sint)^2}$ = 4.

==

Ex.: For the curve x = cost+tsint, y = sint - tcost. Find the tangential and normal components of acceleration at any time t.

Solution: Let a particle moves along the curve x = cost+tsint, y = sint - tcost

∴ The position vector of a particle is

$\bar{r}$ = x$\bar{\imath}$ + y$\bar{\jmath}$ + z$\bar{k}$ = (cost+tsint)$\bar{\imath}$ + (sint - tcost) $\bar{\jmath}$

∴ The velocity and acceleration of a particle at any time t are

$\bar{v} = \frac{d\bar{r}}{dt}$ = (-sint+sint+tcost) $\bar{\imath}$ + (cost − cost + tsint) $\bar{\jmath}$ = tcost$\bar{\imath}$ +tsint$\bar{\jmath}$ and

$\bar{a} = \frac{d\bar{v}}{dt}$ = (cost-tsint) $\bar{\imath}$ + (sint + tcost) $\bar{\jmath}$

Now $\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{(tcost)^2 + (tsint)^2}$ = t

∴ The unit tangent vector is

$\bar{T} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}} = \frac{1}{t}$(tcost$\bar{\imath}$ +tsint$\bar{\jmath}$)= cost$\bar{\imath}$ + sint$\bar{\jmath}$

∴ The tangential component of acceleration at any time t = $\bar{a}.\bar{T}$

= [(cost-tsint) $\bar{\imath}$ + (sint + tcost) $\bar{\jmath}$].(cost$\bar{\imath}$ + sint$\bar{\jmath}$)

= $\cos^2$t – tsintcost + $\sin^2$t + tcostsint

= 1

And the normal component of acceleration at any time t $=\sqrt{|\bar{a}|^2 - (\bar{a}.\bar{T})^2}$

$=\sqrt{(\text{cost} - \text{tsint})^2 + (\text{sint} + \text{tcost})^2 - 1}$

$=\sqrt{cos^2\text{t} - 2\text{tcostsint} + t^2\sin^2 t + sin^2 t + t\text{sintcost} + t^2 cos^2 t - 1}$

$=\sqrt{1 + t^2 - 1}$

= t

==

Ex.: For the curve x = t^3+1, y = t^2, z = t. Find the magnitude of tangential and normal components of acceleration for a particle moving on the curve at t = 1.

Solution: Let a particle moves along the curve x = t^3+1, y = t^2, z = t.

∴ The position vector of a particle at time t is

$\bar{r}$ = x$\bar{\imath}$ + y$\bar{\jmath}$ + z$\bar{k}$ = (t^3+1)$\bar{\imath}$ + $t^2\bar{\jmath}$ + t$\bar{k}$

∴ The velocity and acceleration of a particle at any time t are

$\bar{v} = \frac{d\bar{r}}{dt}$ = $3t^2\bar{\imath}$ + 2t$\bar{\jmath}$ + $\bar{k}$ and $\bar{a} = \frac{d\bar{v}}{dt}$ = 6t$\bar{\imath}$ + 2$\bar{\jmath}$

∴ The velocity and acceleration of a particle at time t = 1 are

$\bar{v} = \frac{d\bar{r}}{dt}$ = 3$\bar{\imath}$ + 2$\bar{\jmath}$ + $\bar{k}$ and $\bar{a} = \frac{d\bar{v}}{dt}$ = 6$\bar{\imath}$ + 2$\bar{\jmath}$

Now $\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right| = \sqrt{9 + 4 + 1} = \sqrt{14}$

∴ The unit tangent vector to the curve at t = 1 is

$\bar{T} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}} = \frac{1}{\sqrt{14}}$(3$\bar{\imath}$ + 2$\bar{\jmath}$ + $\bar{k}$)

∴ The tangential component of acceleration = $\bar{a}.\bar{T}$

= (6$\bar{\imath}$ + 2$\bar{\jmath}$). $\frac{1}{\sqrt{14}}$(3$\bar{\imath}$ + 2$\bar{\jmath}$ + $\bar{k}$)

= $\frac{1}{\sqrt{14}}$(18+4)

= $\frac{22}{\sqrt{14}}$

And the normal component of acceleration at any time t $=\sqrt{|\bar{a}|^2 - (\bar{a}.\bar{T})^2}$

$$= \sqrt{6^2+2^2-\left(\frac{22}{\sqrt{14}}\right)^2}$$

$$= \sqrt{40-\frac{484}{14}}$$

$$= \sqrt{\frac{76}{14}}$$

$$= \sqrt{\frac{38}{7}}$$

==

Vector functions of two and three variables:

i)Let A and B be the non-empty subsets of set of real numbers R and W be a non-empty subset of R^3, then a function $\bar{v}$: A×B→W defined by $\bar{v}=v_1(x, y)\bar{\imath}+v_2(x, y)\,\bar{\jmath}+v_3(x, y)\,\bar{k}$ is called a vector function of two variables x, y.

ii)Let A, B and C be the non-empty subsets of set of real numbers R and W be a non-empty subset of R^3, then a function $\bar{v}$: A×B×C→W defined by $\bar{v} = v_1(x, y, z)\bar{\imath} +v_2(x, y, z)\,\bar{\jmath} +v_3(x, y, z)\,\bar{k}$ is called a vector function of three variables x, y and z.

Limit of Vector Function of Two Variables:

Let $\bar{v}(x,y) = v_1(x, y)\bar{\imath} +v_2(x, y)\,\bar{\jmath} +v_3(x, y)\,\bar{k}$ be a vector function of two variables x, y. If for small ε > 0, there exist $\delta > 0$ depends on ε such that $\left|\bar{v}(x,y)-\bar{l}\right| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$.

Then $\bar{l}$ is said to be limit of $\bar{v}(x,y)$ as (x, y) → (a, b).

Denoted by $\lim_{(x,y)\to(a,b)} \bar{v}(x,y) = \bar{l}$.

Continuity: A vector function $\bar{v} = \bar{v}(x,y)$ of a scalar variables x, y is said to be continuous at (a, b) if $\bar{v}(a,b)$ is defined, $\lim_{(x,y)\to(a,b)} \bar{v}(x,y)$ is exists and $\lim_{(x,y)\to(a,b)} \bar{v}(x,y) = \bar{v}(a,b)$.

Remark: A vector function $\bar{v}(x,y) = v_1(x, y)\bar{\imath} +v_2(x, y)\,\bar{\jmath} +v_3(x, y)\,\bar{k}$ is continuous at (a, b) if $v_1(x, y)$, $v_2(x, y)$, $v_3(x, y)$ are continuous at (a, b).

Partial Derivatives: Let $\bar{v} = \bar{v}(x,y)$ be a vector function of scalar variables x, y and $\overline{\delta v}$ be change in $\bar{v}$ corresponding to small changes δx in x.

If $\lim_{\delta x\to 0}\frac{\overline{\delta v}}{\delta x} = \lim_{\delta x\to 0}\frac{\bar{v}(x+\delta x,y)-\bar{v}(x,y)}{\delta x}$ exist and finite, then $\bar{v}(x,y)$ is said to be partially differentiable w.r.t.x and $\frac{\overline{\partial v}}{\partial x} = \lim_{\delta x\to 0}\frac{\bar{v}(x+\delta x,y)-\bar{v}(x,y)}{\delta x}$ is called partial derivative of $\bar{v}$ w.r.t.x.

Remark: If $\bar{v}(x,y) = v_1(x, y)\bar{\imath} +v_2(x, y)\,\bar{\jmath} +v_3(x, y)\,\bar{k}$, then $\frac{\overline{\partial v}}{\partial x} = \frac{\overline{\partial v_1}}{\partial x}\,\bar{\imath} + \frac{\overline{\partial v_2}}{\partial x}\,\bar{\jmath} + \frac{\overline{\partial v_3}}{\partial x}\,\bar{k}$

Results: i) $\frac{\partial}{\partial x}(\bar{u} \pm \bar{v}) = \frac{\partial \bar{u}}{\partial x} \pm \frac{\partial \bar{v}}{\partial x}$

ii) $\frac{\partial}{\partial x}(\bar{u}.\bar{v}) = \bar{u}.\frac{\partial \bar{v}}{\partial x} + \bar{v}.\frac{\partial \bar{u}}{\partial x}$

iii) $\frac{\partial}{\partial x}(\bar{u} \times \bar{v}) = \bar{u} \times \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial x} \times \bar{v}$

iv) $\frac{\partial}{\partial x}(\phi\bar{u}) = \phi\frac{\partial \bar{u}}{\partial x} + \frac{\partial \phi}{\partial x}\bar{u}$

Total Differential: If $\bar{v} = \bar{v}(x, y, z)$ be a vector function of scalar variables x, y and z, then it's total differential is $d\bar{v} = \frac{\partial \bar{v}}{\partial x}dx + \frac{\partial \bar{v}}{\partial y}dy + \frac{\partial \bar{v}}{\partial z}dz$.

Note: If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ and $d\bar{r} = dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k}$ then $\bar{r}.d\bar{r} = xdx + ydy + zdz$

==

Ex.: If $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$, find i) $\frac{\partial \bar{r}}{\partial x}$ ii) $\frac{\partial \bar{r}}{\partial y}$ iii) $\frac{\partial^2 \bar{r}}{\partial x^2}$ iv) $\frac{\partial^2 \bar{r}}{\partial y^2}$ v) $\frac{\partial^2 \bar{r}}{\partial x\,\partial y}$

Solution: Let $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$,

i) $\frac{\partial \bar{r}}{\partial x} = \cos y\,\bar{\imath} + \sin y\,\bar{\jmath}$

ii) $\frac{\partial \bar{r}}{\partial y} = -x\sin y\,\bar{\imath} + x\cos y\,\bar{\jmath} + ame^{my}\,\bar{k}$

iii) $\frac{\partial^2 \bar{r}}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial \bar{r}}{\partial x}) = \frac{\partial}{\partial x}(\cos y\,\bar{\imath} + \sin y\,\bar{\jmath}) = \bar{0}$

iv) $\frac{\partial^2 \bar{r}}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial \bar{r}}{\partial y}) = \frac{\partial}{\partial y}(-x\sin y\,\bar{\imath} + x\cos y\,\bar{\jmath} + ame^{my}\,\bar{k})$

$= -x\cos y\,\bar{\imath} - x\sin y\,\bar{\jmath} + am^2e^{my}\,\bar{k}$

iv) $\frac{\partial^2 \bar{r}}{\partial x\,\partial y} = \frac{\partial}{\partial x}(\frac{\partial \bar{r}}{\partial y}) = \frac{\partial}{\partial x}(-x\sin y\,\bar{\imath} + x\cos y\,\bar{\jmath} + ame^{my}\,\bar{k})$

$= -\sin y\,\bar{\imath} + \cos y\,\bar{\jmath}$

==

Ex.: If $\bar{r} = \frac{a}{2}(x + y)\,\bar{\imath} + \frac{b}{2}(x - y)\,\bar{\jmath} + \frac{xy}{2}\,\bar{k}$,

find i) $\frac{\partial \bar{r}}{\partial x}$ ii) $\frac{\partial \bar{r}}{\partial y}$ iii) $\frac{\partial^2 \bar{r}}{\partial x^2}$ iv) $\frac{\partial^2 \bar{r}}{\partial y^2}$ v) $\frac{\partial^2 \bar{r}}{\partial x\,\partial y}$

Solution: Let $\bar{r} = \frac{a}{2}(x + y)\,\bar{\imath} + \frac{b}{2}(x - y)\,\bar{\jmath} + \frac{xy}{2}\,\bar{k}$,

ii) $\frac{\partial \bar{r}}{\partial x} = \frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + \frac{y}{2}\,\bar{k}$

ii) $\frac{\partial \bar{r}}{\partial y} = \frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + \frac{x}{2}\,\bar{k}$

iii) $\frac{\partial^2 \bar{r}}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial \bar{r}}{\partial x}) = \frac{\partial}{\partial x}(\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + \frac{y}{2}\,\bar{k}) = \bar{0}$

iv) $\frac{\partial^2 \bar{r}}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial \bar{r}}{\partial y}) = \frac{\partial}{\partial y}(\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + \frac{x}{2}\,\bar{k}) = \bar{0}$

v) $\frac{\partial^2 \bar{r}}{\partial x\,\partial y} = \frac{\partial}{\partial x}(\frac{\partial \bar{r}}{\partial y}) = \frac{\partial}{\partial x}(\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + \frac{x}{2}\,\bar{k}) = \frac{1}{2}\,\bar{k}$

==

Ex.: If $\bar{r} = \frac{a}{2}(x + y)\,\bar{\imath} + \frac{b}{2}(x - y)\,\bar{\jmath} + xy\,\bar{k}$,

find i) $[\frac{\partial \bar{r}}{\partial x}\ \frac{\partial \bar{r}}{\partial y}\ \frac{\partial^2 \bar{r}}{\partial x^2}]$ ii) $[\frac{\partial \bar{r}}{\partial x}\ \frac{\partial \bar{r}}{\partial y}\ \frac{\partial^2 \bar{r}}{\partial x\,\partial y}]$

Solution: Let $\bar{r} = \frac{a}{2}(x+y)\,\bar{\imath} + \frac{b}{2}(x-y)\,\bar{\jmath} + xy\,\bar{k}$,

$$\therefore \frac{\partial \bar{r}}{\partial x} = \frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + y\,\bar{k}$$

$$\frac{\partial \bar{r}}{\partial y} = \frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + x\,\bar{k}$$

$$\frac{\partial^2 \bar{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{r}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + y\,\bar{k}\right) = \bar{0}$$

$$\& \ \frac{\partial^2 \bar{r}}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{r}}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + x\,\bar{k}\right) = \bar{k}$$

$$\text{vi) } \left[\frac{\partial \bar{r}}{\partial x}\ \frac{\partial \bar{r}}{\partial y}\ \frac{\partial^2 \bar{r}}{\partial x^2}\right] = \begin{vmatrix} \frac{a}{2} & \frac{b}{2} & y \\ \frac{a}{2} & -\frac{b}{2} & x \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\text{ii)} \left[\frac{\partial \bar{r}}{\partial x}\ \frac{\partial \bar{r}}{\partial y}\ \frac{\partial^2 \bar{r}}{\partial x\,\partial y}\right] = \begin{vmatrix} \frac{a}{2} & \frac{b}{2} & y \\ \frac{a}{2} & -\frac{b}{2} & x \\ 0 & 0 & 1 \end{vmatrix} = \frac{a}{2}\left(-\frac{b}{2} - 0\right) - \frac{b}{2}\left(\frac{a}{2} - 0\right) + y(0 - 0)$$

$$= -\frac{ab}{4} - \frac{ab}{4}$$

$$= -\frac{ab}{2}$$

==

Ex.: If $\bar{u} = x^2yz\,\bar{\imath} - 2xz^3\,\bar{\jmath} + xz^2\,\bar{k}$ and $\bar{v} = 2z\,\bar{\imath} + y\,\bar{\jmath} - x^2\,\bar{k}$

find $\frac{\partial^2}{\partial x\,\partial y}(\bar{u} \times \bar{v})$ at (1, 0, 2)

Solution: Let $\bar{u} = x^2yz\,\bar{\imath} - 2xz^3\,\bar{\jmath} + xz^2\,\bar{k}$ and $\bar{v} = 2z\,\bar{\imath} + y\,\bar{\jmath} - x^2\,\bar{k}$

$$\therefore \bar{u} \times \bar{v} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\,\bar{\imath} - (-x^4yz - 2xz^3)\,\bar{\jmath} + (x^2y^2z + 4xz^4)\,\bar{k}$$

$$= (2x^3z^3 - xyz^2)\,\bar{\imath} + (x^4yz + 2xz^3)\,\bar{\jmath} + (x^2y^2z + 4xz^4)\,\bar{k}$$

$$\therefore \frac{\partial}{\partial y}(\bar{u} \times \bar{v}) = (0 - xz^2)\,\bar{\imath} + (x^4z + 0)\,\bar{\jmath} + (2x^2yz + 0)\,\bar{k}$$

$$\therefore \frac{\partial}{\partial y}(\bar{u} \times \bar{v}) = -xz^2\,\bar{\imath} + x^4z\,\bar{\jmath} + 2x^2yz\,\bar{k}$$

$$\therefore \frac{\partial^2}{\partial x\,\partial y}(\bar{u} \times \bar{v}) = -z^2\,\bar{\imath} + 4x^3z\,\bar{\jmath} + 4xyz\,\bar{k}$$

$$\therefore \left[\frac{\partial^2}{\partial x\,\partial y}(\bar{u} \times \bar{v})\right]_{(1,\,0,\,2)} = -4\,\bar{\imath} + 8\,\bar{\jmath} + 0\,\bar{k} = -4(\bar{\imath} - 2\,\bar{\jmath})$$

==

Ex.: If $\bar{u} = z^3\,\bar{\imath} - x^2\,\bar{k}$, $\bar{v} = 2xyz\,\bar{\jmath}$ and $\bar{w} = 5xy\,\bar{\imath} + 3z\,\bar{\jmath}$,

then find $\frac{\partial^3}{\partial x\,\partial y\,\partial z}(\bar{u} \times \bar{v}.\bar{w})$

Solution: Let $\bar{u} = z^3\,\bar{\imath} - x^2\,\bar{k}$, $\bar{v} = 2xyz\,\bar{\jmath}$ and $\bar{w} = 5xy\,\bar{\imath} + 3z\,\bar{\jmath}$

$$\therefore \bar{u} \times \bar{v}\,.\bar{w} = \begin{vmatrix} z^3 & 0 & -x^2 \\ 0 & 2xyz & 0 \\ 5xy & 3z & 0 \end{vmatrix}$$

$$= z^3(0 - 0) - 0 - x^2(0 - 10x^2y^2z)$$

$$= 10x^4y^2z$$

$$\therefore \frac{\partial}{\partial z}(\bar{u} \times \bar{v}\,.\bar{w}) = 10x^4y^2$$

$$\therefore \frac{\partial^2}{\partial y\,\partial z}(\bar{u} \times \bar{v}\,.\bar{w}) = 20x^4y$$

$$\therefore \frac{\partial^3}{\partial x\,\partial y\,\partial z}(\bar{u} \times \bar{v}\,.\bar{w}) = 80x^3y$$

==

Ex.: If $\phi = xy^2z$ and $\bar{u} = xz\,\bar{i} - xy^2\,\bar{j} + yz^2\bar{k}$, then find $\frac{\partial^3}{\partial x^2\,\partial z}(\phi\bar{u})$ at (2, -1, 1)

Solution: Let $\phi = xy^2z$ and $\bar{u} = xz\,\bar{i} - xy^2\,\bar{j} + yz^2\bar{k}$

$$\therefore \phi\bar{u} = (xy^2z)(xz\,\bar{i} - xy^2\,\bar{j} + yz^2\bar{k})$$

$$= x^2y^2z^2\,\bar{i} - x^2y^4z\,\bar{j} + xy^3z^3\,\bar{k}$$

$$\therefore \frac{\partial}{\partial z}(\phi\bar{u}) = 2x^2y^2z\,\bar{i} - x^2y^4\,\bar{j} + 3xy^3z^2\,\bar{k}$$

$$\therefore \frac{\partial^2}{\partial x\,\partial z}(\phi\bar{u}) = 4xy^2z\,\bar{i} - 2xy^4\,\bar{j} + 3y^3z^2\,\bar{k}$$

$$\therefore \frac{\partial^3}{\partial x^2\,\partial z}(\phi\bar{u}) = 4y^2z\,\bar{i} - 2y^4\,\bar{j} + 0\bar{k}$$

$$\therefore \frac{\partial^3}{\partial x^2\,\partial z}(\phi\bar{u}) = 4y^2z\,\bar{i} - 2y^4\,\bar{j}$$

$$\therefore \left[\frac{\partial^3}{\partial x^2\,\partial z}(\phi\bar{u})\right]_{(2,-1,1)} = 4\,\bar{i} - 2\,\bar{j} = 2(2\bar{i} - \bar{j})$$

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MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) A function $\bar{v}: R \to R^3$ defined by $\bar{v} = v_1(t)\bar{i} + v_2(t)\,\bar{j} + v_3(t)\,\bar{k}$ is called a ……function of a single variable t.

A) scalar B) vector C) analytic D) None of these

2) If for small ε > 0, there exist $\delta > 0$ depends on ε such that $|\bar{v}(t) - \bar{l}| < \varepsilon$ whenever $0 < |t - a| < \delta$, then $\lim_{t \to a} \bar{v}(t) = \ldots\ldots$

A) $\bar{l}$ B) 0 C) a D) None of these

3) If $\lim_{t \to a} \bar{u}(t) = \bar{l}$ and $\lim_{t \to a} \bar{v}(t) = \bar{m}$, then $\lim_{t \to a}[\bar{u}(t) \pm \bar{v}(t)] = \ldots\ldots$

A) $\frac{\bar{l}}{\bar{m}}$ B) $\bar{l}.\bar{m}$ C) $\bar{l} \pm \bar{m}$ D) None of these

4) If $\lim_{t \to a} \bar{u}(t) = \bar{l}$ and $\lim_{t \to a} \bar{v}(t) = \bar{m}$, then $\lim_{t \to a}[\bar{u}(t).\bar{v}(t)] = \ldots\ldots$

A) $\frac{\bar{l}}{\bar{m}}$ B) $\bar{l}.\bar{m}$ C) $\bar{l} \pm \bar{m}$ D) None of these

5) If $\lim_{t\to a}\bar{u}(t) = \bar{l}$ and $\lim_{t\to a}\bar{v}(t) = \bar{m}$, then $\lim_{t\to a}[\frac{\bar{u}(t)}{\bar{v}(t)}] = \frac{\bar{l}}{\bar{m}}$ provided …..

A) $\bar{m} \neq \bar{0}$ B) $\bar{l} \neq \bar{0}$ C) $\bar{m} = \bar{0}$ D) $\bar{l} = \bar{0}$

6) A vector function $\bar{v} = \bar{v}(t)$ of a scalar variable t is said to be continuous at $t = t_0$ if $\lim_{t\to t_0}\bar{v}(t) = ……$

A) $\bar{v}(t)$ B) $\bar{v}(t_0)$ C) t_0 D) None of these

7) A vector function $\bar{v} = \bar{v}(t)$ of a scalar variable t is said to be continuous in an interval (a, b) if it is continuous at ……point in (a, b)

A) every B) some C) a and b only D) None of these

8) Vector $\bar{v}(t)$ is said to be differentiable w.r.t.t, if $\lim_{\delta t\to 0}\frac{\bar{v}(t+\delta t)-\bar{v}(t)}{\delta t}$ is ……

A) exist and finite B) exist and infinite C) not exist D) None of these

9) If $\lim_{t\to t_0}\frac{\bar{v}(t)-\bar{v}(t_0)}{t-t_0}$ is exists and finite then it is denoted by ……

A) $\overline{v'}(t)$ B) $\overline{v'}(t_0)$ C) $\bar{v}(t_0)$ D) None of these

10) $\frac{d^2\bar{v}}{dt^2} = \frac{d}{dt}(\frac{\overline{dv}}{dt})$ is called …… order derivative of $\bar{v}$ w.r.t.t.

A) first B) second C) third D) None of these

11) $\frac{d^3\bar{v}}{dt^3} = \frac{d}{dt}(\frac{d^2\bar{v}}{dt^2})$ is called …… order derivative of $\bar{v}$ w.r.t.t.

A) first B) second C) third D) None of these

12) Statement 'Every differentiable vector function is continuous' is…

A) true B) false C) both true and false D) None of these

13) Statement 'Every continuous vector function is differentiable' is…

A) true B) false C) both true and false D) None of these

14) At point t = 0, $\bar{v}(t) = t\bar{\imath} + |t|\,\bar{\jmath}$ is …….

A) both continuous and differentiable B) differentiable

C) continuous but not differentiable D) None of these

15) If $\bar{u}$ and $\bar{v}$ are differentiable vector functions of scalar variable t, then $\frac{d}{dt}(\bar{u}.\bar{v}) = ……$

A) $\frac{d\bar{u}}{dt}.\frac{d\bar{v}}{dt}$ B) $\bar{u}.\frac{d\bar{v}}{dt} + \bar{v}.\frac{d\bar{u}}{dt}$ C) $\bar{u}.\frac{d\bar{v}}{dt} - \bar{v}.\frac{d\bar{u}}{dt}$ D) None of these

16) If $\bar{u}$ is differentiable vector function of scalar variable t, then $\frac{d\bar{u}^2}{dt} = ……$

A) $2\bar{u}.\frac{d\bar{u}}{dt}$ B) $2\bar{u} \times \frac{d\bar{u}}{dt}$ C) $2\bar{u} + \frac{d\bar{u}}{dt}$ D) None of these

17) If $\bar{u}$ is differentiable vector function of scalar variable t with $u = |\bar{u}|$, then $\bar{u}.\frac{d\bar{u}}{dt} = ……$

A) $u.\frac{du}{dt}$ B) $u\,\frac{du}{dt}$ C) $u\times\frac{du}{dt}$ D) None of these

18) If $\bar{u}$ and $\bar{v}$ are differentiable vector functions of scalar variable t, then $\frac{d}{dt}(\bar{u} \times \bar{v}) = \ldots\ldots$

A) $\frac{d\bar{u}}{dt} \times \frac{d\bar{v}}{dt}$ B) $\bar{u} \times \frac{d\bar{v}}{dt} + \bar{v} \times \frac{d\bar{u}}{dt}$ C) $\bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$ D) None of these

19) $\frac{d}{dt}\ \bar{u} \times (\bar{v} \times \bar{w}) = \ldots\ldots$

A) $\frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times (\frac{d\bar{v}}{dt} \times \bar{w}) + \bar{u} \times (\bar{v} \times \frac{d\bar{w}}{dt})$

B) $\frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt} + \frac{d\bar{w}}{dt})$ C) $\frac{d\bar{u}}{dt} \times \frac{d\bar{v}}{dt} \times \frac{d\bar{w}}{dt}$ D) None of these

20) $\frac{d}{dt}\ [\bar{u}\ \bar{v}\ \bar{w}] = \ldots\ldots$

A) $\frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt} + \frac{d\bar{w}}{dt}]$ B) $[\frac{d\bar{u}}{dt}\bar{v}\ \bar{w}] + [\bar{u}\frac{d\bar{v}}{dt}\bar{w}] + [\bar{u}\ \bar{v}\frac{d\bar{w}}{dt}]$

C) $[\frac{d\bar{u}}{dt}\frac{d\bar{v}}{dt}\ \frac{d\bar{w}}{dt}]$ D) None of these

21) If a vector function $\bar{u}$ and a scalar function ϕ are differentiable functions of scalar variable t, then $\frac{d}{dt}(\phi\bar{u}) = \ldots\ldots$

A) $\phi.\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}.\bar{u}$ B) $\phi \times \frac{d\bar{u}}{dt} + \frac{d\phi}{dt} \times \bar{u}$ C) $\phi\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}\bar{u}$ D) None of these

22) If k is constant scalar, then $\frac{d}{dt}(k\bar{u}) = \ldots\ldots$

A) $k\frac{d\bar{u}}{dt}$ B) $k\frac{d\bar{u}}{dt} + \frac{dk}{dt}\bar{u}$ C) 0 D) None of these

23) If $\bar{u}$ a differentiable vector function of a scalar s and s is the differentiable scalar function of scalar variable t, then $\frac{d\bar{u}}{dt} = \frac{ds}{dt}\frac{d\bar{u}}{ds}$

A) $\frac{ds}{dt} - \frac{d\bar{u}}{ds}$ B) $\frac{ds}{dt}\frac{d\bar{u}}{ds}$ C) $\frac{ds}{dt} + \frac{d\bar{u}}{ds}$ D) None of these

24) If $\bar{f}(t) = f_1(t)\bar{\imath} + f_2(t)\bar{\jmath} + f_3(t)\bar{k}$ is a differentiable vector function of a scalar variable t, then $\frac{d}{dt}\bar{f}(t) = \ldots\ldots$

A) $f_1(t)\bar{\imath} + f_2(t)\bar{\jmath} + f_3(t)\bar{k}$ B) $\bar{\imath} + \bar{\jmath} + \bar{k}$

C) $\frac{df_1(t)}{dt}\bar{\imath} + \frac{df_2(t)}{dt}\bar{\jmath} + \frac{df_3(t)}{dt}\bar{k}$ D) None of these

25) If $\bar{u}(t)$ is constant vector on [a, b], then …… on [a, b].

A) $\frac{d\bar{u}}{dt} = \bar{0}$ B) $\frac{d\bar{u}}{dt} \neq \bar{0}$ C) $\frac{d\bar{u}}{dt} = \bar{1}$ D) None of these

26) If $\frac{d\bar{u}}{dt} = \bar{0}\ \forall\ t \in [a, b]$, then $\bar{u}(t)$ is a …… on [a, b].

A) of constant magnitude B) of constant direction

C) constant vector D) None of these

27) If a differentiable vector $\bar{u}(t)$ is of constant magnitude, then …… ∀ t ∈ [a, b]

A) $\bar{u}.\frac{d\bar{u}}{dt} \neq 0$ B) $\bar{u}.\frac{d\bar{u}}{dt} = 0$ C) $\bar{u}.\frac{d\bar{u}}{dt} = 1$ D) None of these

28) If $\bar{u}.\frac{d\bar{u}}{dt} = 0 \; \forall \; t \in [a, b]$, then $\bar{u}(t)$ is …… on [a, b]

A) of constant magnitude
B) of constant direction
C) constant vector
D) None of these

29) If a non-constant vector $\bar{u}(t)$ is of constant direction, then …… $\forall \; t \in [a, b]$

A) $\bar{u} \times \frac{d\bar{u}}{dt} \neq \bar{0}$
B) $\bar{u} \times \frac{d\bar{u}}{dt} = \bar{0}$
C) $\bar{u}.\frac{d\bar{u}}{dt} = 0$
D) None of these

30) If $\bar{u} \times \frac{d\bar{u}}{dt} = \bar{0} \; \forall \; t \in [a, b]$, then a non $-$ constant vector $\bar{u}(t)$ is …… on [a, b]

A) of constant magnitude
B) of constant direction
C) constant vector
D) None of these

31) $\lim_{t \to 0}[(t^2 + 1)\bar{\imath} + (\frac{3^{2t}-1}{t})\bar{\jmath} + (1 + 2t)^{\frac{1}{t}}\bar{k}] = ……$

A) $\bar{\imath} + 2\log 3\,\bar{\jmath} + e^2\bar{k}$
B) $\bar{\imath} + \log 3\,\bar{\jmath} + e^2\bar{k}$
C) $\bar{\imath} + 2\log 3\,\bar{\jmath} + e\bar{k}$
D) None of these

32) If $\bar{f}(t) = \frac{\sin 2t}{t}\bar{\imath} + \cos t\,\bar{\jmath}$, $t \neq 0$ and $\bar{f}(0) = x\bar{\imath} + \bar{\jmath}$ is continuous at t = 0, then x = …

A) 0
B) 1
C) 2
D) None of these

33) If $\bar{f}(t) = \cos t\,\bar{\imath} + \sin t\,\bar{\jmath} + \tan t\,\bar{k}$, find $\bar{f}'(t) = ……$

A) $\cos t\,\bar{\imath} + \sin t\,\bar{\jmath} + \tan t\,\bar{k}$
B) $-\sin t\,\bar{\imath} + \cos t\,\bar{\jmath} + \sec^2 t\,\bar{k}$
C) $\cos t\,\bar{\imath} + \sin t\,\bar{\jmath}$
D) None of these

34) If $\bar{r} = (t^2+1)\bar{\imath} + (4t-3)\bar{\jmath} + (2t^2 - 6t)\bar{k}$, then $\frac{d\bar{r}}{dt}$ at t = 2 is ……

A) $4\bar{\imath} + 4\bar{\jmath} + 2\bar{k}$
B) $4\bar{\imath} + \bar{\jmath} + 2\bar{k}$
C) $4\bar{\imath} + 4\bar{\jmath} + \bar{k}$
D) None of these

35) If $\bar{r} = (t^2+1)\bar{\imath} + (4t-3)\bar{\jmath} + (2t^2 - 6t)\bar{k}$, then $\frac{d^2\bar{r}}{dt^2}$ at t = 2 is ……

A) $\bar{\imath} + 4\bar{\jmath} + 2\bar{k}$
B) $2\bar{\imath} + 4\bar{k}$
C) $4\bar{\imath} + \bar{\jmath} + 2\bar{k}$
D) None of these

36) If $\bar{r} = (t+1)\bar{\imath} + (t^2+t+1)\bar{\jmath} + (t^3 + t^2 + t + 1)\bar{k}$, then $\frac{d\bar{r}}{dt} = ……$

A) $\bar{\imath} + 2\bar{\jmath} + (6t + 2)\bar{k}$
B) $\bar{\imath} + 2\bar{\jmath}$
C) $\bar{\imath} + (2t+1)\bar{\jmath} + (3t^2 + 2t + 1)\bar{k}$
D) None of these

37) If $\bar{r} = (t+1)\bar{\imath} + (t^2+t+1)\bar{\jmath} + (t^3+t^2 + t + 1)\bar{k}$, then $\frac{d^2\bar{r}}{dt^2} = ……$

A) $2\bar{\jmath} + (6t + 2)\bar{k}$
B) $2\bar{\jmath} + (6t + 2)\bar{k}$
C) $2\bar{\jmath} + 6t\bar{k}$
D) None of these

38) If $\bar{r} = \sin t\,\bar{\imath} + \cos t\,\bar{\jmath} + t\bar{k}$, then $\frac{d\bar{r}}{dt} = ……$

A) $\cos t\,\bar{\imath} - \sin t\,\bar{\jmath} + \bar{k}$
B) $-\sin t\,\bar{\imath} + \cos t\,\bar{\jmath}$
C) $\cos t\,\bar{\imath} + \sin t\,\bar{\jmath} + \bar{k}$
D) None of these

39) If $\bar{r} = sint\bar{\imath} + \cos t\,\bar{\jmath} + t\bar{k}$, then $\frac{d^2\bar{r}}{dt^2} = \ldots\ldots$

A) $sint\bar{\imath} - \cos t\,\bar{\jmath}$ B) $sint\bar{\imath} + \cos t\,\bar{\jmath}$ C) $-sint\bar{\imath} - \cos t\,\bar{\jmath}$ D) None of these

40) If $\bar{r} = e^{-t}\bar{\imath} + \log(t^2+1)\,\bar{\jmath} - \tan t\bar{k}$, find $\frac{d\bar{r}}{dt}$ at t = 0.

A) $-\bar{\imath} - \bar{k}$ B) $\bar{\imath} + \bar{\jmath} - \bar{k}$ C) $-\bar{\imath} + \bar{\jmath} + \bar{k}$ D) None of these

41) If $\bar{r} = e^{-t}\bar{\imath} + \log(t^2+1)\,\bar{\jmath} - \tan t\bar{k}$, find $\left|\frac{d\bar{r}}{dt}\right|$ at t = 0.

A) $\sqrt{5}$ B) $\sqrt{3}$ C) $\sqrt{2}$ D) None of these

42) $\frac{d}{dt}(\bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}) = \ldots\ldots$

A) $\bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}$ B) $\bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^3\bar{r}}{dt^3}$ C) $\bar{r}.\frac{d\bar{r}}{dt} \times \frac{d^4\bar{r}}{dt^4}$ D) None of these

43) If $\bar{r} = (sinht)\bar{a} + (cosht)\bar{b}$, where $\bar{a}, \bar{b}$ are constant vectors, then $\frac{d^2\bar{r}}{dt^2} = \ldots\ldots$

A) $-\bar{r}$ B) $\bar{r}$ C) $\overline{2r}$ D) None of these

44) If $\bar{r} = cosnt\bar{\imath} + \sin nt\,\bar{\jmath}$, where n is constant, then $\bar{r}.\frac{d\bar{r}}{dt} = \ldots\ldots$

A) 0 B) 1 C) -1 D) None of these

45) Let $\overline{r(t)} = x(t)\bar{\imath} + y(t)\bar{\jmath} + z(t)\bar{k}$ be a position vector of a point P(t), then $\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{\imath} + \frac{dy}{dt}\bar{\jmath} + \frac{dz}{dt}\bar{k}$ is the …… to the curve in space at P.

A) unit tangent B) normal C) tangent D) None of these

46) Let $\overline{r(t)} = x(t)\bar{\imath} + y(t)\bar{\jmath} + z(t)\bar{k}$ be a position vector of a point P(t), then $\frac{d\bar{r}}{ds}$ is the …… to the curve in space at P.

A) unit tangent B) normal C) tangent D) None of these

47) Let $\overline{r(t)} = x(t)\bar{\imath} + y(t)\bar{\jmath} + z(t)\bar{k}$ be a position vector of a point P(t) and $\bar{T}$ is unit tangent vector to the curve at point P(t), then $\frac{d\bar{T}}{ds}$ is the …… to the curve in space at P.

A) unit normal B) normal C) tangent D) None of these

48) If $\frac{d\bar{T}}{ds}$ is normal to the curve at point P(t), then $\left|\frac{d\bar{T}}{ds}\right|$ is the …… of the curve.

A) unit normal B) radius of curvature C) curvature D) None of these

49) If $k = \left|\frac{d\bar{T}}{ds}\right|$ is the curvature of the curve, then $\frac{1}{k}$ is the…… of the curve.

A) unit normal B) radius of curvature C) curvature D) None of these

50) If $\overline{r(t)} = x(t)\bar{\imath} + y(t)\bar{\jmath} + z(t)\bar{k}$ is the position of a particle at time t, then $\frac{d\bar{r}}{dt}$ is the …… of a particle at time t.

A) velocity B) acceleration C) speed D) None of these

51) If $\bar{v} = \frac{d\bar{r}}{dt}$ is the velocity of a particle at time t, then $v = \left|\frac{d\bar{r}}{dt}\right|$ is the …… of a particle at time t.

A) velocity B) acceleration C) speed D) None of these

52) If $\overline{r(t)} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ is the position of a particle at time t, then $\frac{d^2\bar{r}}{dt^2}$ is the …… of a particle at time t.

A) velocity B) acceleration C) speed D) None of these

53) Tangential and normal component of velocity are …… and …… respectively.

A) v and 0 B) 0 and v C) $\frac{dv}{dt}$ and kv^2 D) None of these

54) Velocity of a particle is always along the …… to the curve.

A) normal B) tangent C) both normal and tangent D) None of these

55) Tangential and normal component of acceleration are …… and …… respectively.

A) k and v B) 0 and v C) $\frac{dv}{dt}$ and kv^2 D) None of these

56) Velocity of a particle moving along the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$ at time t = 0 is ……

A) $\bar{i} + \bar{j} + \bar{k}$ B) $\bar{i} + \bar{j} - \bar{k}$ C) $\bar{i} - \bar{j} + \bar{k}$ D) None of these

57) Acceleration of a particle moving along the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = e^t$ at time t = 0 is ……

A) $\bar{i} + \bar{j} + \bar{k}$ B) $\bar{i} + \bar{j} - \bar{k}$ C) $2\bar{j} + \bar{k}$ D) None of these

58) Velocity of a particle moving along the curve x = 4cost, y = 4sint, z = 6t at time t = 0 is ……

A) $\bar{i} + \bar{j} + \bar{k}$ B) $4\bar{j} + 6\bar{k}$ C) $\bar{i} - \bar{j} + \bar{k}$ D) None of these

59) Acceleration of a particle moving along the curve x = 4cost, y = 4sint, z = 6t at time t = 0 is ……

A) $-4\bar{i}$ B) $2\bar{j} + \bar{k}$ C) $\bar{i} + \bar{j} + \bar{k}$ D) None of these

60) If $\bar{T}$ is unit tangent vector to the curve and $\bar{a} = \ddot{\bar{r}}$ is acceleration of a particle, then tangential component of acceleration = ……

A) 0 B) $\sqrt{|\bar{a}|^2 - (\ddot{\bar{r}}.\bar{T})^2}$ C) $\ddot{\bar{r}}.\bar{T}$ D) None of these

61) If $\bar{T}$ is unit tangent vector to the curve and $\bar{a} = \ddot{\bar{r}}$ is acceleration of a particle, then normal component of acceleration = ……

A) 0 B) $\sqrt{|\bar{a}|^2 - (\ddot{\bar{r}}.\bar{T})^2}$ C) $\ddot{\bar{r}}.\bar{T}$ D) None of these

62) $\frac{\partial}{\partial x}(\bar{u}.\bar{v}) = $ ……

A) $\frac{\partial \bar{u}}{\partial x}.\frac{\partial \bar{v}}{\partial x}$ B) $\bar{u}.\frac{\partial \bar{v}}{\partial x} + \bar{v}.\frac{\partial \bar{u}}{\partial x}$ C) $\bar{u}.\frac{\partial \bar{v}}{\partial x} - \bar{v}.\frac{\partial \bar{u}}{\partial x}$ D) None of these

63) $\frac{\partial}{\partial x}(\bar{u} \times \bar{v}) = \ldots\ldots$

A) $\bar{u} \times \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial x} \times \bar{v}$ B) $\bar{u} \times \frac{\partial \bar{v}}{\partial x} + \bar{v} \times \frac{\partial \bar{u}}{\partial x}$ C) $\frac{\partial \bar{u}}{\partial x} \times \frac{\partial \bar{v}}{\partial x}$ D) None of these

64) $\frac{\partial}{\partial x}(\phi \bar{u}) = \ldots\ldots$

A) $\frac{\partial \phi}{\partial x}\frac{\partial \bar{u}}{\partial x}$ B) $\phi \frac{\partial \bar{u}}{\partial x}$ C) $\phi \frac{\partial \bar{u}}{\partial x} + \frac{\partial \phi}{\partial x}\bar{u}$ D) None of these

65) If $\bar{r} = \frac{a}{2}(x+y)\,\bar{\imath} + \frac{b}{2}(x-y)\,\bar{\jmath} + xy\,\bar{k}$, then $\frac{\partial \bar{r}}{\partial x} = \ldots\ldots.$

A) $\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + x\,\bar{k}$ B) $\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + y\,\bar{k}$ C) $\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + x\,\bar{k}$ D) None of these

66) If $\bar{r} = \frac{a}{2}(x+y)\,\bar{\imath} + \frac{b}{2}(x-y)\,\bar{\jmath} + xy\,\bar{k}$, then $\frac{\partial \bar{r}}{\partial y} = \ldots\ldots.$

A) $\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + x\,\bar{k}$ B) $\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath} + y\,\bar{k}$ C) $\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath} + x\,\bar{k}$ D) None of these

67) If $\bar{r} = \frac{a}{2}(x+y)\,\bar{\imath} + \frac{b}{2}(x-y)\,\bar{\jmath} + xy\,\bar{k}$, then $\frac{\partial^2 \bar{r}}{\partial x^2} = \ldots\ldots.$

A) $\bar{0}$ B) $\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath}$ C) $\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath}$ D) None of these

68) If $\bar{r} = \frac{a}{2}(x+y)\,\bar{\imath} + \frac{b}{2}(x-y)\,\bar{\jmath} + xy\,\bar{k}$, then $\frac{\partial^2 \bar{r}}{\partial y^2} = \ldots\ldots.$

A) $\bar{0}$ B) $\frac{a}{2}\,\bar{\imath} + \frac{b}{2}\,\bar{\jmath}$ C) $\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath}$ D) None of these

69) If $\bar{r} = \frac{a}{2}(x+y)\,\bar{\imath} + \frac{b}{2}(x-y)\,\bar{\jmath} + xy\,\bar{k}$, then $\frac{\partial^2 \bar{r}}{\partial x\,\partial y} = \ldots\ldots.$

A) $\bar{0}$ B) $\bar{k}$ C) $\frac{a}{2}\,\bar{\imath} - \frac{b}{2}\,\bar{\jmath}$ D) None of these

70) If $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$, then $\frac{\partial \bar{r}}{\partial x} = \ldots\ldots.$

A) $\cos y\,\bar{\imath} + \sin y\,\bar{\jmath}$ B) $-x\sin y\,\bar{\imath} + x\cos y\,\bar{\jmath} + ame^{my}\,\bar{k}$
C) $\bar{0}$ D) None of these

71) If $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$, then $\frac{\partial \bar{r}}{\partial y} = \ldots\ldots.$

A) $\cos y\,\bar{\imath} + \sin y\,\bar{\jmath}$ B) $-x\sin y\,\bar{\imath} + x\cos y\,\bar{\jmath} + ame^{my}\,\bar{k}$
C) $\bar{0}$ D) None of these

72) If $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$, then $\frac{\partial^2 \bar{r}}{\partial x^2} = \ldots\ldots.$

A) $\cos y\,\bar{\imath} + \sin y\,\bar{\jmath}$ B) $-x\sin y\,\bar{\imath} + x\cos y\,\bar{\jmath} + ame^{my}\,\bar{k}$
C) $\bar{0}$ D) None of these

73) If $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$, then $\frac{\partial^2 \bar{r}}{\partial y^2} = \ldots\ldots.$

A) $-\sin y\,\bar{\imath} + \cos y\,\bar{\jmath}$ B) $-x\cos y\,\bar{\imath} - x\sin y\,\bar{\jmath} + am^2e^{my}\,\bar{k}$
C) $-x\cos y\,\bar{\imath} - x\sin y\,\bar{\jmath}$ D) None of these

74) If $\bar{r} = x\cos y\,\bar{\imath} + x\sin y\,\bar{\jmath} + ae^{my}\,\bar{k}$, then $\frac{\partial^2 \bar{r}}{\partial x\,\partial y} = \ldots\ldots.$

A) $-\sin y\,\bar{\imath} + \cos y\,\bar{\jmath}$ B) $-x\cos y\,\bar{\imath} - x\sin y\,\bar{\jmath} + am^2e^{my}\,\bar{k}$
C) $-x\cos y\,\bar{\imath} - x\sin y\,\bar{\jmath}$ D) None of these

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UNIT-3: THE VECTOR OPERATOR DEL

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Scalar Point Function: A scalar valued function φ defined on a region R of a space is called scalar point function.

Remark: A scalar point function together with region R is called scalar field.

e.g. The temperature at a point in a room is a scalar point function.

Surface: If φ = φ(x, y, z) is a scalar point function φ defined on a region R, then φ(x, y, z) = c, where c is parameter, defines family of surfaces in R, such surfaces are called level surfaces in R w.r.t. φ.

e.g. If φ(x, y, z) denotes the temperature at a point P(x, y, z) in a room, then φ(x, y, z) = 25^0 is a level surfaces in a room at any point on this surface, the temperature will be 25^0.

Vector Point Function: A vector valued function $\bar{v}(P)$ defined on a region R of a space is called vector point function.

Remark: A vector point function together with region R is called vector field.

e.g. The velocity of particle at a time t is a vector point function.

Gradient of a Scalar Point Function: Let φ(x, y, z) be scalar point function defined and differentiable in a region R of a space, then gradient of φ is denoted by ∇φ or grad φ and defined as $\nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k}$

Remark: i) $\nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k}$ is a vector point function with components along x, y, z axis are $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ respectively.

ii) The gradient of a scalar point function is a vector point function.

iii) $\nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})\,\varphi \therefore \nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

iv) If $\nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k}$, then

$$\varphi(x, y, z) = \int_{y,z\text{ constant}} \frac{\partial\varphi}{\partial x}\,dx + \int_{z\text{ constant}} [\text{Terms in } \frac{\partial\varphi}{\partial y} \text{ not containing x}]\,dy$$

$$+ \int [\text{Terms in } \frac{\partial\varphi}{\partial z} \text{ containing neither x nor y}]\,dz + c$$

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Theorem-1: If φ and ψ are scalar point functions and if ∇φ and ∇ψ exist in a given region R, then $\nabla(\varphi \pm \psi) = \nabla\varphi \pm \nabla\psi$ i.e. $\text{grad}(\varphi \pm \psi) = \text{grad}\,\varphi \pm \text{grad}\,\psi$

Proof: Consider

$$\text{grad}(\varphi \pm \psi) = \nabla(\varphi \pm \psi)$$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(\varphi \pm \psi)$$

$$= \bar{\imath}\frac{\partial}{\partial x}(\varphi \pm \psi) + \bar{\jmath}\frac{\partial}{\partial y}(\varphi \pm \psi) + \bar{k}\frac{\partial}{\partial z}(\varphi \pm \psi)$$

$$= \bar{\imath}\left[\frac{\partial \varphi}{\partial x} \pm \frac{\partial \psi}{\partial x}\right] + \bar{\jmath}\left[\frac{\partial \varphi}{\partial y} \pm \frac{\partial \psi}{\partial y}\right] + \bar{k}\left[\frac{\partial \varphi}{\partial z} \pm \frac{\partial \psi}{\partial z}\right]$$

$$= \left[\bar{\imath}\frac{\partial \varphi}{\partial x} + \bar{\jmath}\frac{\partial \varphi}{\partial y} + \bar{k}\frac{\partial \varphi}{\partial z}\right] \pm \left[\bar{\imath}\frac{\partial \psi}{\partial x} + \bar{\jmath}\frac{\partial \psi}{\partial y} + \bar{k}\frac{\partial \psi}{\partial z}\right]$$

$$= \nabla\varphi \pm \nabla\psi$$

$$= \text{grad}\,\varphi \pm \text{grad}\,\psi$$

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Theorem-2: A necessary and sufficient condition for a scalar point function φ to be constant is that $\nabla\varphi = \bar{0}$.

Proof: **Necessary Condition:**

Let φ be a constant function.

$$\therefore \frac{\partial \varphi}{\partial x} = 0,\ \frac{\partial \varphi}{\partial y} = 0,\ \frac{\partial \varphi}{\partial z} = 0$$

$$\therefore \nabla\varphi = \frac{\partial \varphi}{\partial x}\bar{\imath} + \frac{\partial \varphi}{\partial y}\bar{\jmath} + \frac{\partial \varphi}{\partial z}\bar{k} = 0\,\bar{\imath} + 0\,\bar{\jmath} + 0\bar{k} = \bar{0}$$

Sufficient Condition:

Let $\nabla\varphi = \bar{0}$

$$\therefore \frac{\partial \varphi}{\partial x}\bar{\imath} + \frac{\partial \varphi}{\partial y}\bar{\jmath} + \frac{\partial \varphi}{\partial z}\bar{k} = 0\,\bar{\imath} + 0\,\bar{\jmath} + 0\bar{k}$$

$$\therefore \frac{\partial \varphi}{\partial x} = 0,\ \frac{\partial \varphi}{\partial y} = 0,\ \frac{\partial \varphi}{\partial z} = 0$$

∴ φ is independent of x, y, z.

∴ φ is constant.

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Theorem-3: If φ and ψ are scalar point functions and if ∇φ and ∇ψ exist in a given region R, then $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$ i.e. $\text{grad}(\varphi\psi) = \varphi\,\text{grad}\,\psi + \psi\,\text{grad}\,\varphi$

Proof: Consider

$$\text{grad}(\varphi\psi) = \nabla(\varphi\psi)$$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(\varphi\psi)$$

$= \bar{\imath}\frac{\partial}{\partial x}(\varphi\psi) + \bar{\jmath}\frac{\partial}{\partial y}(\varphi\psi) + \bar{k}\frac{\partial}{\partial z}(\varphi\psi)$

$= \bar{\imath}\,[\varphi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\varphi}{\partial x}] + \bar{\jmath}[\varphi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\varphi}{\partial y}] + \bar{k}[\varphi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\varphi}{\partial z}]$

$= \varphi[\bar{\imath}\frac{\partial\psi}{\partial x} + \bar{\jmath}\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z}] + \psi[\bar{\imath}\frac{\partial\varphi}{\partial x} + \bar{\jmath}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}]$

$= \varphi\nabla\psi + \psi\nabla\varphi$

$= \varphi \text{ grad } \psi + \psi \text{ grad } \varphi$

===

Corrolary: If φ is scalar point function and k is constant, then $\nabla(k\varphi) = k\nabla\varphi$

i.e. $\text{grad}(k\varphi) = k \text{ grad } \varphi$

Proof: Consider

$\text{grad}(k\varphi) = \nabla(k\varphi)$

$= \varphi\nabla k + k\nabla\varphi$

$= \varphi(0) + k\nabla\varphi$

$= k\nabla\varphi$

$= k \text{ grad } \varphi$

===

Theorem-3: If φ and ψ are scalar point functions and if ∇φ and ∇ψ exist in a given region R, then $\nabla\left(\frac{\varphi}{\psi}\right) = \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}$ i.e. $\text{grad}(\frac{\varphi}{\psi}) = \frac{\psi \text{ grad } \varphi - \varphi \text{ grad } \psi}{\psi^2}$ provided $\psi \neq 0$

Proof: Consider

$\text{grad}(\frac{\varphi}{\psi}) = \nabla(\frac{\varphi}{\psi})$

$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(\frac{\varphi}{\psi})$

$= \bar{\imath}\frac{\partial}{\partial x}(\frac{\varphi}{\psi}) + \bar{\jmath}\frac{\partial}{\partial y}(\frac{\varphi}{\psi}) + \bar{k}\frac{\partial}{\partial z}(\frac{\varphi}{\psi})$

$= \bar{\imath}\,[\frac{\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}}{\psi^2}] + \bar{\jmath}[\frac{\psi\frac{\partial\varphi}{\partial y} - \varphi\frac{\partial\psi}{\partial y}}{\psi^2}] + \bar{k}[\frac{\psi\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial\psi}{\partial z}}{\psi^2}]$

$= \frac{1}{\psi^2}\,[\psi(\bar{\imath}\frac{\partial\varphi}{\partial x} + \bar{\jmath}\frac{\partial\varphi}{\partial y} + \bar{k}\frac{\partial\varphi}{\partial z}) - \varphi(\bar{\imath}\frac{\partial\psi}{\partial x} + \bar{\jmath}\frac{\partial\psi}{\partial y} + \bar{k}\frac{\partial\psi}{\partial z})]$

$= \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}$

$= \frac{\psi \text{ grad } \varphi - \varphi \text{ grad } \psi}{\psi^2}$

===

Ex.: If $\bar{r} = x\,\bar{i} + y\bar{j} + z\bar{k}$, $|\bar{r}| = r$, then prove that

i) $\nabla\varphi(r) = \varphi'(r)\,\nabla r$ ii) ∇r is the unit vector $\hat{r}$ iii) $\nabla \log r = \frac{\bar{r}}{r^2}$

Proof: Consider

i) $\nabla\varphi(r) = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})\varphi(r)$

$= \bar{i}\frac{\partial}{\partial x}\varphi(r) + \bar{j}\frac{\partial}{\partial y}\varphi(r) + \bar{k}\frac{\partial}{\partial z}\varphi(r)$

$= [\bar{i}\,\varphi'(r)\frac{\partial r}{\partial x} + \bar{j}\varphi'(r)\frac{\partial r}{\partial y} + \bar{k}\varphi'(r)\frac{\partial r}{\partial z}]$

$= \varphi'(r)\,[\bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}]$

$\therefore \nabla\varphi(r) = \varphi'(r)\,\nabla r$

Hence proved.

ii) As $r = |\bar{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}(2x) = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \nabla r = \bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}$

$= \frac{x}{r}\bar{i} + \frac{y}{r}\bar{j} + \frac{z}{r}\bar{k}$

$= \frac{x\,\bar{i} + y\bar{j} + z\bar{k}}{r}$

$= \frac{\bar{r}}{r}$

$= \hat{r}$

i.e. ∇r is the unit vector $\hat{r}$ is proved.

iii) Let $\varphi(r) = \log r$

$\therefore \varphi'(r) = \frac{1}{r}$

$\therefore \nabla\varphi(r) = \varphi'(r)\,\nabla r$ gives

$\nabla \log r = \frac{1}{r}(\frac{\bar{r}}{r}) \quad \because \nabla r = \hat{r} = \frac{\bar{r}}{r}$

$\therefore \nabla \log r = \frac{\bar{r}}{r^2}$ Hence proved.

===

Ex.: Prove that $\nabla r^n = nr^{n-2}\bar{r}$, where $\bar{r} = x\,\bar{i} + y\bar{j} + z\bar{k}$

Proof: Let $\varphi(r) = r^n$

$\therefore \varphi'(r) = nr^{n-1}$

$\therefore \nabla\varphi(r) = \varphi'(r)\,\nabla r$ gives

$\nabla r^n = n r^{n-1}\left(\frac{\bar{r}}{r}\right)$ $\quad \because \nabla r = \hat{r} = \frac{\bar{r}}{r}$

$\therefore \nabla r^n = n r^{n-2}\bar{r}$ Hence proved.

==

Ex.: If $\bar{r} = x\,\bar{\imath} + y\bar{\jmath} + z\bar{k}$, and $\bar{a}$, $\bar{b}$ are constant vectors, then show that

i) $\nabla(\bar{r}.\bar{a}) = \bar{a}$ ii) $\nabla[\bar{r}\ \bar{a}\ \bar{b}\,] = \bar{a} \times \bar{b}$

Proof: Let $\bar{a} = a_1\bar{\imath} + a_2\bar{\jmath} + a_3\bar{k}$

$\therefore \bar{r}.\bar{a} = (x\,\bar{\imath} + y\bar{\jmath} + z\bar{k}).(\,a_1\bar{\imath} + a_2\bar{\jmath} + a_3\bar{k})$

$= xa_1 + ya_2 + za_3$

$\therefore \nabla(\bar{r}.\bar{a}) = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(xa_1 + ya_2 + za_3)$

$= (\bar{\imath}\,a_1 + \bar{\jmath}a_2 + \bar{k}a_3)$

$= a_1\bar{\imath} + a_2\bar{\jmath} + a_3\bar{k}$

$\therefore \nabla(\bar{r}.\bar{a}) = \bar{a}$

ii) As $\bar{a}$, $\bar{b}$ are constant vectors.

$\therefore \bar{a} \times \bar{b}$ is constant vector.

$\therefore \nabla[\bar{r}.(\bar{a} \times \bar{b})] = \bar{a} \times \bar{b}$ by (i)

i.e. $\nabla[\bar{r}\ \bar{a}\ \bar{b}] = \bar{a} \times \bar{b}$ Hence proved.

==

Ex.: If $u = 3x^2y$ and $v = xz^2 - 2y$, then find grad[(gradu).(gradv)]

Solution: Let $u = 3x^2y$ and $v = xz^2 - 2y$

$\therefore$ grad u $= \nabla u = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(3x^2y)$

$= 6xy\bar{\imath} + 3x^2\bar{\jmath} + 0\bar{k}$

& grad v $= \nabla v = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(xz^2 - 2y)$

$= z^2\bar{\imath} - 2\bar{\jmath} + 2xz\bar{k}$

$\therefore$ grad u. grad v $= (6xy\bar{\imath} + 3x^2\bar{\jmath} + 0\bar{k}).(z^2\bar{\imath} - 2\bar{\jmath} + 2xz\bar{k})$

$= 6xyz^2 - 6x^2 + 0$

$= 6xyz^2 - 6x^2$

$\therefore$ grad[(gradu).(gradv)] $= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z})(6xyz^2 - 6x^2)$

$= (6yz^2 - 12x)\bar{\imath} + 6xz^2\bar{\jmath} + 12xyz\bar{k}$

==

Ex.: Find f(x, y, z) if f(0, 0, 0) = 1 and

$\nabla f = (y^2-2xyz^3)\,\bar{i} + (3+2xy-x^2z^3)\bar{j} + (8z^3-3x^2yz^2)\bar{k}$

Solution: Let $\nabla f = (y^2-2xyz^3)\,\bar{i} + (3+2xy-x^2z^3)\bar{j} + (8z^3-3x^2yz^2)\bar{k}$

Comparing it with $\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$, we get,

$\frac{\partial f}{\partial x} = y^2-2xyz^3$, $\frac{\partial f}{\partial y} = 3+2xy-x^2z^3$ and $\frac{\partial f}{\partial z} = 8z^3-3x^2yz^2$

Now $f(x, y, z) = \int_{y,z\text{ constant}} \frac{\partial f}{\partial x}\, dx + \int_{z\text{ constant}} [\text{Terms in } \frac{\partial f}{\partial y} \text{ not containing x}]dy$

$+ \int [\text{Terms in } \frac{\partial f}{\partial z} \text{ containing neither x nor y}]dz + c$, gives

$f(x, y, z) = \int_{y,z\text{ constant}} (y^2 - 2xyz^3)\, dx + \int_{z\text{ constant}} (3)dy + \int (8z^3)dz + c$

i.e. $f(x, y, z) = y^2x - x^2yz^3 + 3y + 2z^4 + c$ …… (i)

But f(0, 0, 0) = 1 i.e. c = 1

Putting c = 1 in (i), we get,

$f(x, y, z) = y^2x - x^2yz^3 + 3y + 2z^4 + 1$

==

Geometric Meaning of the gradient $\nabla\varphi$:

i) Normal to the surface φ(x, y, z) = c at point P(x, y, z) = $(\nabla\varphi)_P$

ii) Unit normal to the surface φ(x, y, z) = c at point P(x, y, z) = $\frac{(\nabla\varphi)_P}{|(\nabla\varphi)_P|}$

iii) $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ are the d.r.s. of normal to the surface φ(x, y, z) = c.

iv) If a, b, c are the d.r.s. of normal, then equation of normal passing through $P(x_1, y_1, z_1)$ is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

v) Equation of tangent plane to the surface φ(x, y, z) = c at $P(x_1, y_1, z_1)$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

==

Ex.: Find the unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point P(1, 2, -1)

Solution: Let $\varphi(x, y, z) = x^3 + y^3 + 3xyz = 3$ be the given surface.

$\therefore \nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k}$

$= (3x^2 + 3yz)\,\bar{i} + (3y^2 + 3xz)\,\bar{j} + 3xy\bar{k}$

At the point P(1, 2, -1), we have

$(\nabla\varphi)_P = (3 - 6)\,\bar{i} + (12 - 3)\,\bar{j} + 6\bar{k} = -3\bar{i} + 9\bar{j} + 6\bar{k} = 3(-\bar{i} + 3\bar{j} + 2\bar{k})$

∴ the unit vector normal to the surface φ = 3 at point P is

$\bar{N} = \frac{(\nabla\varphi)_P}{|(\nabla\varphi)_P|} = \frac{3(-\bar{i} + 3\bar{j} + 2\bar{k})}{3\sqrt{(-1)^2+3^2+2^2}} = \frac{(-\bar{i} + 3\bar{j} + 2\bar{k})}{\sqrt{14}}$

==

Ex.: Find the equation of tangent plane and equation of normal to the surface $xz^2 + x^2y - z + 1 = 0$ at the point P(1, -3, 2)

Solution: Let $\varphi(x, y, z) = xz^2 + x^2y - z = -1$ be the given surface.

$\therefore \frac{\partial\varphi}{\partial x} = z^2 + 2xy, \ \frac{\partial\varphi}{\partial y} = x^2, \ \frac{\partial\varphi}{\partial z} = 2xz - 1$

At the point P(1, -3, 2), we have

$a = (\frac{\partial\varphi}{\partial x})_P = -2, \ b = (\frac{\partial\varphi}{\partial y})_P = 1, \ c = (\frac{\partial\varphi}{\partial z})_P = 3$

i.e -2, 1, 3 i.e. 2, -1, -3 are the d.r.s. of normal at point P.

$\therefore$ Equation of tangent plane to the surface $\varphi(x, y, z) = -1$ at P(1, -3, 2) is

$2(x - 1) - (y + 3) - 3(z - 2) = 0$

i.e. $2x - y - 3z + 1 = 0$

The equation of normal at P(1, -3, 2) is $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{-3}$

==

Divergence of a Vector Point Function: Let $\bar{v} = \bar{v}(x, y, z)$ be a differentiable vector point function defined in a region R, then the divergence of $\bar{v}$ is defined as

$\text{div}.\bar{v} = \bar{i}.\frac{\partial\bar{v}}{\partial x} + \bar{j}.\frac{\partial\bar{v}}{\partial y} + \bar{k}.\frac{\partial\bar{v}}{\partial z}$

Note: i) $\text{div}.\bar{v} = \bar{i}.\frac{\partial\bar{v}}{\partial x} + \bar{j}.\frac{\partial\bar{v}}{\partial y} + \bar{k}.\frac{\partial\bar{v}}{\partial z} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).\bar{v} = \nabla.\bar{v}$

ii) The divergence of vector point function is a scalar point function.

iii) If $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$, then $\text{div}.\bar{v} = \nabla.\bar{v} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(v_1\bar{i} + v_2\bar{j} + v_3\bar{k})$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Solenoidal: A vector point function $\bar{v}$ is called solenoidal if $\text{div}.\bar{v} = 0$.

==

Ex.: Find divergence of $\bar{v} = (x^2+yz)\bar{i} + (y^2+zx)\bar{j} + (z^2+xy)\bar{k}$

Solution: Let $\bar{v} = (x^2+yz)\bar{i} + (y^2+zx)\bar{j} + (z^2+xy)\bar{k}$ be the given surface.

$\therefore \text{div}.\bar{v} = \nabla.\bar{v} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).[(x^2+yz)\bar{i} + (y^2+zx)\bar{j} + (z^2+xy)\bar{k}]$

$$= \frac{\partial}{\partial x}(x^2+yz) + \frac{\partial}{\partial y}(y^2+zx) + \frac{\partial}{\partial z}(z^2+xy)$$

$$= 2x + 2y + 2z$$

$$= 2(x + y + z)$$

==

Ex.: Show that $\bar{v} = x^2z\bar{i} + y^2z\,\bar{j} - (xz^2+yz^2)\bar{k}$ is solenoidal.

Proof: Let $\bar{v} = x^2z\bar{i} + y^2z\,\bar{j} - (xz^2+yz^2)\bar{k}$ be the given surface.

$\therefore$ div.$\bar{v} = \nabla.\bar{v} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).[x^2z\bar{i} + y^2z\,\bar{j} - (xz^2+yz^2)\bar{k}]$

$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(y^2z) - \frac{\partial}{\partial z}(xz^2+yz^2)$

$= 2xz + 2yz - 2xz - 2yz$

$= 0$

$\therefore \bar{v}$ is solenoidal is proved.

==

Ex.: Determine the constant a so that the vector function

$\bar{v} = (x+3y)\bar{i} + (y-2z)\,\bar{j} + (x+az)\bar{k}$ is solenoidal.

Solution: Let $\bar{v} = (x+3y)\bar{i} + (y-2z)\,\bar{j} + (x+az)\bar{k}$ is solenoidal.

$\therefore$ div.$\bar{v} = 0$ i.e. $\nabla.\bar{v} = 0$

$\therefore (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).[(x+3y)\bar{i} + (y-2z)\,\bar{j} + (x+az)\bar{k}] = 0$

$\therefore \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$

$\therefore 1 + 1 + a = 0$

$\therefore a = -2$

==

Laplacian of a Scalar Point Function:

Let φ be scalar point function, then divergence of $\nabla\varphi$

i.e. $\nabla.\nabla\varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ is called Laplacian of scalar point function φ

Laplacian Equation: $\nabla^2\varphi = 0$ is called Laplacian equation of scalar point function φ.

Harmonic Function: A scalar point function φ is said to be Harmonic function if it satisfies Laplacian equation $\nabla^2\varphi = 0$.

==

Curl of a Vector Point Function: Let $\bar{v} = \bar{v}(x, y, z)$ be a differentiable vector point function defined in a region R, then the curl (or rotation) of $\bar{v}$ is defined as

curl.$\bar{v} = \bar{i} \times \frac{\partial\bar{v}}{\partial x} + \bar{j} \times \frac{\partial\bar{v}}{\partial y} + \bar{k} \times \frac{\partial\bar{v}}{\partial z}$

Note: i) curl.$\bar{v} = \bar{i} \times \frac{\partial\bar{v}}{\partial x} + \bar{j} \times \frac{\partial\bar{v}}{\partial y} + \bar{k} \times \frac{\partial\bar{v}}{\partial z} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}) \times \bar{v} = \nabla \times \bar{v}$

ii) Curl of vector point function is again a vector point function.

iii) If $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$, then curl$\times \bar{v} = \nabla \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$

iv) If $\bar{v} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial x}\bar{j} + \frac{\partial\varphi}{\partial x}\bar{k}$, then

$$\varphi(x, y, z) = \int_{y,z\ \text{constant}} \frac{\partial\varphi}{\partial x}\,dx + \int_{z\ \text{constant}} [\text{Terms in } \frac{\partial\varphi}{\partial y} \text{ not containing x}]dy$$
$$+ \int [\text{Terms in } \frac{\partial\varphi}{\partial z} \text{ containing neither x nor y}]dz + c$$

Irrotational: A vector point function $\bar{v}$ is called irrotational if curl.$\bar{v} = \bar{0}$.

==

Ex.: Find curl of $\bar{v} = xz^3\bar{i} - 2x^2yz\bar{j} + 2yz^4\bar{k}$

Solution: Let $\bar{v} = xz^3\bar{i} - 2x^2yz\bar{j} + 2yz^4\bar{k}$

$$\therefore \text{curl}.\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$
$$= \bar{i}\,(2z^4+2x^2y) - \bar{j}\,(0-3xz^2) + \bar{k}(-4xyz - 0)$$
$$= 2(z^4+x^2y)\,\bar{i} + 3xz^2\bar{j} - 4xyz\bar{k}$$

==

Ex.: Show that $\bar{v} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ is irrotational.

Proof: Let $\bar{v} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$

$$\therefore \text{curl}.\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$
$$= \bar{i}\,(0-0) - \bar{j}\,(0-0) + \bar{k}(0-0)$$
$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$
$$= \bar{0}$$

$\therefore \bar{v}$ is irrotational is proved.

==

Ex.: Show that $\bar{v} = (\sin y + z)\bar{i} + (x\cos y - z)\bar{j} + (x - y)\bar{k}$ is irrotational.

Proof: Let $\bar{v} = (\sin y + z)\bar{i} + (x\cos y - z)\bar{j} + (x - y)\bar{k}$

$$\therefore \text{curl}.\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$
$$= \bar{i}\,(-1+1) - \bar{j}\,(1-1) + \bar{k}(\cos y - \cos y)$$
$$= 0\bar{i} + 0\bar{j} + 0\bar{k}$$
$$= \bar{0}$$

∴ $\bar{v}$ is irrotational is proved.

==

Ex.: If $\bar{f} = (y+\sin z)\bar{i} + x\bar{j} + x\cos z\bar{k}$, then show that $\bar{f}$ is irrotational and find φ such that $\nabla\varphi = \bar{f}$.

Proof: Let $\bar{f} = (y+\sin z)\bar{i} + x\bar{j} + x\cos z\bar{k}$

$$\therefore \text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+\sin z & x & x\cos z \end{vmatrix}$$

$= \bar{i}\,(0-0) - \bar{j}\,(\cos z - \cos z) + \bar{k}(1-1)$

$= 0\bar{i} + 0\bar{j} + 0\bar{k}$

$= \bar{0}$

∴ $\bar{f}$ is irrotational is proved.

As $\nabla\varphi = \bar{f}$ i.e. $\frac{\partial\varphi}{\partial x}\bar{i} + \frac{\partial\varphi}{\partial y}\bar{j} + \frac{\partial\varphi}{\partial z}\bar{k} = (y+\sin z)\bar{i} + x\bar{j} + x\cos z\bar{k}$

$\therefore \frac{\partial\varphi}{\partial x} = y+\sin z,\ \frac{\partial\varphi}{\partial y} = x,\ \frac{\partial\varphi}{\partial z} = x\cos z$

$\therefore \varphi(x, y, z) = \int_{y,z\text{ constant}} \frac{\partial\varphi}{\partial x}\,dx + \int_{z\text{ constant}} [\text{Terms in } \frac{\partial\varphi}{\partial y} \text{ not containing x}]dy$

$+ \int [\text{Terms in } \frac{\partial\varphi}{\partial z} \text{ containing neither x nor y}]dz + c$

$\therefore \varphi(x, y, z) = \int_{y,z\text{ constant}} (y+\sin z)\,dx + \int_{z\text{ constant}} 0dy + \int 0dz + c$

$\therefore \varphi(x, y, z) = (y+\sin z)x + c$

==

Ex.: Verify that the vector point function $\bar{a} = (6xy + z^3)\bar{I} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$ is irrotational. Find a scalar point function φ such that $\bar{a} = \nabla\varphi$.

Proof: Let $\bar{a} = (6xy + z^3)\bar{I} + (3x^2 - z)\bar{j} + (3xz^2 - y)\bar{k}$

$$\therefore \text{curl } \bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$= \bar{i}\,(-1+1) - \bar{j}\,(3z^2 - 3z^2) + \bar{k}(6x - 6x)$

$= 0\bar{i} + 0\bar{j} + 0\bar{k}$

$= \bar{0}$

∴ $\bar{a}$ is irrotational is proved.

As $\bar{a} = \nabla\varphi$ i.e. $\frac{\partial\varphi}{\partial x}\bar{\imath} + \frac{\partial\varphi}{\partial y}\bar{\jmath} + \frac{\partial\varphi}{\partial z}\bar{k} = (6xy + z^3)\bar{\imath} + (3x^2 - z)\bar{\jmath} + (3xz^2 - y)\bar{k}$

$\therefore \frac{\partial\varphi}{\partial x} = 6xy + z^3,\ \frac{\partial\varphi}{\partial y} = 3x^2 - z,\ \frac{\partial\varphi}{\partial z} = 3xz^2 - y$

$\therefore \varphi(x, y, z) = \int_{y,z\ \text{constant}} \frac{\partial\varphi}{\partial x}\,dx + \int_{z\ \text{constant}} [\text{Terms in } \frac{\partial\varphi}{\partial y} \text{ not containing } x]\,dy$

$+ \int [\text{Terms in } \frac{\partial\varphi}{\partial z} \text{ containing neither } x \text{ nor } y]\,dz + c$

$\therefore \varphi(x, y, z) = \int_{y,z\ \text{constant}} (6xy + z^3)\,dx + \int_{z\ \text{constant}} (-z)\,dy + \int 0\,dz + c$

$\therefore \varphi(x, y, z) = 3x^2y + xz^3 - yz + c$

==

Ex.: Find the constants a, b, c so that the vector function

$\bar{v} = (x+2y+az)\bar{\imath} + (bx-3y-z)\,\bar{\jmath} + (4x+cy+2z)\bar{k}$ is irrotational.

Solution: Let $\bar{v} = (x+2y+az)\bar{\imath} + (bx-3y-z)\,\bar{\jmath} + (4x+cy+2z)\bar{k}$ is irrotational

$\therefore \text{curl}.\bar{v} = \bar{0}$

$$\therefore \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \bar{0}$$

$\therefore \bar{\imath}\,(c+1) - \bar{\jmath}\,(4-a) + \bar{k}(b-2) = 0\bar{\imath} + 0\bar{\jmath} + 0\bar{k}$

$\therefore c + 1 = 0,\ a - 4 = 0$ and $b - 2 = 0$

$\therefore a = 4,\ b = 2$ and $c = -1$ be the required values.

==

Ex.: If $\bar{f} = x^2y\,\bar{\imath} - 2xz\,\bar{\jmath} + 2yz\,\bar{k}$, then find div $\bar{f}$ and curl $\bar{f}$

Solution: Let $\bar{f} = x^2y\,\bar{\imath} - 2xz\,\bar{\jmath} + 2yz\,\bar{k}$

$\therefore \text{div}\,\bar{f} = \nabla.\bar{f} = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).[x^2y\,\bar{\imath} - 2xz\,\bar{\jmath} + 2yz\,\bar{k}]$

$= \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(2yz)$

$= 2xy - 0 + 2y$

$= 2y(x + 1)$

$$\& \text{ curl } \bar{f} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$= \bar{\imath}\,(2z + 2x) - \bar{\jmath}\,(0 - 0) + \bar{k}(-2z - x^2)$

$= 2(x + z)\bar{\imath} - (x^2 + 2z)\bar{k}$

==

Ex.: If $\bar{f} = (y^2+z^2-x^2)\bar{\imath} + (z^2+x^2-y^2)\bar{\jmath} + (x^2+y^2-z^2)\bar{k}$, then find div $\bar{f}$ and curl $\bar{f}$

Solution: Let $\bar{f} = (y^2+z^2-x^2)\bar{\imath} + (z^2+x^2-y^2)\bar{\jmath} + (x^2+y^2-z^2)\bar{k}$

$$\therefore \text{div } \bar{f} = \nabla.\bar{f} = \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).[(y^2+z^2-x^2)\bar{\imath} + (z^2+x^2-y^2)\bar{\jmath} + (x^2+y^2-z^2)\bar{k}]$$

$$= \frac{\partial}{\partial x}(y^2+z^2-x^2) + \frac{\partial}{\partial y}(z^2+x^2-y^2) + \frac{\partial}{\partial z}(x^2+y^2-z^2)$$

$$= -2x - 2y - 2z$$

$$= -2(x + y + z)$$

$$\& \text{ curl } \bar{f} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$$

$$= \bar{\imath}(2y-2z) - \bar{\jmath}(2x-2z) + \bar{k}(2x-2y)$$

$$= 2[(y-z)\bar{\imath} + (z-x)\bar{\jmath} + (x-y)\bar{k}$$

==

Ex.: If $\bar{a}$ is constant vector, then find div $(\bar{r} \times \bar{a})$ and curl $(\bar{r} \times \bar{a})$.

Solution: Let $\bar{a} = a_1\bar{\imath} + a_2\bar{\jmath} + a_3\bar{k}$ be a constant vector and $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

$$\therefore \bar{r} \times \bar{a} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \bar{\imath}(a_3y - a_2z) - \bar{\jmath}(a_3x - a_1z) + \bar{k}(a_2x - a_1y)$$

$$\therefore \text{div } (\bar{r} \times \bar{a}) = \nabla.(\bar{r} \times \bar{a})$$

$$= \left(\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).[\bar{\imath}(a_3y - a_2z) - \bar{\jmath}(a_3x - a_1z) + \bar{k}(a_2x - a_1y)]$$

$$= \frac{\partial}{\partial x}(a_3y - a_2z) - \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_2x - a_1y)$$

$$= 0 - 0 - 0$$

$$= 0$$

$$\& \text{ curl } (\bar{r} \times \bar{a}) = \nabla \times (\bar{r} \times \bar{a}) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3y - a_2z & a_1z - a_3x & a_2x - a_1y \end{vmatrix}$$

$$= (-a_1 - a_1)\bar{\imath} - (a_2 + a_2)\bar{\jmath} + (-a_3 - a_3)\bar{k}$$

$$= -2(a_1\bar{\imath} + a_2)\bar{\jmath} + a_3\bar{k})$$

$$= -2\bar{a}$$

==

Theorem-1: If $\bar{u}$ and $\bar{v}$ are vector point functions, then

$$div.(\bar{u} \pm \bar{v}) = div.\bar{u} \pm div.\bar{v} \quad \text{i.e } \nabla.(\bar{u} \pm \bar{v}) = \nabla.\bar{u} \pm \nabla.\bar{v}$$

Proof: Consider

$$div.(\bar{u} \pm \bar{v}) = \nabla.(\bar{u} \pm \bar{v})$$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(\bar{u} \pm \bar{v})$$

$$= \bar{\imath}\frac{\partial}{\partial x}.(\bar{u} \pm \bar{v}) + \bar{\jmath}\frac{\partial}{\partial y}.(\bar{u} \pm \bar{v}) + \bar{k}\frac{\partial}{\partial z}.(\bar{u} \pm \bar{v})$$

$$= \bar{\imath}.[\frac{\partial \bar{u}}{\partial x} \pm \frac{\partial \bar{v}}{\partial x}] + \bar{\jmath}.[\frac{\partial \bar{u}}{\partial y} \pm \frac{\partial \bar{v}}{\partial y}] + \bar{k}.[\frac{\partial \bar{u}}{\partial z} \pm \frac{\partial \bar{v}}{\partial z}]$$

$$= [\bar{\imath}.\frac{\partial \bar{u}}{\partial x} + \bar{\jmath}.\frac{\partial \bar{u}}{\partial y} + \bar{k}.\frac{\partial \bar{u}}{\partial z}] \pm [\bar{\imath}.\frac{\partial \bar{v}}{\partial x} + \bar{\jmath}.\frac{\partial \bar{v}}{\partial y} + \bar{k}.\frac{\partial \bar{v}}{\partial z}]$$

$$= \nabla.\bar{u} \pm \nabla.\bar{v}$$

$$= div.\bar{u} \pm div.\bar{v}$$

===

Theorem-2: If $\bar{u}$ and $\bar{v}$ are vector point functions, then

$curl.(\bar{u} \pm \bar{v}) = curl.\bar{u} \pm curl.\bar{v}$ i.e $\nabla \times (\bar{u} \pm \bar{v}) = \nabla \times \bar{u} \pm \nabla \times \bar{v}$

Proof: Consider

$$curl.(\bar{u} \pm \bar{v}) = \nabla \times (\bar{u} \pm \bar{v})$$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}) \times (\bar{u} \pm \bar{v})$$

$$= \bar{\imath}\frac{\partial}{\partial x} \times (\bar{u} \pm \bar{v}) + \bar{\jmath}\frac{\partial}{\partial y} \times (\bar{u} \pm \bar{v}) + \bar{k}\frac{\partial}{\partial z} \times (\bar{u} \pm \bar{v})$$

$$= \bar{\imath} \times \left[\frac{\partial \bar{u}}{\partial x} \pm \frac{\partial \bar{v}}{\partial x}\right] + \bar{\jmath} \times [\frac{\partial \bar{u}}{\partial y} \pm \frac{\partial \bar{v}}{\partial y}] + \bar{k} \times [\frac{\partial \bar{u}}{\partial z} \pm \frac{\partial \bar{v}}{\partial z}]$$

$$= [\bar{\imath} \times \frac{\partial \bar{u}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{u}}{\partial y} + \bar{k} \times \frac{\partial \bar{u}}{\partial z}] \pm [\bar{\imath} \times \frac{\partial \bar{v}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{v}}{\partial y} + \bar{k} \times \frac{\partial \bar{v}}{\partial z}]$$

$$= \nabla \times \bar{u} \pm \nabla \times \bar{v}$$

$$= curl\,\bar{u} \pm curl\,\bar{v}$$

===

Theorem-3: If φ is a scalar point function and $\bar{u}$ is vector point function, then

$div.(\varphi\bar{u}) = (grad\,\varphi).\bar{u} + \varphi div.\bar{u}$

i.e $\nabla.(\varphi\bar{u}) = (\nabla\varphi).\bar{u} + \varphi(\nabla.\bar{u})$

Proof: Consider

$$div.(\varphi\bar{u}) = \nabla.(\varphi\bar{u})$$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(\varphi\bar{u})$$

$$= \bar{\imath}\frac{\partial}{\partial x}.(\varphi\bar{u}) + \bar{\jmath}\frac{\partial}{\partial y}.(\varphi\bar{u}) + \bar{k}\frac{\partial}{\partial z}.(\varphi\bar{u})$$

$$= \bar{\imath}.[\frac{\partial \varphi}{\partial x}\bar{u} + \varphi\frac{\partial \bar{u}}{\partial x}] + \bar{\jmath}.[\frac{\partial \varphi}{\partial y}\bar{u} + \varphi\frac{\partial \bar{u}}{\partial y}] + \bar{k}.[\frac{\partial \varphi}{\partial z}\bar{u} + \varphi\frac{\partial \bar{u}}{\partial z}]$$

$$= [\frac{\partial\varphi}{\partial x}\bar{\iota} + \frac{\partial\varphi}{\partial y}\bar{\jmath} + \frac{\partial\varphi}{\partial z}\bar{k}].\bar{u} + \varphi[\bar{\iota}.\frac{\partial\bar{u}}{\partial x} + \bar{\jmath}.\frac{\partial\bar{u}}{\partial y} + \bar{k}.\frac{\partial\bar{u}}{\partial z}]$$

$$= (\nabla\varphi).\bar{u} + \varphi(\nabla.\bar{u})$$

$$= (grad\ \varphi).\bar{u} + \varphi div.\bar{u}$$

==

Corollary: If k is constant and $\bar{u}$ is vector point function, then

$div.(k\bar{u}) = kdiv.\bar{u}$ i.e $\nabla.(k\bar{u}) = k(\nabla.\bar{u})$

Proof: Consider

$$div.(k\bar{u}) = \nabla.(k\bar{u})$$

$$= (\nabla k).\bar{u} + k(\nabla.\bar{u})$$

$$= (0).\bar{u} + k(\nabla.\bar{u})$$

$$= k(\nabla.\bar{u})$$

$$= kdiv.\bar{u}$$

==

Theorem-4: If φ is a scalar point function and $\bar{u}$ is vector point function, then

$curl(\varphi\bar{u}) = (grad\ \varphi) \times \bar{u} + \varphi curl\bar{u}$

i.e $\nabla \times (\varphi\bar{u}) = (\nabla\varphi) \times \bar{u} + \varphi(\nabla \times \bar{u})$

Proof: Consider

$$curl\ (\varphi\bar{u}) = \nabla \times (\varphi\bar{u})$$

$$= (\bar{\iota}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}) \times (\varphi\bar{u})$$

$$= \bar{\iota}\frac{\partial}{\partial x} \times (\varphi\bar{u}) + \bar{\jmath}\frac{\partial}{\partial y} \times (\varphi\bar{u}) + \bar{k}\frac{\partial}{\partial z} \times (\varphi\bar{u})$$

$$= \bar{\iota} \times [\frac{\partial\varphi}{\partial x}\bar{u} + \varphi\frac{\partial\bar{u}}{\partial x}] + \bar{\jmath} \times [\frac{\partial\varphi}{\partial y}\bar{u} + \varphi\frac{\partial\bar{u}}{\partial y}] + \bar{k} \times [\frac{\partial\varphi}{\partial z}\bar{u} + \varphi\frac{\partial\bar{u}}{\partial z}]$$

$$= [\frac{\partial\varphi}{\partial x}\bar{\iota} + \frac{\partial\varphi}{\partial y}\bar{\jmath} + \frac{\partial\varphi}{\partial z}\bar{k}] \times \bar{u} + \varphi[\bar{\iota} \times \frac{\partial\bar{u}}{\partial x} + \bar{\jmath} \times \frac{\partial\bar{u}}{\partial y} + \bar{k} \times \frac{\partial\bar{u}}{\partial z}]$$

$$= (\nabla\varphi) \times \bar{u} + \varphi(\nabla \times \bar{u})$$

$$= (grad\ \varphi) \times \bar{u} + \varphi curl\ \bar{u}$$

==

Corollary: If k is constant and $\bar{u}$ is vector point function, then

$curl(k\bar{u}) = kdcurl\ \bar{u}$ i.e $\nabla \times (k\bar{u}) = k(\nabla \times \bar{u})$

Proof: Consider

$$curl\ (k\bar{u}) = \nabla \times (k\bar{u})$$

$$= (\nabla k) \times \bar{u} + k(\nabla \times \bar{u})$$

$$= (0) \times \bar{u} + k(\nabla \times \bar{u})$$

$$= k(\nabla \times \bar{u})$$

$$= k curl\, \bar{u}$$

==

Theorem-5: If $\bar{u}$ and $\bar{v}$ are vector point functions, then

$$div.(\bar{u} \times \bar{v}) = \bar{v}.curl\, \bar{u} - \bar{u}.curl \bar{v}$$

i.e $\nabla.(\bar{u} \times \bar{v}) = \bar{v}.(\nabla \times \bar{u}) - \bar{u}.(\nabla \times \bar{v})$

Proof: Consider

$$div.(\bar{u} \times \bar{v}) = \nabla.(\bar{u} \times \bar{v})$$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(\bar{u} \times \bar{v})$$

$$= \bar{\imath}\frac{\partial}{\partial x}.(\bar{u} \times \bar{v}) + \bar{\jmath}\frac{\partial}{\partial y}.(\bar{u} \times \bar{v}) + \bar{k}\frac{\partial}{\partial z}.(\bar{u} \times \bar{v})$$

$$= \bar{\imath}.[\frac{\partial \bar{u}}{\partial x} \times \bar{v} + \bar{u} \times \frac{\partial \bar{v}}{\partial x}] + \bar{\jmath}.[\frac{\partial \bar{u}}{\partial y} \times \bar{v} + \bar{u} \times \frac{\partial \bar{v}}{\partial y}] + \bar{k}.[\frac{\partial \bar{u}}{\partial z} \times \bar{v} + \bar{u} \times \frac{\partial \bar{v}}{\partial z}]$$

$$= \bar{\imath}.(\frac{\partial \bar{u}}{\partial x} \times \bar{v}) + \bar{\imath}.(\bar{u} \times \frac{\partial \bar{v}}{\partial x}) + \bar{\jmath}.(\frac{\partial \bar{u}}{\partial y} \times \bar{v}) + \bar{\jmath}.(\bar{u} \times \frac{\partial \bar{v}}{\partial y})$$

$$+ \bar{k}.(\frac{\partial \bar{u}}{\partial z} \times \bar{v}) + \bar{k}.(\bar{u} \times \frac{\partial \bar{v}}{\partial z})$$

$$= [(\bar{\imath} \times \frac{\partial \bar{u}}{\partial x}).\bar{v} + \left(\bar{\jmath} \times \frac{\partial \bar{u}}{\partial y}\right).\bar{v}) + \left(\bar{k} \times \frac{\partial \bar{u}}{\partial z}\right).\bar{v})]$$

$$- [(\bar{\imath} \times \frac{\partial \bar{v}}{\partial x}).\bar{u} + (\bar{\jmath} \times \frac{\partial \bar{v}}{\partial y}).\bar{u} + (\bar{k} \times \frac{\partial \bar{v}}{\partial x}).\bar{u}]$$

$$= \bar{v}.[\bar{\imath} \times \frac{\partial \bar{u}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{u}}{\partial y} + \bar{k} \times \frac{\partial \bar{u}}{\partial z}] \times \bar{v} - \bar{u}.[\bar{\imath} \times \frac{\partial \bar{v}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{v}}{\partial y} + \bar{k} \times \frac{\partial \bar{v}}{\partial z}]$$

$$= \bar{v}.(\nabla \times \bar{u}) - \bar{u}.(\nabla \times \bar{v})$$

$$= \bar{v}.curl\, \bar{u} - \bar{u}.curl \bar{v}$$

==

Theorem-6: If φ is a scalar point function, then curl (grad φ) = $\bar{0}$ i.e. $\nabla \times (\nabla\varphi) = \bar{0}$

Proof: Let φ is a scalar point function, then $\nabla\varphi = \frac{\partial \varphi}{\partial x}\bar{\imath} + \frac{\partial \varphi}{\partial y}\bar{\jmath} + \frac{\partial \varphi}{\partial z}\bar{k}$

$$\therefore \text{curl (grad } \varphi) = \nabla \times (\nabla\varphi) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$= (\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y})\bar{\imath} - (\frac{\partial^2\varphi}{\partial x\partial z} - \frac{\partial^2\varphi}{\partial z\partial x})\bar{\jmath}\frac{\partial}{\partial y} + (\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x})\bar{k}$

$= 0\bar{\imath} + 0\bar{\jmath} + 0\bar{k} \quad \because \frac{\partial^2\varphi}{\partial y\partial z} = \frac{\partial^2\varphi}{\partial z\partial y}, \frac{\partial^2\varphi}{\partial x\partial z} = \frac{\partial^2\varphi}{\partial z\partial x}$ and $\frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x}$

$= \bar{0}$

Hence proved.

==

Theorem-7: If $\bar{u}$ is a vector point functions, then div (curl $\bar{u}$) = 0 i.e. $\nabla.(\nabla \times \bar{u}) = 0$

Proof: Let $\bar{u} = u_1\bar{\imath} + u_2\bar{\jmath} + u_3\bar{k}$ is a vector point function, then

$\therefore$ curl $\bar{u} = \nabla \times \bar{u} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_1 & u_2 & u_3 \end{vmatrix}$

$= (\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z})\bar{\imath} - (\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z})\bar{\jmath}\frac{\partial}{\partial y} + (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})\bar{k}$

$\therefore$ div (curl $\bar{u}$) = $\nabla.(\nabla \times \bar{u})$

$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).[(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z})\bar{\imath} - (\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z})\bar{\jmath}\frac{\partial}{\partial y} + (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})\bar{k}]$

$= \frac{\partial}{\partial x}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) - \frac{\partial}{\partial y}(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}) + \frac{\partial}{\partial z}(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y})$

$= \frac{\partial^2 u_3}{\partial x\partial y} - \frac{\partial^2 u_2}{\partial x\partial z} - \frac{\partial^2 u_3}{\partial y\partial x} + \frac{\partial^2 u_1}{\partial y\partial z} + \frac{\partial^2 u_2}{\partial z\partial x} - \frac{\partial^2 u_1}{\partial z\partial y}$

$= 0$

Hence proved.

==

Ex.: If $\bar{r}$ = x $\bar{\imath}$ + y $\bar{\jmath}$ + z $\bar{k}$, then find

i) div $\bar{r}$ ii) curl $\bar{r}$ iii) div ($r^n\bar{r}$) iv) curl ($r^n\bar{r}$) v) Laplacian of r^n

Solution: Let $\bar{r}$ = x $\bar{\imath}$ + y $\bar{\jmath}$ + z $\bar{k}$

$\therefore$ i) div $\bar{r} = \nabla.\bar{r} = (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(x\,\bar{\imath} + y\,\bar{\jmath} + z\,\bar{k})$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1 + 1 + 1$

$= 3$

ii) curl $\bar{r} = \nabla \times \bar{r} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$= \bar{\imath}(0 - 0) - \bar{\jmath}(0 - 0) + \bar{k}(0 - 0)$

$= \bar{0}$

iii) div $(r^n\bar{r}) = \nabla.(r^n\bar{r}) = (\nabla r^n).\bar{r} + r^n(\nabla.\bar{r})$

$= (nr^{n-2}\bar{r}).\bar{r} + r^n(3)$

$= nr^{n-2}(\bar{r}.\bar{r}) + 3r^n$

$= nr^{n-2}(r^2) + 3r^n$

$= nr^n + 3r^n$

$= (n + 3)\, r^n$

iii) curl $(r^n\bar{r}) = \nabla \times(r^n\bar{r}) = (\nabla r^n) \times \bar{r} + r^n(\nabla \times \bar{r})$

$= (nr^{n-2}\bar{r}) \times \bar{r} + r^n(\bar{0})$

$= nr^{n-2}(\bar{r} \times \bar{r}) + \bar{0}$

$= nr^{n-2}(\bar{0}) + \bar{0}$

$= \bar{0}$

iii) Laplacian of $r^n = \nabla^2(r^n) = \nabla.(\nabla r^n)$

$= \nabla.(nr^{n-2}\bar{r})$

$= nr^{n-2}(\nabla.\bar{r}) + n\nabla(r^{n-2}).\bar{r}$

$= nr^{n-2}(3) + n(n-2)\, r^{n-4}\bar{r}.\bar{r}$

$= 3nr^{n-2} + n(n-2)\, r^{n-4}r^2$

$= 3nr^{n-2} + n(n-2)\, r^{n-2}$

$= nr^{n-2}(3+n-2)$

$= n(n+1)\, r^{n-2}$

==

Ex.: If $\bar{f} = x^2y\,\bar{i} + xz\,\bar{j} + 2yz\,\bar{k}$, then verify that div (curl $\bar{f}$) = 0

Proof: Let $\bar{f} = x^2y\,\bar{i} + xz\,\bar{j} + 2yz\,\bar{k}$

$$\therefore \text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$= \bar{i}\,(2z - x) - \bar{j}\,(0 - 0) + \bar{k}(z - x^2)$

$= (2z - x)\bar{i} - 0\bar{j} + (z - x^2)\bar{k}$

$\therefore$ div (curl $\bar{f}$) $= \nabla.\bar{r} = (\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).[(2z - x)\bar{i} - 0\bar{j} + (z - x^2)\bar{k}]$

$= \frac{\partial}{\partial x}(2z - x) - \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z - x^2)$

$= -1 - 0 + 1$

$= 0$

Hence verified.

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Ex.: Prove that the vector function $f(r)\bar{r}$ is irrotational

Proof: Consider

$\therefore$ curl $f(r)\bar{r} = \nabla \times [f(r)\bar{r}]$

$= \nabla f(r) \times \bar{r} + f(r)\, \nabla \times \bar{r}$

$= f'(r)\nabla r \times \bar{r} + f(r)(\nabla \times \bar{r})$

$= f'(r)\frac{\bar{r}}{r} \times \bar{r} + f(r)(\bar{0})$

$= \frac{f'(r)}{r}\bar{r} \times \bar{r}$

$= \bar{0}$

Hence $f(r)\bar{r}$ is irrotational is proved.

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MULTIPLE CHOICE QUESTIONS [MCQ'S]

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1) A scalar point function together with region R is called …...

A) vector field B) scalar field C) region D) None of these

2) A vector point function together with region R is called …...

A) vector field B) scalar field C) region D) None of these

3) Del operator $\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ is denoted by …

A) ∂ B) ∇ C) Δ D) None of these

4) The gradient of a scalar point function φ(x, y, z) is denoted by $\nabla\varphi$ or grad φ and defined as $\nabla\varphi$ = ……

A) $\frac{\partial\varphi}{\partial x}\bar{\imath} + \frac{\partial\varphi}{\partial x}\bar{\jmath} + \frac{\partial\varphi}{\partial x}\bar{k}$ B) $\frac{\partial\varphi}{\partial x}.\bar{\imath} + \frac{\partial\varphi}{\partial x}.\bar{\jmath} + \frac{\partial\varphi}{\partial x}.\bar{k}$

C) $\frac{\partial\varphi}{\partial x} \times \bar{\imath} + \frac{\partial\varphi}{\partial x} \times \bar{\jmath} + \frac{\partial\varphi}{\partial x} \times \bar{k}$ D) None of these

5) The gradient of a scalar point function is a ……

A) scalar point function B) vector point function

C) neither scalar nor vector D) None of these

6) A necessary and sufficient condition for a scalar point function φ(x, y, z) is to be constant is that $\nabla\varphi$ = ……

A) $\bar{0}$ B) 0 C) 1 D) None of these

7) If φ and ψ are scalar point functions and if $grad\ \varphi$ and $grad\ \psi$ exist in a given region R, then $grad\ (\varphi\psi)$ = ……

A) $grad\ \varphi + grad\ \psi$ B) $\varphi\ grad\ \psi - \psi\ grad\ \varphi$

C) $\varphi\ grad\ \psi + \psi\ grad\ \varphi$ D) None of these

8) If φ and ψ are scalar point functions and if $\nabla\varphi$ and $\nabla\psi$ exist in a given region R, then $\nabla(\varphi\psi)$ = ……

A) $\nabla\varphi + \nabla\psi$ B) $\varphi\nabla\psi - \psi\nabla\varphi$ C) $\varphi\nabla\psi + \psi\nabla\varphi$ D) None of these

9) If φ is scalar point functions and k is constant, then $grad(k\varphi)$ = ……

A) $k\ grad\ \varphi$ B) $\varphi\ grad\ k - k\ grad\ \varphi$
C) $\varphi\ grad\ k + k\ grad\ \varphi$ D) None of these

10) If φ is scalar point functions and k is constant, then $\nabla(k\varphi)$ = ……

A) $k\nabla\varphi$ B) $\varphi\nabla k - k\nabla\varphi$ C) $\varphi\nabla k + k\nabla\varphi$ D) None of these

11) If φ and ψ are scalar point functions and if $grad\ \varphi$ and $grad\ \psi$ exist in a given region R with $\psi \neq 0$, then $grad(\frac{\varphi}{\psi})$ = ……

A) $\frac{\psi\ grad\ \varphi - \varphi\ grad\ \psi}{\varphi^2}$ B) $\frac{\psi\ grad\ \varphi - \varphi\ grad\ \psi}{\psi^2}$
C) $\frac{\psi\ grad\ \varphi + \varphi\ grad\ \psi}{\psi^2}$ D) None of these

12) If φ and ψ are scalar point functions and if $\nabla\varphi$ and $\nabla\psi$ exist in a given region R with $\psi \neq 0$, then $\nabla\left(\frac{\varphi}{\psi}\right)$ = ……

A) $\frac{\psi\nabla\varphi - \varphi\nabla\psi}{\varphi^2}$ B) $\frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}$ C) $\frac{\psi\nabla\varphi + \varphi\nabla\psi}{\psi^2}$ D) None of these

13) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, $|\bar{r}|$ = r then $\nabla\varphi(r)$ = ……

A) 0 B) $\nabla\varphi'(r)$ C) $\nabla\varphi'(r)\ \nabla r$ D) None of these

14) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, $|\bar{r}|$ = r then ∇r = ……

A) $\hat{r}$ B) $\bar{r}$ C) 0 D) None of these

15) $\nabla\ log\ r$ = ……

A) $\hat{r}$ B) $\bar{r}$ C) $\frac{\bar{r}}{r^2}$ D) None of these

16) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, $\bar{a}, \bar{b}$ are constant vectors, then $\nabla(\bar{r}\cdot\bar{a})$ = ……

A) $\bar{r}$ B) $\bar{a}$ C) 0 D) None of these

17) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, $\bar{a}, \bar{b}$ are constant vectors, then $\nabla[\bar{r}\ \ \bar{a}\ \ \bar{b}]$ = ……

A) $\bar{r}$ B) $\bar{a}$ C) $\bar{b}$ D) $\bar{a} \times \bar{b}$

18) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, $|\bar{r}|$ = r then ∇r^n = ……

A) $nr^{n-1}\bar{r}$ B) $nr^{n-2}\bar{r}$ C) $n(n-1)r^{n-2}\bar{r}$ D) None of these

19) Components along x, y, z axis of a vector point function $\nabla\varphi$ are …… respectively.

A) $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$ B) $\frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}$ C) $\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial x}$ D) $\frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial x}$

20) Normal to the surface φ(x, y, z) = c at point P(x, y, z) is ……

A) $(\nabla\varphi)_P$ B) $\frac{(\nabla\varphi)_P}{|(\nabla\varphi)_P|}$ C) 0 D) None of these

21) Unit normal to the surface φ(x, y, z) = c at point P(x, y, z) is ……

A) $(\nabla\varphi)_P$ B) $\frac{(\nabla\varphi)_P}{|(\nabla\varphi)_P|}$ C) 0 D) None of these

22) The equation of normal with d.r.s. a, b, c and passing through the point P(x_1, y_1, z_1) is……

A) $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$ B) $\frac{x-x_1}{a} + \frac{y-y_1}{b} + \frac{z-z_1}{c} = 0$

C) $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ D) None of these

==

23) If a, b, c are the d.r.s. of normal, then the equation of plane passing through the point P(x_1, y_1, z_1) is……

A) $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$ B) $\frac{x-x_1}{a} + \frac{y-y_1}{b} + \frac{z-z_1}{c} = 0$

C) $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ D) None of these

24) The divergence of a vector point function $\bar{v}$ is denoted by $\nabla.\bar{v}$ or div $\bar{v}$ and defined as $\nabla.\bar{v}$ = ……

A) $\frac{\partial v}{\partial x}\bar{\imath} + \frac{\partial v}{\partial x}\bar{\jmath} + \frac{\partial v}{\partial x}\bar{k}$ B) $\bar{\imath}.\frac{\partial \bar{v}}{\partial x} + \bar{\jmath}.\frac{\partial \bar{v}}{\partial y} + \bar{k}.\frac{\partial \bar{v}}{\partial z}$

C) $\bar{\imath} \times \frac{\partial \bar{v}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{v}}{\partial y} + \bar{k} \times \frac{\partial \bar{v}}{\partial z}$ D) None of these

25) If $\bar{v} = v_1\bar{\imath} + v_2\bar{\jmath} + v_3\bar{k}$, then div.$\bar{v}$ = $\nabla.\bar{v}$ = ……

A) $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ B) $\bar{\imath}\frac{\partial \bar{v}}{\partial x} + \bar{\jmath}\frac{\partial \bar{v}}{\partial y} + \bar{k}\frac{\partial \bar{v}}{\partial z}$ C) $\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ D) None of these

26) If $\bar{f} = x^2y\,\bar{\imath} - 2xz\,\bar{\jmath} + 2yz\,\bar{k}$, then find div $\bar{f}$ = ……

A) 0 B) 2y(x+1) C) 2x(y+1) D) 2z(x+y)

27) If $\bar{f} = (x^2+yz)\,\bar{\imath} + (y^2+zx)\,\bar{\jmath} + (z^2+xy)\,\bar{k}$, then find div $\bar{f}$ = ……

A) 0 B) 1 C) 2xyz D) 2(x+y+z)

28) The divergence of a vector point function is a ……

A) scalar point function B) vector point function

C) neither scalar nor vector D) None of these

29) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, then div.$\bar{r}$ = $\nabla.\bar{r}$ = ……

A) 0 B) $\bar{0}$ C) 3 D) None of these

30) If divergence of a vector point function $\bar{v}$ is 0, then $\bar{v}$ is called ……

A) irrotational B) solenoidal C) rotational D) None of these

31) If a vector point function $\bar{v}$ is solenoidal, then ……

A) div $\bar{v} = 0$ B) curl $\bar{v} = \bar{0}$ C) grad v = $\bar{0}$ D) None of these

32) A vector point function $\bar{v} = x^2z\bar{\imath} + y^2z\bar{\jmath} - (xz^2 + yz^2)\bar{k}$ is ……

A) irrotational B) solenoidal C) rotational D) None of these

33) If a vector point function $\bar{v} = (x+3y)\bar{\imath} + (y-2z)\bar{\jmath} + (x+az)\bar{k}$ is solenoidal, then a = ……

A) 0 B) -1 C) -2 D) -3

34) $\nabla^2\varphi$ is called …… of scalar point function φ.

A) gradient B) divergence C) curl D) Laplacian

==

35) If $\nabla^2\varphi = 0$, then a scalar point function φ is called ……function

A) Homogeneous B) Harmonic C) Regular D) None of these

36) The curl of a vector point function $\bar{v}$ is denoted by $\nabla \times \bar{v}$ or curl $\bar{v}$ and defined as $\nabla \times \bar{v}$ = ……

A) $\frac{\partial v}{\partial x}\bar{\imath} + \frac{\partial v}{\partial x}\bar{\jmath} + \frac{\partial v}{\partial x}\bar{k}$ B) $\bar{\imath} \cdot \frac{\partial \bar{v}}{\partial x} + \bar{\jmath} \cdot \frac{\partial \bar{v}}{\partial y} + \bar{k} \cdot \frac{\partial \bar{v}}{\partial z}$

C) $\bar{\imath} \times \frac{\partial \bar{v}}{\partial x} + \bar{\jmath} \times \frac{\partial \bar{v}}{\partial y} + \bar{k} \times \frac{\partial \bar{v}}{\partial z}$ D) None of these

37) The curl of a vector point function is a ……

A) scalar point function B) vector point function

C) neither scalar nor vector D) None of these

38) If $\bar{v} = v_1\bar{\imath} + v_2\bar{\jmath} + v_3\bar{k}$, then curl $\bar{v} = \nabla \times \bar{v}$ = ……

A) $\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$ B) $\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ v_1 & v_2 & v_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$ C) $\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ D) None of these

39) A vector point function $\bar{v}$, is said to be irrotational if ……

A) grad v = $\bar{0}$ B) div $\bar{v} = 0$ C) curl $\bar{v} = \bar{0}$ D) None of these

40) A vector point function $\bar{v}$, is said to be …… if curl $\bar{v} = \bar{0}$.

A) irrotational B) solenoidal C) rotational D) None of these

41) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$, then curl $\bar{r} = \nabla \times \bar{r}$ = ……

A) 0 B) $\bar{0}$ C) 3 D) None of these

42) A vector point function $\bar{v} = x^2\bar{\imath} + y^2\bar{\jmath} + z^2\bar{k}$ is ……

A) irrotational B) solenoidal C) rotational D) None of these

43) A vector point function $\bar{v} = (\sin y + z)\bar{\imath} + (x\cos y - z)\bar{\jmath} + (x - y)\bar{k}$ is ……

A) irrotational B) solenoidal C) rotational D) None of these

44) A vector point function $\bar{v} = (y + sinz)\bar{\imath} + x\bar{\jmath} + xcosz\bar{k}$ is ……

A) irrotational B) solenoidal C) rotational D) None of these

45) If φ is a scalar point function and $\bar{u}$ is vector point function, then $div(\varphi\bar{u})$= …

A) $(grad\ \varphi) \times \bar{u} + \varphi div.\bar{u}$ B) $(grad\ \varphi).\bar{u} + \varphi div.\bar{u}$

C) $(grad\ \varphi).\bar{u} + \varphi curl\ \bar{u}$ D) None of these

46) If k is constant and $\bar{u}$ is vector point function, then $\nabla.(k\bar{u})$= …

A) $k(\nabla.\bar{u})$ B) $\bar{u}(\nabla.k)$ C) $k(\nabla \times \bar{u})$ D) None of these

==

47) If φ is a scalar point function and $\bar{u}$ is vector point function, then curl$(\varphi\bar{u})$= …

A) $(grad\ \varphi) \times \bar{u} + \varphi curl\bar{u}$ B) $(grad\ \varphi).\bar{u} + \varphi curl\ \bar{u}$

C) $(grad\ \varphi).\bar{u} + \varphi curl\ \bar{u}$ D) None of these

48) If $\bar{u}$ and $\bar{v}$ are vector point functions, then div $(\bar{u} \times \bar{v})$ = ……

A) $\bar{u}.curl\ \bar{v} - \bar{v}curl\ \bar{u}$ B) $\bar{v}.curl\ \bar{u} - \bar{u}.curl\ \bar{v}$

C) $\bar{v}.curl\ \bar{u} + \bar{u}.curl\ \bar{v}$ D) None of these

49) If φ is a scalar point function, then curl(grad φ) = ……

A) grad φ B) 0 C) $\bar{0}$ D) None of these

50) If $\bar{u}$ is a vector point function, then div (curl $\bar{u}$) = ……

A) grad φ B) 0 C) $\bar{0}$ D) None of these

A) B) C) D)

51) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ and $\bar{a}$ is constant then div$(\bar{r} \times \bar{a})$ = ……

A) 0 B) $\bar{a}$ C) $\bar{r}$ D) None of these

52) If $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ and $\bar{a}$ is constant then curl $(\bar{r} \times \bar{a})$ = ……

A) $\bar{r}$ B) $\bar{a}$ C) $-2\bar{a}$ D) None of these

53) div $(\nabla\varphi \times \nabla\psi)$ = ……

A) 0 B) $\nabla\varphi$ C) $\nabla\psi$ D) None of these

54) $f(r)\,\bar{r}$ is ……

A) scalar B) solenoidal C) irrotational D) None of these

55) curl is also called ……

A) scalar B) rotation C) divergence D) None of these

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UNIT-4: VECTOR INTEGRATION

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An infinite Integral of Vector: Let $\bar{f}(t)$ be a vector valued function of a single scalar variable t. If there exists a vector function $\bar{F}(t)$ such that $\frac{d}{dt}[\bar{F}(t)] = \bar{f}(t)$, then $\bar{F}(t)$ is called an infinite integral or antiderivative of $\bar{f}(t)$. Denoted by $\int \bar{f}(t)dt = \bar{F}(t) + \bar{c}$, where $\bar{c}$ is constant of integration.

Finite Integral of Vector: Let $\bar{f}(t)$ be a vector valued function of a single scalar variable t. If there exists a vector function $\bar{F}(t)$ such that $\frac{d}{dt}[\bar{F}(t)] = \bar{f}(t)$, then $\int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$ is called a finite integral.

Remark: i) If $\bar{f}(t) = f_1(t)\,\bar{\imath} + f_2(t)\,\bar{\jmath} + f_3(t)\,\bar{k}$, then $\int \bar{f}(t)dt = \bar{\imath}\int f_1(t)\,dt + \bar{\jmath}\int f_2(t)dt + \bar{k}\int f_3(t)dt$

ii) $\int [\bar{f}(t) \pm \bar{g}(t)]dt = \int [\bar{f}(t)dt \pm \int \bar{g}(t)]dt$

iii) $\int c\bar{f}(t)dt = c\int [\bar{f}(t)dt$

iv) $\int [\overline{\frac{df}{dt}}.\bar{g} + \bar{f}.\overline{\frac{dg}{dt}}]dt = \bar{f}.\bar{g} + c$

v) $\int [\bar{f} \times \overline{\frac{d^2f}{dt^2}}]dt = \bar{f}\times\overline{\frac{df}{dt}} + \bar{c}$

vi) $\int [\bar{a}\times\overline{\frac{df}{dt}}]dt = \bar{a} \times \bar{f} + c$

==

Ex. If $\bar{f}(t) = \sin t\,\bar{\imath} + \cos t\,\bar{\jmath} + 3\bar{k}$, then evaluate $\int_0^{\frac{\pi}{2}} \bar{f}(t)dt$

Solution: Let $\bar{f}(t) = \sin t\,\bar{\imath} + \cos t\,\bar{\jmath} + 3\bar{k}$

$$\therefore \int_0^{\frac{\pi}{2}} \bar{f}(t)dt = \int_0^{\frac{\pi}{2}}[\sin t\,\bar{\imath} + \cos t\,\bar{\jmath} + 3\bar{k}]dt$$

$$= [-\cos t\,\bar{\imath} + \sin t\,\bar{\jmath} + 3t\bar{k}]_0^{\frac{\pi}{2}}$$

$$= [0\bar{\imath} + \bar{\jmath} + \frac{3\pi}{2}\bar{k}] - [-\bar{\imath} + 0\,\bar{\jmath} + 0\bar{k}]$$

$$= \bar{\imath} + \bar{\jmath} + \frac{3\pi}{2}\bar{k}$$

==

Ex. If $\bar{f}(t) = (t - t^2)\bar{\imath} + 2t^3\bar{\jmath} - 3\bar{k}$, then evaluate $\int_1^2 \bar{f}(t)dt$

Solution: Let $\bar{f}(t) = (t - t^2)\bar{\imath} + 2t^3\bar{\jmath} - 3\bar{k}$

$$\therefore \int_1^2 \bar{f}(t)dt = \int_0^{\frac{\pi}{2}}[(t - t^2)\bar{\imath} + 2t^3\bar{\jmath} - 3\bar{k}]dt$$

$$= [\left(\frac{t^2}{2} - \frac{t^3}{3}\right)\bar{\imath} + \frac{t^4}{2}\bar{\jmath} - 3t\bar{k}]_1^2$$

$$= [\left(2 - \frac{8}{3}\right)\bar{\imath} + 8\bar{\jmath} - 6\bar{k}] - [\left(\frac{1}{2} - \frac{1}{3}\right)\bar{\imath} + \frac{1}{2}\bar{\jmath} - 3\bar{k}]$$

$$= [\left(-\frac{2}{3}\right)\bar{\imath} + 8\bar{\jmath} - 6\bar{k}] - [\left(\frac{1}{6}\right)\bar{\imath} + \frac{1}{2}\bar{\jmath} - 3\bar{k}]$$

$$= \left(-\frac{2}{3} - \frac{1}{6}\right)\bar{\imath} + \left(8 - \frac{1}{2}\right)\bar{\jmath} + (-6 + 3)\bar{k}]$$

$$= \frac{-5}{6}\bar{\imath} + \frac{15}{2}\bar{\jmath} - 3\bar{k}$$

==

Ex. Evaluate $\int_0^1 (e^t\,\bar{\imath} + e^{-2t}\bar{\jmath} + t\,\bar{k})dt$

Solution: Consider

$$\int_0^1 (e^t\,\bar{\imath} + e^{-2t}\bar{\jmath} + t\,\bar{k})dt$$

$$= [e^t\bar{\imath} + \frac{e^{-2t}}{-2}\bar{\jmath} + \frac{t^2}{2}\bar{k}]_0^1$$

$$= [e\bar{\imath} - \frac{e^{-2}}{2}\bar{\jmath} + \frac{1}{2}\bar{k}] - [\bar{\imath} - \frac{1}{2}\bar{\jmath} + 0\bar{k}]$$

$$= (e - 1)\,\bar{\imath} - \frac{1}{2}(e^{-2} - 1)\bar{\jmath} + \frac{1}{2}\bar{k}$$

==

Ex. If $\bar{f} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{g} = 2t^2\bar{\imath} + 6t\bar{k}$, then find $\int_0^1 \bar{f}.\bar{g}\,dt$

Solution: Let $\bar{f} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{g} = 2t^2\bar{\imath} + 6t\bar{k}$

$\therefore\ \bar{f}.\bar{g} = t(2t^2) + (-t^2)(0) + (t-1)(6t) = 2t^3 + 6t^2 - 6t$

$\therefore\ \int_0^1 \bar{f}.\bar{g}\,dt = \int_0^1 (2t^3 + 6t^2 - 6t)dt$

$$= [\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}]_0^1$$

$$= [\frac{1}{2} + 2 - 3] - [0]$$

$$= -\frac{1}{2}$$

==

Ex. If $\bar{u} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{v} = 2t^2\bar{\imath} + 6t\bar{k}$, then find $\int_0^2 \bar{u}.\bar{v}\,dt$

Solution: Let $\bar{u} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{v} = 2t^2\bar{\imath} + 6t\bar{k}$

$\therefore\ \bar{u}.\bar{v} = t(2t^2) + (-t^2)(0) + (t-1)(6t) = 2t^3 + 6t^2 - 6t$

$\therefore\ \int_0^2 \bar{u}.\bar{v}\,dt = \int_0^2 (2t^3 + 6t^2 - 6t)dt$

$$= [\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}]_0^2$$

$$= [8 + 16 - 12] - [0]$$

$$= 12$$

==

Ex. If $\bar{f} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{g} = 2t^2\bar{\imath} + 6t\bar{k}$, then find $\int_0^1 \bar{f}x\bar{g}\,dt$

Solution: Let $\bar{f} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{g} = 2t^2\bar{\imath} + 6t\bar{k}$

$$\therefore \bar{f}x\bar{g} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix} = \bar{\imath}(-6t^3 - 0) - \bar{\jmath}(6t^2 - 2t^3 + 2t^2) + \bar{k}(0 + 2t^4)$$

$$= (-6t^3)\bar{\imath} + (2t^3 - 8t^2)\bar{\jmath} + 2t^4\bar{k}$$

$$\therefore \int_0^1 \bar{f}x\bar{g}\, dt = \int_0^1 [(-6t^3)\bar{\imath} + (2t^3 - 8t^2)\bar{\jmath} + 2t^4\bar{k}]dt$$

$$= \left[\left(-\frac{6t^4}{4}\right)\bar{\imath} + \left(\frac{2t^4}{4} - \frac{8t^3}{3}\right)\bar{\jmath} + \frac{2t^5}{5}\bar{k}\right]_0^1$$

$$= \left[-\frac{3}{2}\bar{\imath} + \left(\frac{1}{2} - \frac{8}{3}\right)\bar{\jmath} + \frac{2}{5}\bar{k}\right] - [0\bar{\imath} + 0\bar{\jmath} + 0\bar{k}]$$

$$= -\frac{3}{2}\bar{\imath} - \frac{13}{3}\bar{\jmath} + \frac{2}{5}\bar{k}$$

==

Ex. If $\bar{u} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{v} = 2t^2\bar{\imath} + 6t\bar{k}$, then find $\int_0^2 \bar{u}x\bar{v}\, dt$

Solution: Let $\bar{u} = t\bar{\imath} - t^2\bar{\jmath} + (t-1)\bar{k}$ and $\bar{v} = 2t^2\bar{\imath} + 6t\bar{k}$

$$\therefore \bar{u}x\bar{v} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix} = \bar{\imath}(-6t^3 - 0) - \bar{\jmath}(6t^2 - 2t^3 + 2t^2) + \bar{k}(0 + 2t^4)$$

$$= (-6t^3)\bar{\imath} + (2t^3 - 8t^2)\bar{\jmath} + 2t^4\bar{k}$$

$$\therefore \int_0^2 \bar{u}x\bar{v}\, dt = \int_0^2 [(-6t^3)\bar{\imath} + (2t^3 - 8t^2)\bar{\jmath} + 2t^4\bar{k}]dt$$

$$= \left[\left(-\frac{6t^4}{4}\right)\bar{\imath} + \left(\frac{2t^4}{4} - \frac{8t^3}{3}\right)\bar{\jmath} + \frac{2t^5}{5}\bar{k}\right]_0^2$$

$$= \left[-24\bar{\imath} + \left(8 - \frac{64}{3}\right)\bar{\jmath} + \frac{64}{5}\bar{k}\right] - [0\bar{\imath} + 0\bar{\jmath} + 0\bar{k}]$$

$$= -24\bar{\imath} - \frac{40}{3}\bar{\jmath} + \frac{64}{5}\bar{k}$$

==

Ex. Prove that $\int_0^{\frac{\pi}{2}} (a\sin t\, \bar{\imath} + b\cos t\, \bar{\jmath})dt = a\,\bar{\imath} + b\,\bar{\jmath}$

Proof: Consider

$$\int_0^{\frac{\pi}{2}} (a\sin t\, \bar{\imath} + b\cos t\, \bar{\jmath})dt$$

$$= [-a\cos t\bar{\imath} + b\sin t\bar{\jmath}]_0^{\frac{\pi}{2}}$$

$$= [0\bar{\imath} + b\bar{\jmath}] - [-a\bar{\imath} + 0\bar{\jmath}]$$

$$= a\,\bar{\imath} + b\,\bar{\jmath}$$

==

Ex. The acceleration of a particle at time t is given by $\bar{a} = 12\cos 2t\, \bar{\imath} - 8\sin 2t\, \bar{\jmath} + 16t\, \bar{k}$.

If velocity $\bar{v}$ and displacement $\bar{r}$ are zero at t = 0, find $\bar{v}$ and $\bar{r}$ at time t.

Solution: We have $\bar{a} = \frac{d\bar{v}}{dt} = 12\cos 2t\,\bar{\imath} - 8\sin 2t\,\bar{\jmath} + 16t\,\bar{k}$

$\therefore \bar{v} = \int[12\cos 2t\,\bar{\imath} - 8\sin 2t\,\bar{\jmath} + 16t\,\bar{k}]\,dt$

$= 6\sin 2t\,\bar{\imath} + 4\cos 2t\,\bar{\jmath} + 8t^2\bar{k} + \bar{c}$

When t = 0, $\bar{v} = \bar{0}$

$\therefore 0\,\bar{\imath} + 4\,\bar{\jmath} + 0\bar{k} + \bar{c} = \bar{0}$

$\therefore \bar{c} = -4\,\bar{\jmath}$

$\therefore \bar{v} = 6\sin 2t\,\bar{\imath} + (4\cos 2t - 4)\,\bar{\jmath} + 8t^2\bar{k}$

As $\bar{v} = \frac{d\bar{r}}{dt} = 6\sin 2t\,\bar{\imath} + (4\cos 2t - 4)\,\bar{\jmath} + 8t^2\bar{k}$

$\therefore \bar{r} = \int[6\sin 2t\,\bar{\imath} + (4\cos 2t - 4)\,\bar{\jmath} + 8t^2\bar{k}]\,dt$

$= -3\cos 2t\,\bar{\imath} + (2\sin 2t - 4t)\,\bar{\jmath} + \frac{8}{3}t^3\bar{k} + \bar{c}$

When t = 0, $\bar{r} = \bar{0}$

$\therefore -3\,\bar{\imath} + 0\,\bar{\jmath} + 0\bar{k} + \bar{c} = \bar{0}$

$\therefore \bar{c} = 3\,\bar{\imath}$

$\therefore \bar{r} = 3(1 - \cos 2t)\,\bar{\imath} + 2(\sin 2t - 2t)\,\bar{\jmath} + \frac{8}{3}t^3\bar{k}$

Ex. The acceleration of a particle at time t is given by $\bar{a} = e^{-t}\,\bar{\imath} - 6(t+1)\,\bar{\jmath} + 3\sin t\,\bar{k}$. If velocity $\bar{v}$ and displacement $\bar{r}$ are zero at t = 0, find $\bar{v}$ and $\bar{r}$ at time t.

Solution: We have $\bar{a} = \frac{d\bar{v}}{dt} = e^{-t}\,\bar{\imath} - 6(t+1)\,\bar{\jmath} + 3\sin t\,\bar{k}$

$\therefore \bar{v} = \int[e^{-t}\,\bar{\imath} - 6(t+1)\,\bar{\jmath} + 3\sin t\,\bar{k}]\,dt$

$= -e^{-t}\,\bar{\imath} - 6\left(\frac{t^2}{2} + t\right)\bar{\jmath} - 3\cos t\,\bar{k} + \bar{c}$

When t = 0, $\bar{v} = \bar{0}$

$\therefore -\bar{\imath} - 0\,\bar{\jmath} - 3\bar{k} + \bar{c} = \bar{0}$

$\therefore \bar{c} = \bar{\imath} + 3\bar{k}$

$\therefore \bar{v} = -e^{-t}\,\bar{\imath} - 6\left(\frac{t^2}{2} + t\right)\bar{\jmath} - 3\cos t\,\bar{k} + \bar{\imath} + 3\bar{k}$

$= (1 - e^{-t})\bar{\imath} - (3t^2 + 6t)\bar{\jmath} + 3(1 - \cos t)\,\bar{k}$

As $\bar{v} = \frac{d\bar{r}}{dt} = (1 - e^{-t})\bar{\imath} - (3t^2 + 6t)\bar{\jmath} + 3(1 - \cos t)\,\bar{k}$

$\therefore \bar{r} = \int[(1 - e^{-t})\bar{\imath} - (3t^2 + 6t)\bar{\jmath} + 3(1 - \cos t)\,\bar{k}]\,dt$

$= (t + e^{-t})\bar{\imath} - (t^3 + 3t^2)\bar{\jmath} + 3(t - \sin t)\,\bar{k} + \bar{c}$

When t = 0, $\bar{r} = \bar{0}$

$\therefore\ \bar{\imath} - 0\,\bar{\jmath} + 0\bar{k} + \bar{c} = \bar{0}$

$\therefore \bar{c} = -\,\bar{\imath}$

$\therefore \bar{r} = (t + e^{-t})\bar{\imath} - (t^3 + 3t^2)\bar{\jmath} + 3(t - \sin t)\,\bar{k} - \bar{\imath}$

$= (e^{-t} + t - 1)\bar{\imath} - (t^3 + 3t^2)\bar{\jmath} + 3(t - \sin t)\,\bar{k}$

==

Line Integral : The line integral of $\bar{f}$ along any curve C lies in a region in which $\bar{f}$ is defined, is the integral of tangential component of $\bar{f}$ along C

i.e. Line integral = $\int_C \bar{f}\,.\,\bar{T}\,ds = \int_C \bar{f}\,.\,\frac{d\bar{r}}{ds}\,ds = \int_C \bar{f}\,.\,\overline{dr}$

Remark: i) If $\bar{f} = f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}$, then line integral of $\bar{f}$ along C is

$\int_C \bar{f}.\overline{dr} = \int_C (f_1\bar{\imath} + f_2\bar{\jmath} + f_3\bar{k}).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k}) = \int_C f_1 dx + f_2 dy + f_3 dz$

ii) If $\bar{f}$ represents the force on a particle moving along C, then the line integral represents the **work done** by the force.

iii) If C is simple closed curve, then the line integral of $\bar{f}$ along C is denoted by $\oint_C \bar{f}.\overline{dr}$

iv) Line integral may or may not depend upon the path of integration.

v) If C is any arc APB in a given region, then $\int_{arcAPB} \bar{f}.\overline{dr} = -\int_{arcPPA} \bar{f}.\overline{dr}$

==

Ex. Evaluate $\int_C \bar{f}.\overline{dr}$, where $\bar{f} = x^2\bar{\imath} + y^3\bar{\jmath}$ and curve C is the arc of the parabola $y = x^2$ in the xy plane from (0, 0) to (1, 1).

Solution: Along the curve C, which is the arc of the parabola $y = x^2$ in the xy plane from (0, 0) to (1, 1), we have $y = x^2$ i.e. dy = 2xdx, where x varies from 0 to 1.

$\int_C \bar{f}.\overline{dr} = \int_C (x^2\bar{\imath} + y^3\bar{\jmath}).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$

$= \int_C (x^2 dx + y^3 dy)$

$= \int_{x=0}^{1} [x^2 dx + x^6(2x)dx]$

$= \int_{x=0}^{1} (x^2 + 2x^7)dx]$

$= \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1$

$= \left(\frac{1}{3} + \frac{1}{4}\right) - 0$

$= \frac{7}{12}$

==

Ex. Evaluate $\int_C [(x^2 + y^2)\bar{i} + (x^2 - y^2)\bar{j}].\overline{dr}$, where C is the straight line joining the points (0, 0) to (1, 1)

Solution: Along the straight C, line joining the points (0, 0) to (1, 1) we have y = x i.e. dy = dx, where x varies from 0 to 1.

$\int_C \bar{f}.\overline{dr} = \int_C [(x^2 + y^2)\bar{i} + (x^2 - y^2)\bar{j}].(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$= \int_C (x^2 + y^2)dx + (x^2 - y^2)dy$

$= \int_{x=0}^{1} [2x^2dx + (0)dx]$

$= [\frac{2x^3}{3}]_0^1$

$= \frac{2}{3} - 0$

$= \frac{2}{3}$

==

Ex. Evaluate $\int_C [(x^2 + y^2)\bar{i} + (x^2 - y^2)\bar{j}].\overline{dr}$, where C is the parabola $y^2 = x$ from (0, 0) to (1, 1)

Solution: Along the straight C, the parabola $y^2 = x$ from (0, 0) to (1, 1) we have $x = y^2$ i.e. dx = 2ydy, where y varies from 0 to 1.

$\int_C \bar{f}.\overline{dr} = \int_C [(x^2 + y^2)\bar{i} + (x^2 - y^2)\bar{j}].(dx\bar{i} + dy\bar{j} + dz\bar{k})$

$= \int_C (x^2 + y^2)dx + (x^2 - y^2)dy$

$= \int_C (y^4 + y^2)(2ydy) + (y^4 - y^2)dy$

$= \int_{x=0}^{1} (2y^5 + 2y^3 + y^4 - y^2)dy$

$= [\frac{2y^6}{6} + \frac{2y^4}{4} + \frac{y^5}{5} - \frac{y^3}{3}]_0^1$

$= (\frac{1}{3} + \frac{1}{2} + \frac{1}{5} - \frac{1}{3}) - 0$

$= \frac{7}{10}$

==

26) If $\bar{F} = \sqrt{y}\,\bar{i} + 2x\,\bar{j} + 3y\,\bar{k}$ and curve C is given by $\bar{r} = t\,\bar{i} + t^2\,\bar{j} + t^3\,\bar{k}$ from t = 0 to t =1, then $\int_C \bar{F}.\overline{dr}$ = ……

A) $\frac{109}{30}$ B) $-\frac{109}{30}$ C) 0 D) None of these

27) $\int(xdy - ydx)$ around the circle $x^2 + y^2 = 1$ is ……

A) -2π B) 2π C) -π D) π

28) If $\bar{f} = 2xy\,\bar{i} + x^2\bar{j}$ and curve C is the straight line joining the points (0, 0) to (1, 1), then $\int_C \bar{f}.\overline{dr}$ = ……

A) 1 B) -1 C) 0 D) None of these

29) If $\bar{f} = 2xy\,\bar{i} + x^2\bar{j}$ and curve C is the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1), then $\int_C \bar{f}.\overline{dr}$ = ……

A) 1 B) -1 C) 0 D) None of these

30) The total work done by a particle moving in a force field $\bar{F} = 3xy\,\bar{i} - 5z\,\bar{j} + 10x\,\bar{k}$ along the curve C: $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from t = 0 to t = 2 is ……

A) 101 B) 202 C) 303 D) None of these

31) If the line integral of a vector field $\bar{f}$ is independent of path of integration in a given region, then $\bar{f}$ is said to be ……

A) non conservative B) conservative C) solenoidal D) None of these

32) If a vector field $\bar{f}$ conservative, then the circulation of $\bar{f}$ about any closed curve in the

region is ……

A) zero B) not zero C) 1 D) None of these

33) If the circulation of $\bar{f}$ about any closed curve in the region is zero, then a vector field $\bar{f}$ is ……

A) non conservative B) conservative C) solenoidal D) None of these

34) If a continuously differentiable vector field $\bar{f}$ is the gradient of some scalar point function φ i.e. $\bar{f} = \nabla\varphi$, then $\bar{f}$ is ……in the given region R.

A) conservative B) not conservative C) solenoidal D) None of these

35) If $\bar{f} = \nabla\varphi$, then φ is called ……of $\bar{f}$.

A) normal B) scalar potential C) vector potential D) None of these

36) If a continuously differentiable vector field $\bar{f}$ is conservative, then $\bar{f}$ is ……

A) solenoidal B) rotational C) irrotational D) None of these

37) If a continuously differentiable vector field $\bar{f}$ is irrotational i.e. curl $\bar{f} = \bar{0}$, then $\bar{f}$ is ……

A) non conservative B) conservative C) solenoidal D) None of these

38) $\bar{f} = (y^2\cos x + z^3)\bar{i} + (2y\sin x - 4)\bar{j} + (3xz^2 + 2)\bar{k}$ is a ……force field.

A) conservative B) non conservative C) solenoidal D) None of these

39) A vector field $\bar{f} = (2xz^3 + 6y)\bar{i} + (6x - 2yz)\bar{j} + (3x^2z^2 - y^2)\bar{k}$ is ……

A) non conservative B) conservative C) solenoidal D) None of these

40) If $\hat{n}$ is the unit normal vector to an element ds, then the surface integral of a vector point function $\bar{F}$ over the surface S is ……

A) $\iint_S (\bar{F}.\hat{n})ds$ B) $\iint_S (\bar{F} \times \hat{n})ds$ C) $\iint_S \bar{F}ds$ D) $\iint_S \hat{n}ds$

41) If $\bar{F}$ represents the velocity of a liquid, then the surface integral of $\bar{F}$ over the surface S i.e. $\iint_S (\bar{F}.\hat{n})ds$ is called……

A) velocity B) acceleration C) flux D) None of these

42) If $\iint_S (\bar{F}.\hat{n})ds = 0$, then $\bar{F}$ is said to be …… vector point function.

A) rotational B) solenoidal C) irrotational D) None of these

43) If $\emptyset(x, y)$, $\psi(x, y)$, $\frac{\partial\emptyset}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ are continuous functions over a region R bounded by simple closed curve C in xy plane, then $\oint_C \emptyset dx + \psi dy = \iint_R (\frac{\partial\psi}{\partial x} - \frac{\partial\emptyset}{\partial y})dxdy$ is the statement of …….

A) Lagrange's theorem B) Euler's theorem

C) Green's theorem D) Stokes theorem

44) By Green's theorem, if $\emptyset(x, y)$, $\psi(x, y)$, $\frac{\partial\emptyset}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ are continuous functions over a region R bounded by simple closed curve C in xy plane, then $\oint_C \emptyset dx + \psi dy =$ …...

A) $\iint_R (\frac{\partial\psi}{\partial x} - \frac{\partial\emptyset}{\partial y})dxdy$ B) $\iint_R (\frac{\partial\emptyset}{\partial y} - \frac{\partial\psi}{\partial x})dxdy$

C) $\iint_R (\frac{\partial\psi}{\partial x} + \frac{\partial\emptyset}{\partial y})dxdy$ D) $\iint_R (\frac{\partial\psi}{\partial y} - \frac{\partial\emptyset}{\partial x})dxdy$

45) If S is a surface bounded by a simple closed curve C and $\bar{F}$ is continuously differentiable vector function, then $\oint_C \bar{F}.\overline{dr} = \iint_S (\text{curl } \bar{F}).\hat{n}ds = \iint_S (\nabla \times \bar{F}).\hat{n}ds$ is the statement of …….

A) Lagrange's theorem B) Euler's theorem

C) Green's theorem D) Stokes theorem

46) If S is a surface bounded by a simple closed curve C and $\bar{F}$ is continuously differentiable vector function, then $\oint_C \bar{F}.\overline{dr} =$ ……

A) $\iint_S (\nabla.\bar{F}).\hat{n}ds$ B) $\iint_S (\nabla \times \bar{F}).\hat{n}ds$

C) $\iint_S (\nabla \times \hat{n}).ds$ D) $\iint_S (\nabla.\hat{n})ds$

47) Unit normal vector to the plane x = 0 is…

A) $\bar{\imath}$ B) $\bar{\jmath}$ C) $\bar{k}$ D) None of these

48) Unit normal vector to the plane y = 0 is…

A) $\bar{\imath}$ B) $\bar{\jmath}$ C) $\bar{k}$ D) None of these

49) Unit normal vector to the plane z = 0 is…

A) $\bar{\imath}$ B) $\bar{\jmath}$ C) $\bar{k}$ D) None of these

50) Unit normal vector to the surface S defined by φ = c is $\hat{n}$ = …

A) $\frac{\nabla\varphi}{|\nabla\varphi|}$ B) $\nabla\varphi$ C) $|\nabla\varphi|$ D) None of these

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच ह्रदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."