Pimpalner Education Society's

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CLASS: S.Y.B.SC SEM.-IV

SUBJECT: MTH-403: PRACTICAL COURSE

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PRACTICAL NO.-1: COMPLEX NUMBERS

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1) Find the modulus and principle value of the argument of $\frac{(1+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}}$

Solution:Let $z = \frac{(1+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}} = \frac{(-i^2+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}}$

$= \frac{i^{13}(\sqrt{3}-i)^{13}}{(\sqrt{3}-i)^{11}}$

$= (i^2)^6 i(\sqrt{3}-i)^2$

$= i(3\text{-}2\sqrt{3}i - 1)$

$= i(\text{-}2\sqrt{3}i + 2)$

$= 2\sqrt{3} + 2i$

$\therefore x = 2\sqrt{3}$ and $y = 2$

$\therefore r = |z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{12+4} = 4$

$\therefore \theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{2\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \in (-\pi, \pi)$ is the principal argument.

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2) If z_1, z_2, z_3 represents vertices of an equilateral triangle, prove that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

Proof: Let A, B and C are the vertices of an equilateral triangle represented by the complex numbers z_1, z_2 and z_3 respectively,

$\therefore l(AB) = |z_2 - z_1|, l(BC) = |z_3 - z_2|, l(AC) = |z_3 - z_1|$ and

$m\angle A = \arg\left(\frac{z_3-z_1}{z_2-z_1}\right)$, $m\angle B = \arg\left(\frac{z_1-z_2}{z_3-z_2}\right)$, $m\angle C = \arg\left(\frac{z_2-z_3}{z_1-z_3}\right)$

As ΔABC is an equilateral triangle

$\therefore l(AB) = l(BC) = l(AC)$ i.e. $|z_2 - z_1| = |z_3 - z_2| = |z_3 - z_1|$

$\therefore \left|\frac{z_3-z_1}{z_2-z_1}\right| = \left|\frac{z_1-z_2}{z_3-z_2}\right| = 1$ ……(1)

and $m\angle A = m\angle B = m\angle C = \frac{\pi}{3}$

i.e. $\arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = \arg\left(\frac{z_1-z_2}{z_3-z_2}\right) = \arg\left(\frac{z_2-z_3}{z_1-z_3}\right) = \frac{\pi}{3}$ ……(2)

By (1) and (2)

$\frac{z_3-z_1}{z_2-z_1} = \frac{z_1-z_2}{z_3-z_2}$

i.e. $z_3^2 - z_3z_2 - z_1z_3 + z_1z_2 = z_1z_2 - z_1^2 - z_2^2 + z_2z_1$

$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

Hence proved.

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3) If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then show that

i) $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$ and

$\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

ii) $\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and

$\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

Proof: Given $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$ … … (1)

Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$

$\therefore\ a + b + c = \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0$ by (1)

$\therefore\ a + b + c = 0$ … … (2)

and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$\therefore \frac{bc + ac + ab}{abc} = 0 + i0$ by (1)

$\therefore ab + bc + ac = 0$ … … (3)

i) As $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$\therefore a^3 + b^3 + c^3 - 3abc = 0$ by (2)

$\therefore a^3 + b^3 + c^3 = 3abc$

$\therefore \cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma$

$= 3[\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

$\therefore (\cos3\alpha + \cos3\beta + \cos3\gamma) + i(\sin3\alpha + \sin3\beta + \sin3\gamma$

$= 3\cos(\alpha + \beta + \gamma) + i3\sin(\alpha + \beta + \gamma)$

Equating real and imaginary parts, we get,

$\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$ and

$\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

ii) As $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 0$ by (2) and (3)

$\therefore \cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0$

$\therefore (\cos2\alpha + \cos2\beta + \cos2\gamma) + i(\sin2\alpha + \sin2\beta + \sin2\gamma) = 0$

Equating real and imaginary parts, we get,

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and

$\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

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4) Find all the values of $(1 + i)^{1/5}$. Show that their continued product is 1 + i.

Proof: Let z =1+ i

$= \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$

$= 2^{1/2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$= 2^{1/2}[\cos(\frac{\pi}{4} + 2k\pi) + i\sin(\frac{\pi}{4} + 2k\pi)]$

$= 2^{1/2}[\cos(\frac{\pi+8k\pi}{4}) + i\sin(\frac{\pi+8k\pi}{4})]$

$\therefore \omega_k = z^{1/5} = (1 + i)^{1/5} = 2^{1/10}\,[\cos(\frac{\pi+8k\pi}{4}) + i\sin(\frac{\pi+8k\pi}{4})]^{1/5}$

$= 2^{1/10}\,[\cos(\frac{\pi+8k\pi}{20}) + i\sin(\frac{\pi+8k\pi}{20})]$, where k = 0, 1, 2, 3, 4.

Puting k = 0, 1, 2, 3, 4. we get all the values of $(1 + i)^{1/5}$ as

$\omega_0 = 2^{1/10}\,[\cos(\frac{\pi}{20}) + i\sin(\frac{\pi}{20})]$,

$\omega_1 = 2^{1/10}\,[\cos(\frac{9\pi}{20}) + i\sin(\frac{9\pi}{20})]$,

$\omega_2 = 2^{1/10}\,[\cos(\frac{17\pi}{20}) + i\sin(\frac{17\pi}{20})]$,

$\omega_3 = 2^{1/10}\,[\cos(\frac{25\pi}{20}) + i\sin(\frac{25\pi}{20})]$,

& $\omega_4 = 2^{1/10}[\cos(\frac{33\pi}{20}) + i\sin(\frac{33\pi}{20})]$.

The continued product of these values is

$\omega_0.\omega_1.\omega_2.\omega_3.\omega_4 = 2^{5/10}\,[\cos(\frac{\pi}{20}+\frac{9\pi}{20}+\frac{17\pi}{20}+\frac{25\pi}{20}+\frac{33\pi}{20}) + i\sin(\frac{\pi}{20}+\frac{9\pi}{20}+\frac{17\pi}{20}+\frac{25\pi}{20}+\frac{33\pi}{20})]$

$= 2^{1/2}\,[\cos(\frac{85\pi}{20}) + i\sin(\frac{85\pi}{20})]$

$= \sqrt{2}\,[\cos(\frac{17\pi}{4}) + i\sin(\frac{17\pi}{4})]$

$= \sqrt{2}\,[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})]$ $\quad \because \frac{17\pi}{4} = 4\pi + \frac{\pi}{4}$

$= \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$

$= 1 + i$

Hence proved

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5) Solve the equation $x^8 - x^4 + 1 = 0$.

Solution: Let $x^8 - x^4 + 1 = 0$ …… (1) be the given equation.

Put $x^4 = z$, we get,

$z^2 - z + 1 = 0$ having roots $z = \frac{1\pm\sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore x^4 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = \cos(\frac{\pi}{3} + 2k\pi) \pm i\sin(\frac{\pi}{3} + 2k\pi)$

$\therefore x_k = [\cos(\frac{\pi+6k\pi}{3}) \pm i\sin(\frac{\pi+6k\pi}{3})]^{1/4}$

$= \cos(\frac{\pi+6k\pi}{12}) \pm i\sin(\frac{\pi+6k\pi}{12})$, where k = 0, 1, 2, 3.

Puting k = 0, 1, 2, 3. we get,

$x_0 = \cos(\frac{\pi}{12}) \pm i\sin(\frac{\pi}{12})$, $x_1 = \cos(\frac{7\pi}{12}) \pm i\sin(\frac{7\pi}{12})$,

$x_2 = \cos(\frac{13\pi}{12}) \pm i\sin(\frac{13\pi}{12})$, and $x_3 = \cos(\frac{19\pi}{12}) \pm i\sin(\frac{19\pi}{12})$

are the roots of given equation.

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6) Determine the region in the z-plane represented by $|z - 3|+|z + 3|=10$

Proof: Let z = x + iy

$\therefore |z - 3|+|z + 3|=10$ gives

$|x + iy - 3|+|x + iy + 3|=10$

i.e. $|(x - 3) + iy|+|(x + 3) + iy|=10$

$\therefore \sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10$

$\therefore \sqrt{(x+3)^2 + y^2} = 10 - \sqrt{(x-3)^2 + y^2}$

Squaring both sides, we get,

$(x + 3)^2 + y^2 = 100 - 20\sqrt{(x-3)^2 + y^2} + (x - 3)^2 + y^2$

$\therefore x^2+6x+9+y^2 = 100-20\sqrt{(x-3)^2 + y^2} + x^2-6x+9+y^2$

$\therefore 12x-100 = -20\sqrt{(x-3)^2 + y^2}$

$\therefore -5\sqrt{x^2 - 6x + 9 + y^2} = 3x-25$

Again squaring both sides, we get,

$25(x^2 - 6x + 9 + y^2)=9x^2-150x+625$

$\therefore 25x^2-150x+225 + 25y^2=9x^2-150x+625$

$\therefore 16x^2+25y^2 = 400$

$\therefore \frac{x^2}{25} + \frac{y^2}{16} = 1$

i.e.The region in the z-plane is the ellipse.

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7) Express $\cos^6\theta$ in terms of cosines of multiples of θ.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$\therefore x + \frac{1}{x} = 2\cos\theta$ and $x^m + \frac{1}{x^m} = 2\cos m\theta$

$\therefore (2\cos\theta)^6 = (x + \frac{1}{x})^6$

$\therefore 64\cos^6\theta = x^6 + 6x^5(\frac{1}{x}) + 15x^4(\frac{1}{x})^2 + 20x^3(\frac{1}{x})^3 + 15x^2(\frac{1}{x})^4 + 6x(\frac{1}{x})^5 + (\frac{1}{x})^6$

$= x^6 + 6x^4 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$

$= (x^6 + \frac{1}{x^6}) + 6(x^4 + \frac{1}{x^4}) + 15(x^2 + \frac{1}{x^2}) + 20$

$\therefore 64\cos^6\theta = (2\cos6\theta) + 6(2\cos4\theta) + 15(2\cos2\theta) + 20$

$\therefore \cos^6\theta = \frac{1}{32}(\cos6\theta + 6\cos4\theta + 15\cos2\theta + 10)$

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8) If $|z_1|=|z_2|=|z_3|=5$ and $z_1 + z_2 + z_3 = 0$ then prove that $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$

Proof: Let $|z_1|=|z_2|=|z_3|=5$ and $z_1+z_2+z_3=0$……(1)

Consider $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}$

$= \frac{\overline{z_1}}{z_1\overline{z_1}} + \frac{\overline{z_2}}{z_2\overline{z_2}} + \frac{\overline{z_3}}{z_3\overline{z_3}}$

$= \frac{\overline{z_1}}{|z_1|^2} + \frac{\overline{z_2}}{|z_2|^2} + \frac{\overline{z_3}}{|z_3|^2}$

$= \frac{\overline{z_1}}{25} + \frac{\overline{z_2}}{25} + \frac{\overline{z_3}}{25}$ by (1)

$= \frac{1}{25}[\overline{z_1}+\overline{z_2}+\overline{z_3}]$

$= \frac{1}{25}[\overline{z_1+z_2+z_3}]$

$= \frac{1}{25}[\overline{0}]$ by (1)

$= 0$

Hence proved.

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PRACTICAL NO.-2: FUNCTIONS OF COMPLEX VARIABLES

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1. Evaluate $\lim_{z\to 1+i}\frac{z^4+4}{z-1-i}$

Sol. Consider $\lim_{z\to 1+i}\frac{z^4+4}{z-1-i}$

$= \lim_{z\to 1+i}\frac{(z^2)^2-(2i)^2}{z-1-i}$

$= \lim_{z\to 1+i}\frac{(z^2-2i)(z^2+2i)}{z-1-i}$

$= \lim_{z\to 1+i}\frac{[z^2-(1+i)^2](z^2+2i)}{z-1-i}$

$= \lim_{z\to 1+i}\frac{(z-1-i)(z+1+i)(z^2+2i)}{z-1-i}$

$= \lim_{z\to 1+i}(z+1+i)(z^2+2i) \quad \because z-1-i\neq 0$

$= (1+i+1+i)[(1+i)^2+2i)]$

$= 2(1+i)[1+2i-1+2i]$

$= 8i(1+i)$

$= 8i - 8$

$= -8(1-i)$

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2. If $(z)=\frac{3z^4-2z^3+8z^2-2z+5}{z-i}$, $z\neq i$ is continuous at $z=i$, then find the value of $f(i)$.

Sol. Let f (z) = $\frac{3z^4-2z^3+8z^2-2z+5}{z-i}$, $z\neq i$ is continuous at $z=i$

$\therefore \lim_{z\to i} f(z) = f(i)$

$\therefore\ f(i) = \lim_{z\to i}\frac{3z^4-2z^3+8z^2-2z+5}{z-i}$

$= \lim_{z\to i}\frac{3z^4+3z^2-2z^3-2z+5z^2+5}{z-i}$

$= \lim_{z\to i}\frac{3z^2(z^2+1)-2z(z^2+1)+5(z^2+1)}{z-i}$

$= \lim_{z\to i}\frac{(z^2+1)(3z^2-2z+5)}{z-i}$

$= \lim_{z\to i}\frac{(z-i)(z+i)(3z^2-2z+5)}{z-i}$

$= \lim_{z\to i}(z+i)(3z^2-2z+5) \quad \because z-i\neq 0$

$= 2i(-3-2i+5)$

$= 2i(-2i+2)$

$= 4i(-i+1)$

$\therefore f(i) = 4(1+i)$

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3. Find an analytic function f(z) = u + iv and express it in terms of z, if $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Solution. Let f(z) = u + iv is an analytic function.

∴ u and v are satisfies C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

As $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is given

∴ $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ and $\frac{\partial u}{\partial y} = -6xy - 6y$…… (2)

Now to find an analytic function f(z) = u + iv, we have to find v.

Consider

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

∴ $dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ by (1)

∴ $dv = (6xy + 6y)\, dx + (3x^2 - 3y^2 + 6x)dy$ by (2)

which is an exact equation.

∴ It's G. S. is

$v = \int_{y-const.} (6xy + 6y)dx + \int(-3y^2)dy + c'$

i.e. $v = 3x^2y + 6xy - y^3 + c'$.

∴ By using this v and given u, an analytic function is

$f(z) = u + iv = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y + 6xy - y^3 + c')$

∴ $f(z) = z^3 + 3z^2 + c$ obtained by putting x = z and y = 0 and taking 1+ic' = c

Which is the required analytic function in z.

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4. Find an analytic function f(z) = u + iv whose imaginary part is $v = e^x(x\sin y + y\cos y)$ using Milne Thomson Method.

Solution. Let $v = e^x(x\sin y + y\cos y)$

∴ $v_x = e^x(x\sin y + y\cos y) + e^x \sin y = e^x(x\sin y + y\cos y + \sin y)$

and $v_y = e^x(x\cos y + \cos y - y\sin y)$

∴ $v_1(z, 0) = v_x(z, 0) = 0$

and $v_2(z, 0) = v_y(z, 0) = e^z(z + 1)$

By Milne Thomson Method, we get,

$f(z) = \int[v_2(z,0) + iv_1(z,0)]dz + c$

$= \int[e^z(z + 1) + 0]dz + c$

$= \int e^z(z + 1)dz + c$

$= ze^z + c$

Which is the required analytic function.

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5. Show that the real and imaginary part of the function e^z satisfy C-R equations and they are harmonic.

Proof. Let $f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y = u + iv$

be a given function with real and imaginary parts are

$u = e^x\cos y$ and $v = e^x\sin y$

Differentiating partially w.r.t. x and y, we get

$\therefore u_x = e^x\cos y,\ u_y = -e^x\sin y,\ v_x = e^x\sin y$ and $v_y = e^x\cos y$

We observe that $u_x = v_y$ and $u_y = -v_x$

Thus, u and v satisfies C-R equations.

Now $u_{xx} = e^x\cos y,\ u_{yy} = -e^x\cos y,\ v_{xx} = e^x\sin y$ and $v_{yy} = -e^x\sin y$

$\therefore u_{xx} + u_{yy} = e^x\cos y - e^x\cos y = 0$ and $v_{xx} + v_{yy} = e^x\sin y - e^x\sin y = 0$

i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$

i.e. u and v satisfies Laplace differential equation

$\therefore$ u and v are satisfies C-R equations and they are harmonic.

Hence proved.

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6. Show that $\frac{1}{2}\log(x^2+y^2)$ satisfies Laplace equation. Finds its harmonic conjugates.

Proof. Let $u = \frac{1}{2}\log(x^2+y^2)$ is an analytic function of z, then

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) = \frac{x}{x^2+y^2} \text{ and } \frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) = \frac{y}{x^2+y^2} \quad \ldots\ldots (1)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \text{ and } \frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0 \text{ i.e. } \nabla^2 u = 0$$

Hence u satisfies Laplace equation is proved.

Now to find harmonic conjugate of u,

Consider

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$\therefore dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ by using C-R equations $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}$

$\therefore dv = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ which is an exact equation.

$\therefore$ It's G. S. is

$$v = \int_{y-const.}\left(-\frac{y}{x^2+y^2}\right)dx + \int 0\,dy + c$$

i.e. $v = -\tan^{-1}\left(\frac{x}{y}\right) + c$ is the harmonic conjugate of u.

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7. If f(z) is analytic function with constant modulus, then show that f(z) is a constant function.

Proof. Let f (z) = u + iv is analytic function with constant modulus.

∴ u and v are satisfies C-R equations

$u_x = v_y$ and $u_y = -v_x$ …… (1)

and $|f(z)| = \sqrt{u^2 + v^2}$ is constant say k.

i.e. $\sqrt{u^2 + v^2} = k$

∴ $u^2 + v^2 = k^2$ …… (2)

Differentiating equation (2) partially w.r.t. x and y, we get,

$2uu_x + 2vv_x = 0$ i.e. $uu_x - vu_y = 0$ …… (3) by (1) $v_x = -u_y$

and $2uu_y + 2vv_y = 0$ i.e. $uu_y + vu_x = 0$ …… (4) by (1) $v_y = u_x$

Consider u(3) + v(4), we get,

$u^2u_x - uvu_y + vuu_y + v^2u_x = 0$

i.e. $(u^2 + v^2)u_x = 0$

Similarly u(4)-v(3) gives $(u^2 + v^2)u_y = 0$.

If $u^2 + v^2 = 0$, then u = v = 0 and hence f(z) = 0 is constant function.

But if $u^2 + v^2 \neq 0$, then $u_x = 0$ and $u_y = 0$

∴ $f'(z) = u_x + iv_x = u_x - iu_y = 0 - i0 = 0$.

∴ f(z) is a constant function is proved.

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8. Evaluate $\lim_{z \to e^{i\pi/3}} \frac{(z - e^{\frac{i\pi}{3}})z}{z^3 + 1}$

Sol. Consider $\lim_{z \to e^{i\pi/3}} \frac{(z - e^{\frac{i\pi}{3}})z}{z^3 + 1}$

$$= \lim_{z \to e^{i\pi/3}} \frac{(z - e^{\frac{i\pi}{3}})z}{z^3 - (e^{i\pi/3})^3}$$

$$= \lim_{z \to e^{i\pi/3}} \frac{(z - e^{\frac{i\pi}{3}})z}{(z - e^{i\pi/3})\,[z^2 + ze^{i\pi/3} + (e^{i\pi/3})^2]}$$

$$= \lim_{z \to e^{i\pi/3}} \frac{z}{z^2 + ze^{i\pi/3} + (e^{i\pi/3})^2} \qquad \because z - e^{i\pi/3} \neq 0$$

$$= \frac{e^{i\pi/3}}{e^{i2\pi/3} + e^{i2\pi/3} + e^{i2\pi/3}}$$

$$= \frac{e^{i\pi/3}}{3e^{i2\pi/3}}$$

$$= \frac{1}{3} e^{-i\pi/3}$$

$$= \frac{1}{3} (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$$

$$= \frac{1}{3} (\frac{1}{2} - i\frac{\sqrt{3}}{2})$$

$$= \frac{1}{6} (1 - i\sqrt{3})$$

PRACTICAL NO.-3: COMPLEX INTEGRATION

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1) Evaluate $\int_C f(y - x - 3x^2 i)dz$, where C is:

i) The straight line joining z = 0 to z = 1 + i

ii) The straight line joining z = 0 to z = i first and then from z = i to z = 1 + i

Solution: i) Parametric equation of the line segment C: z = 0 to z = 1 + i is x = t, y = t, so that z = x + iy = t + it = (1+i)t, $0 \le t \le 1$.

$\therefore$ f(z) = y – x – $3x^2$i = t – t – $3t^2$i = – $3t^2$i and dz = (1+i)dt

$\therefore \int_C f(z)dz = \int_{t=0}^{1}(-3t^2 i)\,(1 + i)dt$

$= -i(1 + i)\,[t^3]_0^1$

$= (-i + 1)[1 – 0]$

$= 1 - i$

ii) Let C = $C_1 + C_2$, where C_1 is the straight line segments from z = 0 to z = i and C_2 is the straight line segments from z = i to z = 1 + i

$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ …… (1)

Parametric equation of the line segment C_1: z = 0 to z = i is x = 0, y = t, so that z = x + iy = 0 + it = ti, $0 \le t \le 1$.

$\therefore$ f(z) = y – x – $3x^2$i = t – 0 – 0i = t and dz = idt

$\therefore \int_C f(z).\,dz = \int_{t=0}^{1} tidt$

$= i[\frac{t^2}{2}]_0^1$

$= i[\frac{1}{2} - 0]$

$= \frac{1}{2}$ i

Again parametric equation of the line segment C_2: z = i to z = 1 + i is x = t, y =1 so that z = x + iy = t + i , $0 \le t \le 1$.

$\therefore$ f(z) = y – x – $3x^2$i = 1 – t – $3t^2$i and dz = dt

$\therefore \int_C f(z)dz = \int_{t=0}^{1}(1 - t - 3t^2 i)dt$

$= [t - \frac{t^2}{2} - t^3 i]_0^1$

$= [1 - \frac{1}{2} - i - 0]$

$= \frac{1}{2} - i$

Putting in (1), we get,

$\int_C f(z)dz = \frac{1}{2}$ i + $\frac{1}{2}$ – i = $\frac{1}{2}$ (1 – i)

==

2) Use Cauchy Goursat Theorem to obtain the value $\int_C e^z dz$, where C is the circle $|z| = 1$ and hence deduce that
i) $\int_0^{2\pi} e^{cos\theta} \sin(\theta + \sin\theta)\, d\theta = 0$ and ii) $\int_0^{2\pi} e^{cos\theta} \cos(\theta + \sin\theta)\, d\theta = 0$

Proof: Take f(z) = e^z which is analytic everywhere in the complex plane, hence it is analytic inside and on the circle C: $|z| = 1$

∴ By Cauchy's Integral Theorem, $\int_C f(z)dz = 0$.

i.e. $\int_C e^z dz = 0$ …… (1)

Now parametric equation of C is z = $e^{i\theta}$, $0 \le \theta \le 2\pi$.

∴ dz = $e^{i\theta} i d\theta$

$$\therefore \int_C f(z)dz = \int_{|z|=1} e^{e^{i\theta}}\, e^{i\theta} i d\theta$$
$$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\, e^{i\theta} i d\theta$$
$$= i\int_0^{2\pi} e^{\cos\theta + i(\theta + sin\theta)}\, d\theta$$
$$= i\int_0^{2\pi} e^{\cos\theta}\, e^{i(\theta + sin\theta)} d\theta$$
$$= i\int_0^{2\pi} e^{\cos\theta}\, [\cos(\theta + sin\theta) + i\sin(\theta + sin\theta)]\, d\theta$$

But $\int_C f(z)dz = 0$

∴ $i\int_0^{2\pi} e^{\cos\theta} \cos(\theta + sin\theta) - \int_0^{2\pi} e^{\cos\theta} \sin(\theta + sin\theta)\, d\theta = 0 = 0 + i0$

Equating real and imaginary parts, we get,

i) $\int_0^{2\pi} e^{cos\theta} \sin(\theta + \sin\theta)\, d\theta = 0$ and ii) $\int_0^{2\pi} e^{cos\theta} \cos(\theta + \sin\theta)\, d\theta = 0$

Hence proved.

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3) Using Cauchy's Integral formula, evaluate $\int_C \frac{dz}{z^3(z+4)} dz$, where C is the circle $|z| = 2$

Solution: We observe that $\frac{1}{z^3(z+4)}$ is not analytic at z = 0 and z = -4, out of these only the point z = 0 lies inside circle C: $|z| = 2$.

∴ We take f(z) = $\frac{1}{(z+4)}$ which is analytic inside and on the circle C: $|z| = 2$ and the point z = 0 lies inside C.

∴ By Cauchy's integral formula for f ''(a), we have,

$$f''(0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-0)^3} dz$$
$$\therefore \int_C \frac{f(z)}{z^3} dz = \pi i\, f''(0)$$

As f(z) = $\frac{1}{(z+4)}$ ∴ f '(z) = $\frac{-1}{(z+4)^2}$ & f ''(z) = $\frac{2}{(z+4)^3}$ ∴ f ''(0) = $\frac{2}{64} = \frac{1}{32}$

$$\therefore \int_C \frac{1}{z^3(z+4)} dz = \frac{\pi i}{32}$$

==

4) Obtain the expansion of $(z) = \frac{z^2-1}{(z+2)(z+3)}$, in the powers of z in the region

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$.

Solution: First we express $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ into partial fractions as follows

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{A}{(z+2)}+\frac{B}{(z+3)} \;\ldots\ldots (1)$$

i.e. $z^2 - 1 = (z + 2)(z + 3) + A(z + 3) + B(z + 2)$ ……(2)

Putting z = -2 in (2), we get,

$4 - 1 = 0 + A + 0 \quad \therefore A = 3$

Again putting z = -3 in (2), we get,

$9 - 1 = 0 + 0 - B \quad \therefore B = -8$

From (1), we have,

$$f(z) = 1+\frac{3}{(z+2)} - \frac{8}{(z+3)}$$

(i) $|z| < 2 \Rightarrow |z| < 3 \Rightarrow |\frac{z}{2}| < 1$ & $|\frac{z}{3}| < 1$

$$\therefore f(z) = 1+\frac{3}{(z+2)} - \frac{8}{(z+3)} = 1+\frac{3}{2}\frac{1}{(1+\frac{z}{2})} - \frac{8}{3}\frac{1}{(1+\frac{z}{3})}$$

$$= 1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z}{3}\right)^n$$

by Taylor’s series expansion

$$= 1+ 3\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{2^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^{n+1}}$$

(ii) $2 < |z| < 3 \Rightarrow 2 < |z|$ & $|z| < 3 \Rightarrow |\frac{2}{z}| < 1$ & $|\frac{z}{3}| < 1$

$$\therefore f(z) = 1+\frac{3}{(z+2)} - \frac{8}{(z+3)} = 1+\frac{3}{z}\frac{1}{(1+\frac{2}{z})} - \frac{8}{3}\frac{1}{(1+\frac{z}{3})}$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z}{3}\right)^n$$

by Taylor’s series expansion

$$= 1+ 3\sum_{n=0}^{\infty}(-1)^n \frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^{n+1}}$$

(iii) $|z| > 3 \Rightarrow |z| > 2 \Rightarrow |\frac{3}{z}| < 1$ & $|\frac{2}{z}| < 1$

$$\therefore f(z) = 1+\frac{3}{(z+2)} - \frac{8}{(z+3)} = 1+\frac{3}{z}\frac{1}{(1+\frac{2}{z})} - \frac{8}{z}\frac{1}{(1+\frac{3}{z})}$$

$$= 1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{3}{z}\right)^n$$

by Taylor’s series expansion

$$= 1+ 3\sum_{n=0}^{\infty}(-1)^n \frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n \frac{3^n}{z^{n+1}}$$

==

5) Prove that $\frac{1}{4z-z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$, where $0 < |z| < 4$.

Proof : $0 < |z| < 4 \Longrightarrow |\frac{z}{4}| < 1$

Consider L.H.S. $= \frac{1}{4z-z^2}$

$= \frac{1}{4z(1-\frac{z}{4})}$

$= \frac{1}{4z} \sum_{n=0}^{\infty} (\frac{z}{4})^n$ by Taylor's series expansion

$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$

= R.H.S.

Hence proved.

==

6) Verify is Cauchy's Integral Theorem for $f(z) = z^2$ around the circle $|z| = 1$.

Proof: Here the closed contour C is the circle $|z| = 1$, which is simple closed curve.

As $f(z) = z^2$ is analytic everywhere in the complex plane, hence it is analytic inside and on C.

∴ By Cauchy's Integral Theorem, $\int_C f(z)dz = 0$.

i.e. $\int_C z^2 dz = 0$ …… (1)

Now parametric equation of C is $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

∴ $dz = e^{i\theta} i d\theta$

∴ $\int_C f(z)dz = \int_{|z|=1} z^2 dz$

$= \int_0^{2\pi} (e^{i\theta})^2 e^{i\theta} i d\theta$

$= \int_0^{2\pi} (e^{3i\theta})\, i d\theta$

$= i[\frac{e^{3i\theta}}{3i}]_0^{2\pi}$

$= (\frac{e^{6\pi i}}{3} - \frac{e^0}{3})$

$= \frac{1}{3} - \frac{1}{3}$

∴ $\int_C f(z)dz = 0$

Hence Cauchy's theorem is verified.

==

7) Evaluate $\int_{|z|=2} \frac{e^{2z}}{(z-1)^4} dz$, Using Cauchy's Integral formula.

Solution: We take f(z) = e^{2z} which is analytic inside and on the circle C: $|z|$ = 2 and the point z = 1 lies inside C.

∴ By Cauchy's integral formula for f '''(a), we have,

$$f'''(1) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-1)^4} dz$$

$$\therefore \int_C \frac{f(z)}{(z-1)^4} dz = \frac{1}{3} \pi i \, f'''(0)$$

As f(z) = e^{2z} ∴ f '(z) = $2e^{2z}$, f ''(z) = $4e^{2z}$ & f '''(z) = $8e^{2z}$ ∴ f '''(1) = $8e^2$

$$\therefore \int_{|z|=2} \frac{e^{2z}}{(z-1)^4} dz = \frac{8}{3} \pi e^2 i$$

==

8) Find the expansion of $f(z) = \frac{1}{(z^2+1)(z^2+2)}$ in powers of z, when $|z| < 1$

Solution: $|z| < 1 \Longrightarrow |z^2| < 1$

Now $f(z) = \frac{1}{(z^2+1)(z^2+2)} = \frac{1}{(z^2+1)} - \frac{1}{(z^2+2)}$

$$= \frac{1}{(1+z^2)} - \frac{1}{2(1+\frac{z^2}{2})}$$

$$= \sum_{n=0}^{\infty} (-1)^n (z^2)^n - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (\frac{z^2}{2})^n$$

by Taylor's series expansion

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^n}$$

$\therefore f(z) = \sum_{n=0}^{\infty} (-1)^n (1 - \frac{1}{2^n}) z^{2n}$ be the required expansion, when $|z| < 1$

==

PRACTICAL NO.-4: CALCULUS OF RESIDUES

1) Find the residue of $f(z) = \frac{z^2+2z}{(z+1)^2(z+4)}$ at its poles.

Solution: Given function $f(z) = \frac{z^2+2z}{(z+1)^2(z+4)}$ has double pole at z = -1 and simple pole at z = -4.

$$\therefore \operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} \frac{d}{dz}[(z+1)^2 f(z)]$$
$$= \lim_{z\to -1} \frac{d}{dz}\left[\frac{z^2+2z}{(z+4)}\right]$$
$$= \lim_{z\to -1} \left[\frac{(z+4)(2z+2)-(z^2+2z)(1)}{(z+4)^2}\right]$$
$$= \lim_{z\to -1} \left[\frac{z^2+8z+8}{(z+4)^2}\right]$$
$$= \frac{1-8+8}{(3)^2}$$
$$= \frac{1}{9}$$

$$\& \operatorname*{Res}_{z=-4} f(z) = \lim_{z\to -4} [(z+4)f(z)]$$
$$= \lim_{z\to -4} \left[\frac{z^2+2z}{(z+1)^2}\right]$$
$$= \frac{16-8}{(-3)^2}$$
$$= \frac{8}{9}$$

2) Evaluate $\int_{|z|=3} \frac{e^z}{z(z-1)^2} dz$ by Cauchy's residue

Solution: Given integrant $f(z) = \frac{e^z}{z(z-1)^2}$ has simple pole at z = 0 and double pole at z = 1. Both these poles lies inside circle C: $|z| = 3$ and f(z) is analytic inside and on C except these poles.

∴ By Cauchy's Residue Theorem,

$$\int_C f(z)dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right] \;\ldots\ldots (1)$$

Now $\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} [(z-0)f(z)]$

$$= \lim_{z\to 0} \left[\frac{e^z}{(z-1)^2}\right]$$
$$= \frac{1}{(-1)^2}$$
$$= 1$$

$$\& \operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}[(z-1)^2 f(z)]$$
$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{e^z}{z}\right]$$
$$= \lim_{z\to 1} \left[\frac{ze^z - e^z(1)}{z^2}\right]$$
$$= \frac{e-e}{(1)^2}$$
$$= 0$$

Putting in (1), we get,

$$\int_C f(z)dz = 2\pi i\,[1+0]$$
$$\therefore \int_{|z|=3} \frac{e^z}{z(z-1)^2} dz = 2\pi i$$

==

3) Evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz$ by Cauchy's residue theorem, where C is

(i) The circle $|z-2| = 2$ (ii) The circle $|z| = 4$

Solution: Given integrant $f(z) = \frac{3z^2+2}{(z-1)(z^2+9)} = \frac{3z^2+2}{(z-1)(z-3i)(z+3i)}$ has simple poles at z = 1, z = 3i and z = -3i.

Now $\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} [(z-1)f(z)]$

$$= \lim_{z\to 1} \left[\frac{3z^2+2}{z^2+9}\right]$$
$$= \frac{5}{10}$$
$$= \frac{1}{2}$$

$\& \operatorname{Res}_{z=3i} f(z) = \lim_{z\to 3i} [(z-3i)f(z)]$

$$= \lim_{z\to 3i} \left[\frac{3z^2+2}{(z-1)(z+3i)}\right]$$
$$= \frac{-27+2}{(3i-1)(6i)}$$
$$= \frac{-25}{6(-3-i)}$$
$$= \frac{25}{6(3+i)} \times \frac{(3-i)}{(3-i)}$$
$$= \frac{25(3-i)}{6(9+1)}$$
$$= \frac{5}{12}(3-i)$$
$$= \frac{5}{4} - \frac{5}{12} i$$

Similarly, $\underset{z=-3i}{\text{Res}} f(z) = \frac{5}{4} + \frac{5}{12}i$

i) Let C is the circle $|z-2| = 2$, then only the pole z = 1 lies inside circle C and f(z) is analytic inside and on C except this pole.

∴ By Cauchy's Residue Theorem,

$\int_C f(z)dz = 2\pi i\,[\underset{z=1}{\text{Res}} f(z)]$

$\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz = 2\pi i\,[\frac{1}{2}] = \pi i$

ii) Let C is the circle $|z| = 4$, then all the poles z = 1, z = 3i and z = -3i lies inside circle C and f(z) is analytic inside and on C except these poles.

∴ By Cauchy's Residue Theorem,

$\int_C f(z)dz = 2\pi i\,[\underset{z=1}{\text{Res}} f(z) + \underset{z=3i}{\text{Res}} f(z) + \underset{z=-3i}{\text{Res}} f(z)]$

$= 2\pi i\,[\frac{1}{2} + \frac{5}{4} - \frac{5}{12}i + \frac{5}{4} + \frac{5}{12}i]$

$= 2\pi i\,(3)$

$\therefore \int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz = 6\pi i$

==

4) Use the contour integration to evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Put $z = e^{i\theta}$ $\therefore d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, where $0 \leq \theta \leq 2\pi$

$\therefore I = \int_C \frac{1}{5+\frac{3}{2}(z+\frac{1}{z})} \frac{dz}{iz}$ where C is the unit circle $|z| = 1$

$= \int_C \frac{-2i}{5+\frac{3}{2}(z+\frac{1}{z})} \frac{dz}{2z}$

$= \int_C \frac{-2i}{10z+3z^2+3} dz$

$\therefore I = \int_C f(z)dz$

where $f(z) = \frac{-2i}{3z^2+10z+3} = \frac{-2i}{(3z+1)(z+3)}$ has simple poles at $z = \frac{-1}{3}$ and z = -3.

Out of these only the pole $z = \frac{-1}{3}$ lies inside the unit circle C: $|z| = 1$ and f(z) is analytic inside and on C except this pole.

∴ By Cauchy's residue theorem,

$\int_C f(z)dz = 2\pi i\,[\underset{z=\frac{-1}{3}}{\text{Res}} f(z)]$

$\therefore I = 2\pi i \lim_{z\to\frac{-1}{3}} [(z+\frac{1}{3})f(z)]$

$= \frac{2}{3}\pi i \lim_{z\to\frac{-1}{3}} [(3z+1)f(z)]$

$= \frac{2}{3}\pi i \lim_{z \to \frac{-1}{3}} [\frac{-2i}{(z+3)}]$

$= \frac{2}{3}\pi i \, [\frac{-2i}{(\frac{-1}{3}+3)}]$

$= \frac{4\pi}{(-1+9)}$

$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$

==

5) Evaluate by contour integration $\int_{-\infty}^{\infty} \frac{1}{x^4+13x^2+36} dx$.

Solution: Let $I = \int_{-\infty}^{\infty} \frac{1}{x^4+13x^2+36} dx$

Then, here $P(x) = 1$ and $Q(x) = x^4 + 13x^2 + 36$ and $f(x) = \frac{P(x)}{Q(x)}$.

i) P(x) and Q(x) are polynomials in x.

ii) degree of Q(x) - degree of P(x) = 4 – 0 = 4 ≥ 2

iii) Q(x) = 0 gives $x^4 + 13x^2 + 36 = 0$ i.e. $(x^2 + 4)(x^2 + 9) = 0$

$\therefore$ $\pm 2i$ and $\pm 3i$ are the roots of Q(x) = 0 i.e. Q(x) = 0 has no real roots.

$\therefore$ I = 2πi [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

$\therefore I = 2\pi i \, [\operatorname{Res}_{z=2i} f(z) + \operatorname{Res}_{z=3i} f(z)]$ …… (1)

Now $f(z) = \frac{1}{z^4+13z^2+36} = \frac{1}{(z^2+4)(z^2+9)} = \frac{1}{(z-2i)(z+2i)(z-3i)(z+3i)}$

$\therefore \operatorname{Res}_{z=2i} f(z) = \lim_{z\to 2i} [(z-2i)f(z)]$

$= \lim_{z\to 2i} [\frac{1}{(z+2i)(z^2+9)}]$

$= \frac{1}{4i(-4+9)}$

$= \frac{1}{20i}$

& $\operatorname{Res}_{z=3i} f(z) = \lim_{z\to 3i} [(z-3i)f(z)]$

$= \lim_{z\to 3i} [\frac{1}{(z+3i)(z^2+4)}]$

$= \frac{1}{6i(-9+4)}$

$= \frac{-1}{30i}$

Putting in (1), we get,

$I = 2\pi i \, [\frac{1}{20i} - \frac{1}{30i}] = \pi \, [\frac{1}{10} - \frac{1}{15}]$

$\therefore \int_{-\infty}^{\infty} \frac{1}{x^4+13x^2+36} dx = \frac{\pi}{30}$

==

6) Find the sum of residue of $f(z) = \frac{e^z}{z^2+a^2}$ at its poles.

Solution: Given function $f(z) = \frac{e^z}{z^2+a^2} = \frac{e^z}{(z-ai)(z+ai)}$ has simple poles at z = ai and z = -ai.

$\therefore \underset{z=ai}{\text{Res}}\, f(z) = \lim_{z\to ai} [(z-ai)f(z)]$

$= \lim_{z\to ai} [\frac{e^z}{(z+ai)}]$

$= \frac{e^{ai}}{2ai}$

Similarly $\underset{z=-ai}{\text{Res}}\, f(z) = \frac{e^{-ai}}{-2ai}$

$\therefore$ The sum of residues $= \underset{z=ai}{\text{Res}}\, f(z) + \underset{z=-ai}{\text{Res}}\, f(z)$

$= \frac{e^{ai}}{2ai} - \frac{e^{-ai}}{2ai}$

$= \frac{1}{a}(\frac{e^{ai}-e^{-ai}}{2i})$

$= \frac{\sin a}{a}$

==

7) Evaluate $\int_{|z|=2} \frac{dz}{z^3(z+4)}$ by Cauchy's residue theorem.

Solution: Given function $f(z) = \frac{1}{z^3(z+4)}$ has pole of order 3 at z = 0 and simple pole at z = -4. Out of these only the pole z = 0 lies inside the circle C: |z| = 2 and f(z) is analytic inside and on C except this pole.

$\therefore$ By Cauchy's residue theorem,

$\int_C f(z)dz = 2\pi i\, [\underset{z=0}{\text{Res}}\, f(z)]$

$\therefore \int_{|z|=2} \frac{dz}{z^3(z+4)} = 2\pi i\{\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}[(z-0)^3 f(z)]\}$

$= \pi i \lim_{z\to 0}\frac{d}{dz}\{\frac{d}{dz}[\frac{1}{(z+4)}]\}$

$= \pi i \lim_{z\to 0}\frac{d}{dz}[\frac{-1}{(z+4)^2}]$

$= \pi i \lim_{z\to 0}[\frac{2}{(z+4)^3}]$

$= \pi i\, [\frac{2}{(4)^3}]$

$= \frac{\pi i}{32}$

==

8) Evaluate $\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}$ by contour integration.

Solution: Let I = $\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}$ …… (1)

Then, here P(x) = x^2 and Q(x) = $(x^2+1)(x^2+4)$ and f(x) = $\frac{P(x)}{Q(x)}$.

i) P(x) and Q(x) are polynomials in x.

ii) degree of Q(x) - degree of P(x) = 4 – 2 = 2 ≥ 2

iii) Q(x) = 0 gives $(x^2+1)(x^2+4) = 0$

∴ $\pm$i and $\pm$2i are the roots of Q(x) = 0 i.e. Q(x) = 0 has no real roots.

∴ I = 2πi [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

∴ I = 2πi $[\underset{z=i}{\text{Res}} f(z) + \underset{z=2i}{\text{Res}} f(z)]$ …… (1)

Now f(z) = $\frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)}$

∴ $\underset{z=i}{\text{Res}} f(z) = \lim_{z\to i} [(z-i)f(z)]$

$= \lim_{z\to i} [\frac{z^2}{(z+i)(z^2+4)}]$

$= \frac{-1}{2i(-1+4)}$

$= \frac{-1}{6i}$

& $\underset{z=2i}{\text{Res}} f(z) = \lim_{z\to 2i} [(z-2i)f(z)]$

$= \lim_{z\to 2i} [\frac{z^2}{(z+2i)(z^2+1)}]$

$= \frac{-4}{4i(-4+1)}$

$= \frac{1}{3i}$

Putting in (1), we get,

I = 2πi $[\frac{-1}{6i} + \frac{1}{3i}]$ = π $[-\frac{1}{3} + \frac{2}{3}]$

∴ $\int_{-\infty}^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$

From (1), we get,

$\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{1}{2}(\frac{\pi}{3}) = \frac{\pi}{6}$

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PRACTICAL NO.-5: THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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1) Show that the function f(x, y) = xy^2 satisfies Lipchitz's condition on the rectangle $R: |x| \leq 1, |y| \leq 1$, but does not satisfy Lipchitz's condition on strip $S: |x| \leq 1, |y| \leq \infty$.

Proof: Let f(x, y) = xy^2 …… (1)

i) Let R is a rectangle given by $|x| \leq 1, |y| \leq 1$ …… (2)

Clearly f(x, y) = xy^2 is continuous function on R and hence bounded on R

with $\frac{\partial f}{\partial y} = 2xy \Rightarrow \left|\frac{\partial f}{\partial y}\right| = 2|x||y| \leq 2(1)(1) \leq 2\ \forall\ (x, y) \in R$

∴ f(x, y) satisfies Lipchitz's condition on R and Lipchitz's constant K = 2.

i) Let R is a strip given by $|x| \leq 1, |y| \leq \infty$ …… (2)

Here f(x, y) = xy^2 is continuous function on S and hence bounded on S

with $\frac{\partial f}{\partial y} = 2xy \Rightarrow \left|\frac{\partial f}{\partial y}\right| = 2|x||y| \leq 2(1)(\infty) < \infty\ \ \forall\ (x, y) \in S$

$\Rightarrow \frac{\partial f}{\partial y}$ is unbounded on strip S.

∴ f(x, y) does not satisfy Lipchitz's condition on strip S is proved.

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2) Prove that sin2x and cos2x are solutions of the y"+ 4y = 0 and these solutions are linearly independent.

Proof: Let y_1 = sin2x and y_2 = cos2x …… (1)

∴ y_1' = 2cos2x and y_2' = -2sin2x

∴ y_1'' = -4sin2x and y_2'' = -4cos2x

∴ $y_1'' = -4y_1$ and $y_2'' = -4y_2$ by (1)

∴ $y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$

∴ y_1 = sin2x and y_2 = cos2x are the solutions of the differential equation y"+ 4y = 0 is proved.

The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -2sin^2 2x - 2cos^2 2x$$

∴ W(x) = -2 ≠ 0

∴ Given solutions are linearly independent is proved.

==

3) Prove that 1, x, x^2 are linearly independent. Hence form the differential equation whose solutions are 1, x, x^2.

Proof: Let $y_1 = 1$, $y_2 = x$ and $y_3 = x^2$ are the given functions.

$\therefore y_1' = 0, y_2' = 1$ and $y_3' = 2x$

$\therefore y_1'' = 0, y_2'' = 0$ and $y_3'' = 2$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$= (2 - 0) – x(0 - 0) + x^2(0 - 0)$

$\therefore W(x) = 2 \neq 0$.

$\therefore$ 1, x, x^2 are linearly independent solutions.

To find differential equation, let $y = c_1 + c_2x + c_3x^2$ …… (i)

where c_1, c_2, c_3 are constants.

Differentiating equation (i) thrice, we get,

$\frac{dy}{dx} = c_2 + 2c_3x$

$\frac{d^2y}{dx^2} = 2c_3$

$\frac{d^3y}{dx^3} = 0$ which is free from constants c_1, c_2 and c_3

$\therefore \frac{d^3y}{dx^3} = 0$ be the required differential equation.

==

4) Examine whether the set of functions 1, x^2, x^3 are linearly independent or not.

Solution: Let $y_1 = 1$, $y_2 = x^2$ and $y_3 = x^3$ are the given functions.

$\therefore y_1' = 0, \quad y_2' = 2x$ and $y_3' = 3x^2$

$\therefore y_1'' = 0, \quad y_2'' = 2$ and $y_3'' = 6x$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$= (12x^2 - 6x^2) - x^2 (0 - 0) + x^3(0 - 0)$

$\therefore W(x) = 6x^2 \neq 0$

$\therefore$ Given set of functions are linearly independent.

==

5) Solve by method of variation of parameters $\frac{d^2y}{dx^2}$ + a^2y = cosec(ax)

Solution: Let $\frac{d^2y}{dx^2}$ + a^2y = cosec(ax) i.e. (D^2 + a^2)y = cosec(ax) …… (i)

be the given equation is

∴ Its A.E. is $D^2 + a^2 = 0$ which has roots D = $\pm$ai.

∴ C.F. is y = Acosax + Bsinax

By method of variation of parameter assume that y = Acosax + Bsinax …… (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and cosax $\frac{dA}{dx}$ + sinax $\frac{dB}{dx}$ = 0 …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -aAsinax +$ cosax $\frac{dA}{dx} + aBcosax +$ sinax $\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -aAsinax + aBcosax$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -a^2Acosax - asinax\frac{dA}{dx} - a^2Bsinax + acosax\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -a^2(Acosax + Bsinax) - asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -a^2y - asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + a^2y = -asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$

$\therefore -asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$ = cosec(ax) …… (v) by (i)

To solve (iii) and (v), consider asinax(iii)+cosax(v), we get,

asinaxcosax $\frac{dA}{dx}$ + $asin^2ax$ $\frac{dB}{dx}$ - asinaxcosax $\frac{dA}{dx}$ + $acos^2ax$ $\frac{dB}{dx}$ = 0 + cosaxcosec(ax)

∴ a $\frac{dB}{dx}$ = cot(ax) $\Longrightarrow \frac{dB}{dx} = \frac{1}{a}$ cot(ax)

Putting value of $\frac{dB}{dx}$ in (iii), we get,

cosax $\frac{dA}{dx}$ + sinax[$\frac{1}{a}$ cot(ax)] = 0

∴ cosax $\frac{dA}{dx} = -\frac{1}{a}$ cosax $\Longrightarrow \frac{dA}{dx} = -\frac{1}{a}$

Now $\frac{dA}{dx} = -\frac{1}{a} \Longrightarrow A = \int(-\frac{1}{a})dx = -\frac{x}{a} + c_1$ and

$\frac{dB}{dx} = \frac{1}{a}$ cot(ax) $\Longrightarrow B = \int(\frac{1}{a}cotax)dx = \frac{1}{a^2}$ logsinax+ c_2

Putting these values of A and B in (iii), we get G.S. of given equation (i) as

y = $(-\frac{x}{a} + c_1)$cosax + $(\frac{1}{a^2}$ logsinax+ $c_2)$sinax

$\therefore$ y = $c_1$cosax + $c_2$sinax$-\frac{x}{a}$ cosax + $\frac{1}{a^2}$ sinax(logsinax).

6) Solve by method of variation of parameters y'' + y – x = 0

Solution: Let y'' + y – x = 0 i.e. $(D^2 + 1)y = x$ …… (i)

be the given equation is

$\therefore$ Its A.E. is $D^2 + 1 = 0$ which has roots D = $\pm$i.

$\therefore$ C.F. is y = Acosx + Bsinx

By method of variation of parameter assume that y = Acosx + Bsinx …… (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and cosx $\frac{dA}{dx}$ + sinx $\frac{dB}{dx}$ = 0 …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

y' = $-Asinx$+ cosx $\frac{dA}{dx}$ + $Bcosx$ + sinx $\frac{dB}{dx}$

$\Rightarrow$ y' = $-Asinx + Bcosx$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

y'' = $-Acosx - sinx\frac{dA}{dx} - Bsinx + cosx\frac{dB}{dx}$

$\therefore$ y'' = $-(Acosx + Bsinx) - sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$

$\therefore$ y'' = $-y - sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$by (ii)

$\therefore$ y'' + $y = -sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$

$\therefore$ $-sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$ = x …… (v) by (i)

To solve (iii) and (v), consider sinx(iii) + cosx(v), we get,

sinxcosx $\frac{dA}{dx}$ + $\sin^2$x $\frac{dB}{dx}$ - sinxcosx $\frac{dA}{dx}$ + $\cos^2$x $\frac{dB}{dx}$ = 0 + xcosx

$\therefore$ $\frac{dB}{dx}$ = xcosx

Putting value of $\frac{dB}{dx}$ in (iii), we get,

cosx $\frac{dA}{dx}$ + sinx$[xcosx]$ = 0

$\therefore$ cosx $\frac{dA}{dx}$ = – xsinxcosx $\Rightarrow$ $\frac{dA}{dx} = -xsinx$

Now $\frac{dA}{dx} = -xsinx \Rightarrow$ A = $\int(-xsinx)$dx = xcosx-$\int$cosxdx+c_1 = xcosx- $sinx$ + c_1 &

$\frac{dB}{dx}$ = xcosx $\Rightarrow$ B = $\int$ xcosxdx = xsinx-$\int$$sinxdx$ + c_2 = xsinx + cos$x$ + c_2

Putting these values of A and B in (iii), we get G.S. of given equation (i) as

y = (xcosx- $sinx$+ c_1)cosx + (xsinx + cos$x$ + c_2)sinx

$\therefore y = c_1\cos x + c_2\sin x + x\cos^2 x - \sin x\cos x + x\sin^2 x + \cos x\sin x$

$\therefore y = c_1\cos x + c_2\sin x + x.$

7) Show that the functions $1+x$, x^2 and $1+2x$ are linearly independent.

Proof: Let $y_1 = 1+x$, $y_2 = x^2$ and $y_3 = 1+2x$ are the given functions.

$\therefore y_1' = 1, y_2' = 2x$ and $y_3' = 2$

$\therefore y_1'' = 0, y_2'' = 2$ and $y_3'' = 0$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1+x & x^2 & 1+2x \\ 1 & 2x & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$= (1 + x)(0 - 4) - x^2(0 - 0)+(1 + 2x)(2 - 0)$

$= - 4 - 4x + 2 + 4x$

$\therefore W(x)= -2 \neq 0.$

$\therefore$ Given functions are linearly independent.

==

8) Examine whether e^{2x} and e^{3x} are linearly independent solutions of the differential equation $y'' - 5y' + 6y = 0$ or not?

Solution: Let $y_1 = e^{2x}$ and $y_2 = e^{3x}$ …… (1)

$\therefore y_1' = 2e^{2x}$ and $y_2' = 3e^{3x}$

$\therefore y_1'' = 4e^{2x}$ and $y_2'' = 9e^{3x}$

Consider $y_1'' - 5y_1' + 6y_1 = 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$ and

$y_2'' - 5y_2' + 6y_2 = 9e^{3x} - 15e^{3x} + 6e^{3x} = 0$

$\therefore y_1 = e^{2x}$ and $y_2 = e^{3x}$ are the solutions of the differential equation $y'' - 5y' + 6y = 0$.

Now the Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$

$= 3e^{5x} - 2e^{5x}$

$\therefore W(x) = e^{5x} \neq 0$

$\therefore y_1 = e^{2x}$ and $y_2 = e^{3x}$ are linearly independent solutions of the differential equation y''- 5y'+ 6y = 0.

==

9) Solve by method of variation of parameters $\frac{d^2y}{dx^2}$ + 9y = sec3x

Solution: Let $\frac{d^2y}{dx^2}$ + 9y = sec3x i.e. $(D^2 + 9)$y = sec3x …… (i)

be the given equation is

∴ Its A.E. is $D^2 + 9 = 0$ which has roots D = $\pm$3i.

∴ C.F. is y = Acos3x + Bsin3x

By method of variation of parameter assume that y = Acos3x + Bsin3x …… (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and cos3x $\frac{dA}{dx}$ + sin3x $\frac{dB}{dx}$ = 0 …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -3Asin3x$+ cos3x $\frac{dA}{dx}$ + $3Bcos3x$ + sin3x $\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -3Asin3x + 3Bcos3x$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -9Acos3x - 3sin3x\frac{dA}{dx} - 9Bsin3x + 3cos3x\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -9(Acos3x + Bsin3x) - 3sin3x\frac{dA}{dx} + 3cos3x\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -9y - 3sin3x\frac{dA}{dx} + 3cos3x\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + 9y = -3sin3x\frac{dA}{dx} + 3cos3x\frac{dB}{dx}$

$\therefore -3sin3x\frac{dA}{dx} + 3cos3x\frac{dB}{dx}$ = sec3x …… (v) by (i)

To solve (iii) and (v), consider 3sin3x(iii) + cos3x(v), we get,

3sin3xcos3x $\frac{dA}{dx}$ + $3\sin^2 3x$ $\frac{dB}{dx}$ - 3sin3xcos3x $\frac{dA}{dx}$ + $3\cos^2 3x$ $\frac{dB}{dx}$ = 0 + cos3xsec3x

$\therefore 3\frac{dB}{dx} = 1 \Rightarrow \frac{dB}{dx} = \frac{1}{3}$

Putting value of $\frac{dB}{dx}$ in (iii), we get,

cos3x $\frac{dA}{dx}$ + sin3x$(\frac{1}{3})$ = 0

∴ cos3x $\frac{dA}{dx} = -\frac{1}{3}$ sin3x $\Rightarrow$ $\frac{dA}{dx} = -\frac{1}{3}$ tan3x

Now $\frac{dA}{dx} = -\frac{1}{3}$ tan3x $\Rightarrow$ A = $\int(-\frac{1}{3}tan3x)$dx = $\frac{1}{9}$ logcos3x + c_1 and

$\frac{dB}{dx} = \frac{1}{3} \Rightarrow$ B = $\int(\frac{1}{3})$dx = $\frac{x}{3}$ + c_2

Putting these values of A and B in (iii), we get G.S. of given equation (i) as

$$y = \left(\frac{1}{9}\log\cos 3x + c_1\right)\cos 3x + \left(\frac{x}{3} + c_2\right)\sin 3x$$

$$\therefore y = c_1\cos 3x + c_2\sin 3x + \frac{1}{9}\cos 3x(\log\sin 3x) + \frac{x}{3}\sin 3x$$

PRACTICAL NO.-6: SIMULTANEOUS DIFFERENTIAL EQUATIONS

===

1) i) Solve $\frac{dx}{x^2z} = \frac{dy}{0} = \frac{dz}{-x^2}$

Solution: Let $\frac{dx}{x^2z} = \frac{dy}{0} = \frac{dz}{-x^2}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{x^2z} = \frac{dy}{0} \Longrightarrow dy = 0$

Integrating, we get, $y = c_1$ i.e. $y - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{x^2z} = \frac{dz}{-x^2} \Longrightarrow dx = -zdz \Longrightarrow 2dx + 2zdz = 0$

Integrating, we get, $2x + z^2 = c_2$ i.e. $2x + z^2 - c_2 = 0$ …… (iii)

∴ By (i) and (ii),

$(y - c_1)(2x + z^2 - c_2) = 0$

be the required general solution of given equation.

===

ii) Solve $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$

Solution: Let $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$ …(i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{tanx} = \frac{dy}{tany} \Longrightarrow$ cotxdx = cotydy

Integrating, we get, logsinx = logsiny + $logc_1$

i.e. sinx = $c_1$siny i.e. sinx - $c_1$siny = 0…… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{tanx} = \frac{dz}{tanz} \Longrightarrow$ cotxdx = cotzdz

Integrating, we get, logsinx = logsinz + $logc_2$

i.e. sinx = $c_2$sinz i.e. sinx − $c_2$sinz = 0 …… (iii)

∴ By (i) and (ii),

(sinx - $c_1$siny)(sinx − $c_2$sinz) = 0

be the required general solution of given equation.

===

2) i) Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - zx^2}$

Solution: Let $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - zx^2}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{xy} = \frac{dy}{y^2} \Longrightarrow \frac{dx}{x} = \frac{dy}{y}$

Integrating, we get, $logx = logy + logc_1$ i.e. $x = c_1y$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{dy}{y^2} = \frac{dz}{xyz - zx^2} \Longrightarrow \frac{dy}{y^2} = \frac{dz}{c_1y^2z - zc_1^2y^2}$ by (ii)

$\Longrightarrow dy = \frac{dz}{(c_1 - c_1^2)z}$

Integrating, we get, $y = \frac{1}{(c_1 - c_1^2)} \log z + c_2$

i.e. $y = \frac{1}{[\frac{x}{y} - \left(\frac{x}{y}\right)^2]} \log z + c_2$ by (ii)

i.e. $y = \frac{y^2}{(xy - x^2)} \log z + c_2$

be the required general solution of given equation.

==

ii) Solve $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$

Solution: Let $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{y} = \frac{dy}{x} \Longrightarrow xdx = ydy \Longrightarrow 2xdx - 2ydy = 0$

Integrating, we get, $x^2 - y^2 = c_1$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{y} = \frac{dz}{xyz^2(x^2 - y^2)} \Longrightarrow xdx = \frac{dz}{c_1z^2}$ by (ii)

Integrating, we get, $\frac{x^2}{2} = -\frac{1}{c_1z} + c_2$

i.e. $\frac{x^2}{2} = -\frac{1}{z(x^2 - y^2)} + c_2$ by (ii)

be the required general solution of given equation.

==

3) Solve $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$

Solution: Let $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ …… (i)

be the given simultaneous differential equation.

By taking multipliers 1, 1, 1, we get,

Each Ratio of (i) = $\frac{dx+dy+dz}{y+z+z+x+x+y}$

i.e. Each Ratio of (i) = $\frac{dx+dy+dz}{2x+2y+2z}$

i.e. Each Ratio of (i) = $\frac{d(x+y+z)}{2(x+y+z)}$

Again by taking multipliers 1, -1, 0 and 0, 1, -1 we get,

Each Ratio of (i) = $\frac{dx-dy+0}{y+z-z-x+0} = \frac{0+dy-dz}{0+z+x-x-y}$

i.e. Each Ratio of (i) = $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

i.e. Each Ratio of (i) = $\frac{d(x+y+z)}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dx-dz}{z-x}$

Consider $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

$\Longrightarrow \frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$

Integrating, we get,

$\log(x-y) = \log(y-z) + \log c_1$

i.e. $(x-y) = c_1(y-z)$

i.e. $(x-y) - c_1(y-z) = 0$ …… (ii)

Again consider $\frac{d(x+y+z)}{2(x+y+z)} = \frac{dx-dy}{y-x}$

$\Longrightarrow \frac{d(x+y+z)}{(x+y+z)} = -2\frac{d(x-y)}{(x-y)}$

$\Longrightarrow \frac{d(x+y+z)}{(x+y+z)} + 2\frac{d(x-y)}{(x-y)} = 0$

Integrating, we get,

$\log(x+y+z)+2\log(x-y) = \log c_2$

i.e. $(x+y+z)(x-y)^2 = c_2$

i.e. $(x+y+z)(x-y)^2 - c_2 = 0$…… (iii)

By (ii) and (iii),

$[(x-y) - c_1(y-z)][(x+y+z)(x-y)^2 - c_2] = 0$

be the required general solution of given equation.

===

4) Solve $\frac{adx}{yz(b-c)} = \frac{bdy}{zx(c-a)} = \frac{cdz}{xy(a-b)}$

Solution: Let $\frac{adx}{yz(b-c)} = \frac{bdy}{zx(c-a)} = \frac{cdz}{xy(a-b)}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers x, y, z, we get,

Each Ratio of (i) = = $\frac{ax\mathrm{dx}+by\mathrm{dy}+cz\mathrm{dz}}{xyz(b-c+c-a+a-b)} = \frac{ax\mathrm{dx}+by\mathrm{dy}+cz\mathrm{dz}}{0}$

$\Longrightarrow ax\mathrm{dx} + by\mathrm{dy} + cz\mathrm{dz} = 0$

$\Longrightarrow 2ax\mathrm{dx} + 2by\mathrm{dy} + 2cz\mathrm{dz} = 0$

Integrating, we get,

$ax^2 + by^2 + cz^2 = c_1$

i.e. $ax^2 + by^2 + cz^2 - c_1 = 0$ ...... (ii)

Again by taking multipliers ax, by, cz, we get,

Each Ratio of (i) = = $\frac{a^2\mathrm{xdx}+b^2\mathrm{ydy}+c^2\mathrm{zdz}}{xyz(ab-ac+bc-ba+ca-cb)} = \frac{a^2\mathrm{xdx}+b^2\mathrm{ydy}+c^2\mathrm{zdz}}{0}$

$\Longrightarrow a^2\mathrm{xdx} + b^2\mathrm{ydy} + c^2\mathrm{zdz} = 0$

$\Longrightarrow a^2 2\mathrm{xdx} + b^2 2\mathrm{ydy} + c^2 2\mathrm{zdz} = 0$

Integrating, we get,

$a^2x^2 + b^2y^2 + c^2z^2 = c_2$

i.e. $a^2x^2 + b^2y^2 + c^2z^2 - c_2 = 0$ ...(iii)

By (ii) and (iii),

$(ax^2 + by^2 + cz^2 - c_1)(a^2x^2 + b^2y^2 + c^2z^2 - c_2) = 0$

be the required general solution of given equation.

===

5) Solve $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

Solution: Let $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ …… (i)

be the given simultaneous differential equation.

Taking second and third ratios of (i), we have

$\frac{dy}{2xy} = \frac{dz}{2xz} \Longrightarrow \frac{dy}{y} = \frac{dz}{z}$

Integrating, we get,

$\log y = \log z + \log c_1$

i.e. $y = c_1 z$

i.e. $y - c_1 z = 0$...... (ii)

Now by taking multipliers x, y, z, we get, Each Ratio of (i) = $\frac{\text{xdx+ydy+zdz}}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{\text{xdx+ydy+zdz}}{x^3+xy^2+xz^2} = \frac{\text{xdx+ydy+zdz}}{x(x^2+y^2+z^2)}$

$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{\text{xdx+ydy+zdz}}{x(x^2+y^2+z^2)}$ …… (iii)

Taking second and fourth ratios of (iii), we have,

$\frac{dy}{2xy} = \frac{\text{xdx+ydy+zdz}}{x(x^2+y^2+z^2)}$

$\Rightarrow \frac{dy}{y} = \frac{\text{2xdx+2ydy+2zdz}}{(x^2+y^2+z^2)}$

$\Rightarrow \frac{dy}{y} = \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)}$

Integrating, we get,

logy = $\log(x^2 + y^2 + z^2) + \log c_2$

i.e. y = c_2 ($x^2 + y^2 + z^2$)

i.e. y - c_2 ($x^2 + y^2 + z^2$) = 0...... (iv)

By (ii) and (iv),

(y - c_1z)[y - c_2 ($x^2 + y^2 + z^2$)] = 0

be the required general solution of given equation.

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6) Solve $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$

Solution: Let $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i), we have

$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)}$

$\Rightarrow \frac{dx}{(x+y)} = \frac{dy}{(x-y)}$

$\Rightarrow xdx - ydx = xdy + ydy$

$\Rightarrow xdx - ydx - xdy - ydy = 0$

$\Rightarrow 2xdx - 2ydx - 2xdy - 2ydy = 0$

$\Rightarrow d(x^2 - 2xy - y^2) = 0$

Integrating, we get,

$x^2 - 2xy - y^2 = c_1$

i.e. $x^2 - 2xy - y^2$ - c_1 = 0...... (ii)

Now by taking multipliers x, -y, -z, we get,

Each Ratio of (i) = $\frac{\text{xdx−ydy−zdz}}{x^2z+xyz-xyz+y^2z-zx^2-zy^2} = \frac{\text{xdx−ydy−zdz}}{0}$

$\therefore$ xdx − ydy − zdz = 0

$\Rightarrow 2xdx - 2ydy - 2zdz = 0$

$\Rightarrow d(x^2 - y^2 - z^2) = 0$

Integrating, we get,

$x^2 - y^2 - z^2 = c_2$

i.e. $x^2 - y^2 - z^2 - c_2 = 0$...... (iii)

By (ii) and (iii),

$(x^2 - 2xy - y^2 - c_1)(x^2 - y^2 - z^2 - c_2) = 0$

be the required general solution of given equation.

==

7) Solve $\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z}$

Solution: Let $\frac{dx}{\sin(x+y)} = \frac{dy}{\cos(x+y)} = \frac{dz}{z}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers 1, 1, 0 and 1, -1, 0 we get,

Each Ratio of (i) = $\frac{dx+dy}{\sin(x+y)+\cos(x+y)} = \frac{dx-dy}{\sin(x+y)-\cos(x+y)}$

$\therefore \frac{dz}{z} = \frac{dx+dy}{\sin(x+y)+\cos(x+y)} = \frac{dx-dy}{\sin(x+y)-\cos(x+y)}$ …… (ii)

Taking first and second ratio of (ii), we have,

$\frac{dz}{z} = \frac{dx+dy}{\sin(x+y)+\cos(x+y)}$

$\Rightarrow \frac{dz}{z} = \frac{dx+dy}{\sqrt{2}\sin\left(x+y+\frac{\pi}{4}\right)}$

$\Rightarrow \sqrt{2}\frac{dz}{z} = \operatorname{cosec}\left(x+y+\frac{\pi}{4}\right) d\left(x+y+\frac{\pi}{4}\right)$

Integrating, we get,

$\sqrt{2}\log z = \log[\tan\frac{1}{2}\left(x+y+\frac{\pi}{4}\right)] + \log c_1$

i.e. $z^{\sqrt{2}} = c_1\tan\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right)$

i.e. $z^{\sqrt{2}} - c_1\tan\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right) = 0$ ...... (iii)

Taking second and third ratio of (ii), we have,

$\frac{dx+dy}{\sin(x+y)+\cos(x+y)} = \frac{dx-dy}{\sin(x+y)-\cos(x+y)}$

$\Rightarrow \frac{dx+dy}{\sin(x+y)+\cos(x+y)} = \frac{dy-dx}{\cos(x+y)-\sin(x+y)}$

$\Rightarrow \frac{\cos(x+y)-\sin(x+y)}{\sin(x+y)+\cos(x+y)} d(x+y) = d(y-x)$

Integrating, we get,

$\log[\sin(x+y)+\cos(x+y)] = y - x + \log c_2$

i.e. $\log[\sin(x+y)+\cos(x+y)] = \log e^{y-x} + \log c_2$

i.e. $\log[\sin(x+y)+\cos(x+y)] = \log c_2e^{y-x}$

i.e. $\sin(x+y)+\cos(x+y) = c_2e^{y-x}$

i.e. $\sin(x+y)+\cos(x+y) - c_2e^{y-x} = 0$ …… (iv)

∴ By (iii) and (iv),

$[z^{\sqrt{2}} - c_1\tan\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right)][\sin(x+y)+\cos(x+y) - c_2e^{y-x}] = 0$

be the required general solution of given equation.

==

8) Solve $\frac{dx}{z^2} = \frac{ydy}{xz^2} = \frac{dz}{xy}$

Solution: Let $\frac{dx}{z^2} = \frac{ydy}{xz^2} = \frac{dz}{xy}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{z^2} = \frac{ydy}{xz^2} \Longrightarrow xdx = ydy \Longrightarrow 2xdx - 2ydy = 0$

Integrating, we get,

$x^2 - y^2 = c_1$ i.e. $x^2 - y^2 - c_1 = 0$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{ydy}{xz^2} = \frac{dz}{xy} \Longrightarrow y^2dy = z^2dz \Longrightarrow 3y^2dy - 3z^2dz = 0$

Integrating, we get,

$y^3 - z^3 = c_2$ i.e. $y^3 - z^3 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^2 - y^2 - c_1)(y^3 - z^3 - c_2) = 0$

be the required general solution of given equation.

==

9) Solve $\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$

Solution: Let $\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratio of (i), we have

$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} \Longrightarrow x^2dx = -y^2dy \Longrightarrow 3x^2dx + 3y^2dy = 0$

Integrating, we get,

$x^3 + y^3 = c_1$ i.e. $x^3 + y^3 - c_1 = 0$ …… (ii)

By taking multipliers 1, -1, 0, we get,

Each ratio of (i) = $\frac{dx-dy}{y^2(x-y)+x^2(x-y)}$

$\therefore \frac{dz}{z(x^2+y^2)} = \frac{dx-dy}{(y^2+x^2)(x-y)}$

$\Longrightarrow \frac{dz}{z} = \frac{d(x-y)}{(x-y)}$

Integrating, we get,

$\log z = \log(x-y) + \log c_2$

i.e. $z = c_2 (x-y)$

i.e. $z - c_2 (x-y) = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^3 + y^3 - c_1)[z - c_2 (x-y)] = 0$

be the required general solution of given equation.

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PRACTICAL NO.-7: TOTAL DIFFERENTIAL OR PFAFFIAN DIFFERENTIAL EQUATIONS

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1) Show that the following differential equations are integrable. Hence solve them

i) $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$ ii) $2yzdx + zxdy - xy(1+z)dz = 0$

Proof: i) Let $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = y^2 + z^2 - x^2$, $Q = -2xy$ and $R = -2xz$

$\therefore \frac{\partial P}{\partial y} = 2y$, $\frac{\partial P}{\partial z} = 2z$, $\frac{\partial Q}{\partial x} = -2y$, $\frac{\partial Q}{\partial z} = 0$, $\frac{\partial R}{\partial x} = -2z$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (y^2 + z^2 - x^2)(0 - 0) - 2xy(-2z - 2z) - 2xz(2y+2y)$

$= 0 + 8xyz - 8xyz$

$= 0$

$\therefore$ The given equation integrable.

Now we rearrange the terms as:

$(x^2 + y^2 + z^2)dx - 2x^2dx - 2xydy - 2xzdz = 0$

i.e. $(x^2 + y^2 + z^2)dx - x(2xdx + 2ydy + 2zdz = 0$

i.e. $(x^2 + y^2 + z^2)dx - xd(x^2 + y^2 + z^2) = 0$

Dividing by $x(x^2 + y^2 + z^2)$, we get,

$\therefore \frac{dx}{x} - \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)} = 0$

i.e. $\frac{dx}{x} = \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)}$

Integrating, we get,

$\log x = \log(x^2 + y^2 + z^2) + \log c$

$\therefore x = c(x^2 + y^2 + z^2)$

be the solution of given equation.

ii) Let $2yzdx + zxdy - xy(1+z)dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = 2yz$, $Q = zx$ and $R = -xy(1+z)$

$\therefore \frac{\partial P}{\partial y} = 2z$, $\frac{\partial P}{\partial z} = 2y$, $\frac{\partial Q}{\partial x} = z$, $\frac{\partial Q}{\partial z} = x$, $\frac{\partial R}{\partial x} = -y(1+z)$ and $\frac{\partial R}{\partial y} = -x(1+z)$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (2yz)[x + x(1+z)] + zx[-y(1+z) - 2y) - xy(1+z)(2z-z)$

$= (2yz)(2x + xz) + zx(-yz - 3y) - xyz(1+z)$

$= 4xyz + 2xyz^2 - xyz^2 - 3xyz - xyz - xyz^2$

$= 0$

∴ The given equation integrable.

Divide the given equation by xyz, we get,

$\frac{2dx}{x} + \frac{dy}{y} - (\frac{1}{z} + 1)dz = 0$

Integrating, we get,

$2\log x + \log y - \log z - z = \log c$

i.e. $\log x^2 + \log y - \log z - \log e^z = \log c$

i.e. $\log(\frac{x^2y}{ze^z}) = \log c$

∴ $\frac{x^2y}{ze^z} = c$

i.e. $x^2y = cze^z$

be the solution of given equation.

==

2) Solve $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2-xy)\,dz = 0$

Proof: Let $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2-xy)\,dz = 0$

be the given homogeneous equation, which is integrable with

$P = yz^2(x^2 - yz)$, $Q = zx^2(y^2 - xz)$ and $R = xy^2(z^2-xy)$

∴ $Px + Qy + Rz = xyz^2(x^2 - yz) + yzx^2(y^2 - xz) + zxy^2(z^2-xy)$

$= xyz\,(x^2z - yz^2 + xy^2 - x^2z + yz^2 - xy^2)$

$= 0$

∴ To solve the given equation put x = zu and y = zv,

∴ dx = udz + zdu and dy = vdz + zdv

∴ the given equation becomes

$vz^3(u^2z^2 - vz^2)(udz + zdu) + u^2z^3(v^2z^2 - uz^2)(vdz + zdv) + uv^2z^3(z^2 - uvz^2)dz = 0$

i.e. $z^5[(u^2v - v^2)(udz + zdu) + (u^2v^2 - u^3)(vdz + zdv) + (uv^2 - u^2v^3)dz] = 0$

i.e. $(u^2v - v^2)zdu + (u^2v^2 - u^3)zdv + (u^3v - uv^2 + u^2v^3 - u^3v + uv^2 - u^2v^3)dz = 0$

i.e. $(u^2 - v)vzdu + (v^2 - u)u^2zdv + (0)dz = 0$

i.e. $u^2vdu - v^2du + u^2v^2dv - u^3dv = 0$

i.e. $u^2(vdu - udv) + u^2v^2dv - v^2du = 0$

Dividing by u^2v^2, we get,

i.e. $\frac{vdu-udv}{v^2} + dv - \frac{du}{u^2} = 0$

i.e. $d(\frac{u}{v}) + dv + d(\frac{1}{u}) = 0$

Integrating, we get,

$\frac{u}{v} + v + \frac{1}{u} = c$

$\therefore u^2 + uv^2 + v = cuv$

i.e. $(\frac{x^2}{z^2}) + \frac{x}{z}(\frac{y^2}{z^2}) + \frac{y}{z} = c(\frac{x}{z})(\frac{y}{z})$

i.e. $x^2z + xy^2 + yz^2 = cxyz$

be the solution of given equation.

==

3) Solve $\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0$

Proof: Let $\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0$ be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

$P = \frac{yz}{x^2+y^2}$, $Q = -\frac{xz}{x^2+y^2}$ and $R = -\tan^{-1}\frac{y}{x}$

$\therefore \frac{\partial P}{\partial y} = z\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{z(x^2-y^2)}{(x^2+y^2)^2}, \frac{\partial P}{\partial z} = \frac{y}{x^2+y^2},$

$\frac{\partial Q}{\partial x} = -z\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{z(x^2-y^2)}{(x^2+y^2)^2}, \frac{\partial Q}{\partial z} = -\frac{x}{x^2+y^2},$

$\frac{\partial R}{\partial x} = -\frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{y}{x^2+y^2}$ and $\frac{\partial R}{\partial y} = -\frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{-x}{x^2+y^2}$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= \frac{yz}{x^2+y^2}[-\frac{x}{x^2+y^2} + \frac{x}{x^2+y^2}] - \frac{xz}{x^2+y^2}[\frac{y}{x^2+y^2} - \frac{y}{x^2+y^2}]$

$- \tan^{-1}(\frac{y}{x})[\frac{z(x^2-y^2)}{(x^2+y^2)^2} - \frac{z(x^2-y^2)}{(x^2+y^2)^2}]$

$= 0$

$\therefore$ The given equation is integrable.

Rearrange the given equation as:

$z[\frac{ydx-xdy}{x^2+y^2}] - \tan^{-1}\frac{y}{x}dz = 0$

i.e. $z[\frac{xdy-ydx}{x^2+y^2}] + \tan^{-1}\frac{y}{x}dz = 0$

i.e. $\frac{1}{\tan^{-1}\frac{y}{x}}[\frac{xdy-ydx}{x^2+y^2}] + \frac{dz}{z} = 0$

i.e. $\frac{d(\tan^{-1}\frac{y}{x})}{\tan^{-1}\frac{y}{x}} + \frac{dz}{z} = 0$

Integrating, we get,

$\log\tan^{-1}\frac{y}{x} + \log z = \log c$

$\therefore z\tan^{-1}\frac{y}{x} = c$

be the solution of given equation.

4) Solve zydx = zxdy + y^2dz .

Proof: Let zydx = zxdy + y^2dz

i.e. zydx - zxdy - y^2dz = 0 be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

P = zy , Q = -zx and R = – y^2

$\therefore \frac{\partial P}{\partial y} = z, \frac{\partial P}{\partial z} = y, \frac{\partial Q}{\partial x} = -z, \frac{\partial Q}{\partial z} = -x, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = -2y$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (zy)(-x + 2y) - zx(0 - y) - y^2(z + z)$

$= -xyz + 2y^2z + xyz - 2y^2z$

$= 0$

∴ The given equation is integrable.

Divide the given equation by y^2z, we get,

$\frac{ydx - xdy}{y^2} - \frac{dz}{z} = 0$

i.e. $d(\frac{x}{y}) - \frac{dz}{z} = 0$

Integrating, we get,

$\frac{x}{y} - \log z = c$

$\therefore x - y\log z = cy$

be the solution of given equation.

5) Solve $(x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz = 0$.

Proof: Let $(x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz = 0$ be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

$P = x^2 - yz, Q = y^2 - zx$ and $R = z^2 - xy$

$\therefore \frac{\partial P}{\partial y} = -z, \frac{\partial P}{\partial z} = -y, \frac{\partial Q}{\partial x} = -z, \frac{\partial Q}{\partial z} = -x, \frac{\partial R}{\partial x} = -y$ and $\frac{\partial R}{\partial y} = -x$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

∴ The given equation exact and hence integrable.

Now we rearrange the terms as:

$(x^2\,dx + y^2\,dy + z^2\,dz) - (yzdx + zxdy + xydz) = 0$

$\therefore (3x^2\,dx + 3y^2\,dy + 3z^2\,dz) - 3(yzdx + zxdy + xydz) = 0$

$\therefore d(x^3 + y^3 + z^3) - 3d(xyz) = 0$

Integrating, we get,

$x^3 + y^3 + z^3 - 3xyz = c$

be the solution of given equation.

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6) Solve $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$.

Proof: Let $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$ be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

$P = 2x^2 + 2xy + 2xz^2 + 1$, Q = 1 and R = 2z

$\therefore \frac{\partial P}{\partial y} = 2x, \frac{\partial P}{\partial z} = 4xz, \frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (2x^2 + 2xy + xz^2 + 1)(0 - 0) + (0 - 4xz) + 2z(2x - 0)$

$= 0 - 4xz + 4xz$

$= 0$

$\therefore$ The given equation integrable.

Rearrange the given terms as:

$2x(x + y + z^2)dx + dx + dy + 2zdz = 0$

Divide the given equation by $(x+y+z^2)$, we get,

$2xdx + \frac{dx+dy+2zdz}{x+y+z^2} = 0$

i.e. $d(x^2) + \frac{d(x+y+z^2)}{x+y+z^2} = 0$

Integrating, we get,

$x^2 + \log(x + y + z^2) = c$

be the solution of given equation.

==

7) Solve $(y + z)\, dx + dy + dz = 0$.

Proof: Let $(y + z)\, dx + dy + dz = 0$ be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

P = y + z, Q = 1 and R = 1

$\therefore \frac{\partial P}{\partial y} = 1, \frac{\partial P}{\partial z} = 1, \frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (y + z)(0 - 0) + (0 - 1) + (1 - 0)$

$= 0 - 1 + 1$

$= 0$

$\therefore$ The given equation is integrable.

Divide the given equation by (y + z), we get,

$dx + \frac{dy+dz}{y+z} = 0$ i.e. $dx + \frac{d(y+z)}{y+z} = 0$

Integrating, we get,

$x + \log(y + z) = \log c$

i.e. $\log e^x + \log(y + z) = \log c$

$\therefore e^x(y + z) = c$

be the solution of given equation.

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8) Show that the equation $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$ is integrable. Is it exact? Verify.

Proof: Let $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$ be the given equation, comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = yz^2(x^2 - yz) = x^2yz^2 - y^2z^3$, $Q = zx^2(y^2 - xz) = x^2zy^2 - x^3z^2$ and

$R = xy^2(z^2 - xy) = xy^2z^2 - x^2y^3$

$\therefore \frac{\partial P}{\partial y} = x^2z^2 - 2yz^3$, $\frac{\partial P}{\partial z} = 2x^2yz - 3y^2z^2$, $\frac{\partial Q}{\partial x} = 2xzy^2 - 3x^2z^2$, $\frac{\partial Q}{\partial z} = x^2y^2 - 2x^3z$,

$\frac{\partial R}{\partial x} = y^2z^2 - 2xy^3$ and $\frac{\partial R}{\partial y} = 2xyz^2 - 3x^2y^2$

$\therefore P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (x^2yz^2 - y^2z^3)(x^2y^2 - 2x^3z - 2xyz^2 + 3x^2y^2) + (x^2zy^2 - x^3z^2)(y^2z^2 - 2xy^3 - 2x^2yz + 3y^2z^2) + (xy^2z^2 - x^2y^3)(x^2z^2 - 2yz^3 - 2xzy^2 + 3x^2z^2)$

$= (x^2yz^2 - y^2z^3)(4x^2y^2 - 2x^3z - 2xyz^2) + (x^2zy^2 - x^3z^2)(4y^2z^2 - 2xy^3 - 2x^2yz) + (xy^2z^2 - x^2y^3)(4x^2z^2 - 2yz^3 - 2xzy^2)$

$= (x^2yz^2 - y^2z^3)(4x^2y^2 - 2x^3z - 2xyz^2) + (x^2zy^2 - x^3z^2)(4y^2z^2 - 2xy^3 - 2x^2yz) + (xy^2z^2 - x^2y^3)(4x^2z^2 - 2yz^3 - 2xzy^2)$

$= 4x^4y^3z^2 - 4x^2y^4z^3 - 2x^5yz^3 + 2x^3y^2z^4 - 2x^3y^2z^4 + 2xy^3z^5 + 4x^2y^4z^3 - 4x^3y^2z^4 - 2x^3y^5z + 2x^4y^3z^2 - 2x^4y^3z^2 + 2x^5yz^3 + 4x^3y^2z^4 - 4x^4y^3z^2 - 2xy^3z^5 + 2x^2y^4z^3 - 2x^2y^4z^3 + 2x^3y^5z$

$= 0$

Hence the given equation is integrable is proved.

But it is not exact $\because \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} \neq \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} \neq \frac{\partial P}{\partial z}$

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PRACTICAL NO.-8: DIFFERENCE EQUATIONS

1) Form the difference equation corresponding to the following general solution:

a) $y = c_1x^2 + c_2x + c_3$ b) $y = (c_1 + c_2n)(-2)^n$

Solution: a) Given solution $y_x = c_1x^2 + c_2x + c_3$ …… (1)

contain three arbitrary constants c_1, c_2 and c_3, so we operate Δ thrice on this y_x, we get

$\Delta y_x = y_{x+1} - y_x = c_1(x+1)^2 + c_2(x+1) + c_3 - c_1x^2 - c_2x - c_3$

$= 2c_1x + c_1 + c_2$ …… (2)

$\Delta^2 y_x = [2c_1(x+1) + c_1 + c_2] - [2c_1x + c_1 + c_2]$

$= 2c_1$ …… (3)

& $\Delta^3 y_x = 2c_1 - 2c_1$

∴ $(E-1)^3 y_x = 0$

∴ $(E^3 - 3E^2 + 3E - 1)y_x = 0$

∴ $y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x = 0$ be the required difference equation.

b) Given solution $y_n = (c_1 + c_2n)(-2)^n$ i.e. $y_n = c_1(-2)^n + c_2n(-2)^n$ …… (1)

contain two arbitrary constants c_1 and c_2.

∴ $y_{n+1} = c_1(-2)^{n+1} + c_2(n+1)(-2)^{n+1} = -2c_1(-2)^n - 2c_2(n+1)(-2)^n$ …… (ii)

& $y_{n+2} = c_1(-2)^{n+2} + c_2(n+2)(-2)^{n+2} = 4c_1(-2)^n + 4c_2(n+2)(-2)^n$ …… (iii)

Eliminating c_1 and c_2 from equations (i), (ii), (iii), we get,

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -2 & -2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

i.e. $y_n[-8n-16+8n+8] - y_{n+1}[4n+8-4n] + y_{n+2}[-2n-2+2n] = 0$

i.e. $-2y_{n+2} - 8y_{n+1} - 8y_n = 0$

i.e. $y_{n+2} + 4y_{n+1} + 4y_n = 0$ be the required difference equation.

2) Show that $y_x = c_1 + c_2 2^x - x$ is a solution of the difference equation

$y_{x+2} - 3y_{x+1} + 2y_x = 1$

Proof: We have $y_x = c_1 + c_2 2^x - x$

∴ $y_{x+1} = c_1 + c_2 2^{x+1} - (x+1) = c_1 + 2c_2 2^x - x - 1$

& $y_{x+2} = c_1 + c_2 2^{x+2} - (x+2) = c_1 + 4c_2 2^x - x - 2$

Consider

LHS = $y_{x+2} - 3y_{x+1} + 2y_x$

$= c_1 + 4c_2\, 2^x - x - 2 - 3[c_1 + 2c_2\, 2^x - x - 1] + 2[c_1 + c_2\, 2^x - x]$

$= c_1 + 4c_2\, 2^x - x - 2 - 3c_1 - 6c_2\, 2^x + 3x + 3 + 2c_1 + 2c_2\, 2^x - 2x$

$= 1$

$=$ RHS

$\therefore$ $y_x = c_1 + c_2\, 2^x - x$ is a solution of the given difference equation is proved.

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3) Formulate the Fibonacci difference equation and solve it.

Solution: A sequence of type 0, 1, 1, 2, 3, 5, 8, ……. is called Fibonacci sequence which is formulated in difference equation form as $y_{x+1} = y_x + y_{x-1}$ with $y_0 = 0$ and $y_1 = 1$

To solve Fibonacci difference equation $y_{x+1} = y_x + y_{x-1}$ with $y_0 = 0$ and $y_1 = 1$

i.e. $y_{x+2} = y_{x+1} + y_x$ i.e. $(E^2 - E - 1)y_x = 0$

we take $y_x = m^x$, then the A.E. is

$m^2 - m - 1 = 0$

$\therefore m = \frac{1\pm\sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ are the roots of an A.E.

$\therefore$ The G. S. of the given Fibonacci difference equation is

$y_x = c_1\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^x + c_2\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^x$

i.e. $y_x = \frac{1}{2^x}\left[c_1(1 + \sqrt{5})^x + c_2(1 - \sqrt{5})^x\right]$

Now $y_0 = 0$ and $y_1 = 1$ gives

$0 = c_1 + c_2$ …… (i) and

$1 = \frac{1}{2}\left[c_1(1 + \sqrt{5}) + c_2(1 - \sqrt{5})\right]$

$= \frac{1}{2}\left[c_1 + c_1\sqrt{5} + c_2 - c_2\sqrt{5}\right]$

$1 = \frac{\sqrt{5}}{2}\left[c_1 - c_2\right]$

i.e. $c_1 - c_2 = \frac{2}{\sqrt{5}}$ …… (ii)

Adding equation (i) and (ii), we get,

$2c_1 = \frac{2}{\sqrt{5}}$ i.e. $c_1 = \frac{1}{\sqrt{5}}$

Putting in (i), we get, $c_2 = -\frac{1}{\sqrt{5}}$

$\therefore$ Required particular solution of Fibonacci difference equation is

$y_x = \frac{1}{2^x}[\frac{1}{\sqrt{5}}(1+\sqrt{5})^x - \frac{1}{\sqrt{5}}(1-\sqrt{5})^x]$

i.e. $y_x = \frac{1}{\sqrt{5}}[(1+\sqrt{5})^x - (1-\sqrt{5})^x].2^{-x}$

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4) Solve the following difference equations:

a) $y_{x+1} - 3y_x = 1$ b) $y_{x+1} - 3y_x = 0,\ y_0 = 2$

Solution: a) Let $y_{x+1} - 3y_x = 1$ i.e. $(E - 3)\ y_x = 1$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m - 3 = 0$

$\therefore$ m = 3 is the roots of an A.E.

$\therefore$ The G. S. of reduced homogeneous difference equation is

$y_x = c\ 3^x$

Now particular solution given non-homogeneous equation is

$$\text{P.S.} = \frac{1}{(E-3)}1$$

$$= \frac{1}{(E-3)}1^x$$

$$= \frac{1}{(1-3)}$$

$$= -\frac{1}{2}$$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = c\ 3^x - \frac{1}{2}$

b) Let $y_{x+1} - 3y_x = 0$ i.e. $(E - 3)\ y_x = 0$

be the given homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E.is

$m - 3 = 0$

$\therefore$ m = 3 is the roots of an A. E.

$\therefore$ The G. S. of given homogeneous difference equation is

$y_x = c\ 3^x$

Now $y_0 = 2$ gives $c\ 3^0 = 2$ i.e. c = 2

Hence particular solution of given equation is

$y_x = 2.3^x$

5) Solve the following non-homogeneous linear difference equations:

i) $y_{x+2} - 4y_x = 9x^2$ b) $\Delta y_x + \Delta^2 y_x = \sin x$

Solution: i) Let $y_{x+2} - 4y_x = 9x^2$ i.e. $(E^2 - 4)\ y_x = 9x^2$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 4 = 0$

i.e. $(m - 2)(m + 2) = 0$

$\therefore$ m = 2, -2 are the roots of an A. E.

$\therefore$ The G. S. of reduced homogeneous difference equation is

$y_x = C_1 2^x + C_2(-2)^x$

Now particular solution of given non-homogeneous equation is

P.S. $\frac{1}{(E^2-4)}(9x^2)$

$= \frac{9}{(1+\Delta)^2-4}(x^2)$

$= \frac{9}{-3+2\Delta+\Delta^2}(x^2)$

$= \frac{-3}{[1-(\frac{2}{3}\Delta+\frac{1}{3}\Delta^2)]}(x^2)$

` $= -3[1+(\frac{2}{3}\Delta + \frac{1}{3}\Delta^2) + (\frac{2}{3}\Delta + \frac{1}{3}\Delta^2)^2 + \ldots](x^2)$

$= -3[1+\frac{2}{3}\Delta + \frac{7}{9}\Delta^2 + \frac{4}{9}\Delta^3 + \ldots](x^2)$

$= -3[x^2 + \frac{2}{3}(2x) + \frac{7}{9}(2) + 0]$

$= -3x^2 - 4x - \frac{14}{3}$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 2^x + C_2(-2)^x - 3x^2 - 4x - \frac{14}{3}$

ii) Let $\Delta y_x + \Delta^2 y_x = \sin x$

i.e. $(\Delta + \Delta^2) y_x = \sin x$

i.e. $(E - 1 + E^2 - 2E + 1) y_x = \sin x$

i.e. $(E^2 - E) y_x = \sin x$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - m = 0$

i.e. $m(m - 1) = 0$

$\therefore$ m = 0, 1 are the roots of an A. E.

$\therefore$ The G. S. of reduced homogeneous difference equation is

$y_x = C_1 0^x + C_2(1)^x$

i.e. $y_x = C$, where $C_2 = C$

Now particular solution of given non-homogeneous equation is

P.S. = $\frac{1}{(E^2-E)}(\sin x)$

= Imaginary part of $\frac{1}{(E^2-E)}(e^{ix})$

= Imaginary part of $\frac{1}{(E^2-E)}(e^{i})^x$

= Imaginary part of $\frac{e^{ix}}{(e^{2i}-e^{i})}$

= Imaginary part of $\frac{e^{i(x-1)}}{(e^{i}-1)}$

= Imaginary part of $\frac{e^{i(x-1)}}{(e^{i}-1)} \times \frac{(e^{-i}-1)}{(e^{-i}-1)}$

= Imaginary part of $\frac{e^{i(x-2)}-e^{i(x-1)}}{(1-e^{i}-e^{-i}+1)}$

= Imaginary part of $\left[\frac{\cos(x-2)+i\sin(x-2)-\cos(x-1)-i\sin(x-1)}{2-\cos 1-i\sin 1-\cos 1+i\sin 1}\right]$

= $\frac{\sin(x-2)-\sin(x-1)}{2-2\cos 1}$

= $\frac{\sin(x-2)-\sin(x-1)}{2(1-\cos 1)}$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C + \frac{\sin(x-2)-\sin(x-1)}{2(1-\cos 1)}$

===

6) Solve $y_{x+2} - 4y_{x+1} + 3y_x = 3^x + 1$.

Solution: Let $y_{x+2} - 4y_{x+1} + 3y_x = 3^x + 1$.

i.e. $(E^2 - 4E + 3)y_x = 3^x + 1$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 4m + 3 = 0$

$(m - 1)(m - 3) = 0$

$\therefore$ m = 1, 3 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 + C_2 3^x$

Now particular solution of given non-homogeneous equation is

P.S. = $\frac{1}{(E^2-4E+3)}(3^x + 1)$

$= \frac{1}{(E-1)(E-3)}(3^x + 1^x)$

$= \frac{1}{(E-1)(E-3)}(3^x) + \frac{1}{(E-1)(E-3)}(1^x)$

$= \frac{x3^{x-1}}{1!(3-1)} + \frac{x1^{x-1}}{1!(1-3)}$

` $= \frac{x3^{x-1}}{2} - \frac{x}{2}$

$= \frac{1}{2}x(3^{x-1}-1)$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 + C_2 3^x + 3^x + \frac{1}{2}x(3^{x-1}-1)$

===

7) Solve $y_{x+2} - 4y_{x+1} + 4y_x = 3x + 2^x$

Solution: Let $y_{x+2} - 4y_{x+1} + 4y_x = 3x + 2^x$

i.e. $(E^2 - 4E + 4)y_x = 3x + 2^x$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E.is

$m^2 - 4m + 4 = 0$

$(m - 2)^2 = 0$

$\therefore$ m = 2, 2 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = (C_1 + C_2 x)\, 2^x$

Now particular solution given non-homogeneous equation is

P.S. = $\frac{1}{(E^2-4E+4)}(3x + 2^x)$

$= \frac{1}{(E-2)^2}(3x + 2^x)$

$= \frac{1}{(1+\Delta-2)^2}(3x) + \frac{1}{(E-2)^2}(2^x)$

$= \frac{3}{(\Delta-1)^2}x + \frac{x(x-1)2^{x-2}}{2!}$

` $= 3(1-\Delta)^{-2}x + \frac{x(x-1)2^{x-2}}{2}$

$= 3(1+2\Delta + \cdots)x + x(x-1)2^{x-3}$

$= 3(x+2(1) + 0) + x(x-1)2^{x-3}$

$= 3x + 6 + x(x-1)2^{x-3}$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = (C_1 + C_2x)\,2^x + 3x + 6 + x(x-1)2^{x-3}$

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8) Solve $u_{x+2} - 5u_{x+1} + 6u_x = 36$

Solution: Let $u_{x+2} - 5u_{x+1} + 6u_x = 36$

i.e. $(E^2 - 5E + 6)u_x = 36$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E.is

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

$\therefore$ m = 2, 3 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 2^x + C_2 3^x$

Now particular solution given non-homogeneous equation is

P.S. $= \frac{1}{(E^2 - 5E + 6)}(36)$

$= \frac{36}{(E-2)(E-3)}(1^x)$

$= \frac{36}{(1-2)(1-3)}$

$= 18$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 2^x + C_2 3^x + 18$

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."