Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: S.Y.B.SC SEM.-IV

SUBJECT: MTH-402(A): DIFFERENTIAL EQUATIONS

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MTH-402(A): DIFFERENTIAL EQUATIONS

Unit-1: Theory of ordinary differential equations **Marks-15**

1.1 Lipschitz condition
1.2 Existence and uniqueness theorem
1.3 Linearly dependent and independent solutions
1.4 Wronskian definition
1.5 Linear combination of solutions
1.6 Theorems on i) Linear combination of solutions ii) Linearly independent solutions iii) Wronskian is zero iv) Wronskian is non-zero
1.7 Method of variation of parameters for second order L.D.E.

Unit-2: Simultaneous Differential Equations **Marks-15**

2.1 Simultaneous linear differential equations of first order
2.2 Simultaneous D.E. of the form $dxP=dyQ=dzR$.
2.3 Rule I: Method of combinations
2.4 Rule II: Method of multipliers
2.5 Rule III: Properties of ratios
2.6 Rule IV: Miscellaneous

Unit-3: Total Differential or Pfaffian Differential Equations **Marks-15**

3.1 Pfaffian differential equations
3.2 Necessary and sufficient conditions for the integrability
3.3 Conditions for exactness
3.4 Method of solution by inspection
3.5 Solution of homogenous equation

Unit-4: Difference Equations **Marks-15**

4.1 Introduction, Order of difference equation, degree of difference equations
4.2 Solution to difference equation and formation of difference equations
4.3 Linear difference equations, Linear homogeneous difference equations with constant coefficients
4.4 Non-homogenous linear difference equation with constant coefficients

Recommended books:

1. Ordinary and Partial Differential Equation by M. D. Rai Singhania, S. Chand & Co. 18th Edition. (Chapter 1 and Chapter 2)
2. Numerical Methods by V. N. Vedamurthy and N. Ch. S. N. Iyengar, Vikas Publishing House, New Delhi. (Chapter 10).

Reference Book:

1. Introductory course in Differential Equations by D. A. Murray, Longmans Green and co. London and Mumbai, 5th Edition 1997.

Learning Outcomes:

a) Students will aware of formation of differential equations and their solutions
b) Students will understand the concept of Lipschitz condition
c) Students will understand method of variation of parameters for second order L.D.E.
d) Students will understand simultaneous linear differential equations and method of their solutions
e) Students will understand Pfaffian differential equations and method of their solutions
f) Students will understand difference equations and their solutions

UNIT-1: THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

Initial Value Problem: $\frac{dy}{dx}$ = f(x, y) with $y(x_0) = y_0$ is called initial value problem.

Remark: Initial value problem may have one solution or more than one solution or no solution.

Lipschitz Condition: A function f(x, y) defined in a region D in xy-plane is said to satisfy Lipschitz condition in D if for (x, y_1) and (x, y_2) in D, there exist a positive constant K such that $|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|$.

Here the constant K is called Lipschitz constant for the function f(x, y).

Existence Theorem: If the function f(x, y) is continuous and bounded for all values of x in a domain D and there exist a positive constants M & K such that $|f(x,\ y)| \le M$ and satisfies Lipschitz's condition $|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|$ for all points in domain D, then initial value problem $\frac{dy}{dx}$ = f(x, y) with $y(x_0) = y_0$ has at least one solution y(x).

Uniqueness Theorem:

If the function f(x, y) is continuous and bounded for all values of x in a domain D and there exist a positive constants M & K such that $|f(x,\ y)| \le M$ and satisfies Lipschitz's condition $|f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|$ for all points in domain D, then initial value problem $\frac{dy}{dx}$ = f(x, y) with $y(x_0) = y_0$ has a unique solution.

Theorem: If S is either a rectangle $|x - x_0| \le h, |y - y_0| \le k$ (h, k > 0) or a strip $|x - x_0| \le h, |y| < \infty$ (h > 0) and f(x, y) is a real valued function defined on S such that $\frac{\partial f}{\partial y}$ exits and continuous on S with $\left|\frac{\partial f}{\partial y}\right| \le K\ \forall\ (x, y) \in S$ for a positive constant K, then f(x, y) satisfies Lipschitz's condition on S with Lipschitz's constant K.

Proof: As
$$|f(x,y_2) - f(x,y_1)| = \left|\{f(x,y)\}_{y=y_1}^{y_2}\right|$$
$$= \left|\int_{y_1}^{y_2} \frac{\partial f}{\partial y} dy\right|$$
$$= \int_{y_1}^{y_2} \left|\frac{\partial f}{\partial y}\right| |dy|$$
$$\le \int_{y_1}^{y_2} K\, |dy|$$
$$\therefore |f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1| \text{ for } (x, y_1), (x, y_2) \in S$$

i.e. f(x, y) satisfies Lipschitz's condition on S with Lipschitz's constant K.

Ex.: Let the function $f(x, y) = x^2 + y^2 \ \forall (x, y) \in S$, S is the rectangle defined by $|x| \le a, |y| \le b$. Show that f(x, y) satisfies Lipschitz's condition. Find Lipschitz's constant.

Proof: Let $(x, y_1), (x, y_2)$ be any two points in the rectangle S which is defined by $|x| \le a, |y| \le b$ and $f(x, y) = x^2 + y^2$ …… (1)

$$\therefore |f(x, y_2) - f(x, y_1)| = |x^2 + {y_2}^2 - x^2 - {y_1}^2|$$
$$= |{y_2}^2 - {y_1}^2|$$
$$= |y_2 - y_1||y_2 + y_1|$$
$$\le [|y_2| + |y_1|]|y_2 - y_1|$$
$$\le 2b|y_2 - y_1| \qquad \because |y_2| \text{ and } |y_1| \le b$$

$\therefore$ f(x, y) satisfies Lipschitz's condition and Lipschitz's constant K = 2b.

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Ex.: If S is defined on the rectangle $|x| \le a, |y| \le b$, then show that the function f(x, y) = xsiny + ycosx satisfies Lipschitz's condition. Find the Lipschitz's constant.

Proof: Let f(x, y) = xsiny + ycosx

$\therefore \frac{\partial f}{\partial y}$ = xcosy + cosx

Here f(x, y) is real valued function defined on S where S is rectangle $|x| \le a, |y| \le b$

$\therefore \frac{\partial f}{\partial y}$ exists and continuous and hence bounded in S, with

$\left|\frac{\partial f}{\partial y}\right| = |x\cos y + \cos x| \le |x\cos y| + |\cos x| \le |x| + 1$

$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \le a + 1$

$\therefore$ f(x, y) satisfies Lipschitz's condition and Lipschitz's constant K = a + 1.

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Ex.: Show that the function $f(x, y) = xy^2$ satisfies Lipschitz's condition on the rectangle $|x| \le 1, |y| \le 1$. But does not satisfy Lipschitz's condition on strip $|x| \le 1, |y| \le \infty$.

Proof: Let $f(x, y) = xy^2$ …… (1)

i) Let S is a rectangle given by $|x| \le 1, |y| \le 1$ …… (2)

Clearly $f(x, y) = xy^2$ is continuous function on S and hence bounded on S

with $\frac{\partial f}{\partial y} = 2xy \Rightarrow \left|\frac{\partial f}{\partial y}\right| = 2|x||y| \le 2(1)(1) \le 2 \ \forall (x, y) \in S$

$\therefore$ f(x, y) satisfies Lipschitz's condition on S and Lipschitz's constant K = 2.

ii) Let R is a strip given by $|x| \leq 1, |y| \leq \infty$ …… (2)

Here f(x, y) = xy^2 is continuous function on R and hence bounded on R

with $\frac{\partial f}{\partial y} = 2xy \Rightarrow \left|\frac{\partial f}{\partial y}\right| = 2|x||y| \leq 2(1)(\infty) < \infty \quad \forall (x, y) \in S$

$\Rightarrow \frac{\partial f}{\partial y}$ is unbounded on strip R.

∴ f(x, y) does not satisfy Lipschitz's condition on strip R is proved.

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Ex.: Examine the existence and uniqueness of solutions of the initial value problem $\frac{dy}{dx} = y^{1/3}$ with y(0) = 0

Solution: Let $\frac{dy}{dx} = y^{1/3}$ with y(0) = 0 …… (i)

Comparing with $\frac{dy}{dx}$ = f(x, y), we get,

f(x, y) = $y^{1/3}$ …… (ii)

Clearly f(x, y) = $y^{1/3}$ is continuous and hence bounded.

Now f(x, y) = $y^{1/3} \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$

$\therefore \left|\frac{\partial f}{\partial y}\right| = \frac{1}{3} \frac{1}{|y^{2/3}|}$

For x = 0, y = 0 $\Rightarrow \left|\frac{\partial f}{\partial y}\right| = \frac{1}{3} \frac{1}{|0|} \to \infty$

$\Rightarrow \frac{\partial f}{\partial y}$ is unbounded at origin.

∴ f(x, y) does not satisfy Lipschitz's condition at origin.

∴ Uniqueness and existence is not applicable to given initial value problem.

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Remark: A continuous function may not satisfy Lipschitz's condition.

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Linear Differential Equation of Second Order:

An equation $a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$ is called a second order linear differential equation, where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous on an interval (a, b) and $a_0(x) \neq 0 \ \forall \ x \in (a, b)$.

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Linearly Dependent Solutions:

Two solutions $y_1(x)$ and $y_2(x)$ of linear differential equation of second order are said to be linearly dependent solutions if there exists two constants c_1 and c_2 not both zero such that $c_1y_1(x) + c_2y_2(x) = 0\ \forall\ x \in (a, b)$.

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Linearly Independent Solutions:

Two solutions $y_1(x)$ and $y_2(x)$ of linear differential equation of second order are said to be linearly independent solutions if for any two constants c_1 and c_2, $c_1y_1(x) + c_2y_2(x) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0\ \forall\ x \in (a, b)$.

Linearly Combination of Solutions:

Let $y_1(x)$ and $y_2(x)$ be any two solutions of linear differential equation of second order, then $c_1y_1(x) + c_2y_2(x) = 0\ \forall\ x \in (a, b)$ is called linear combination of two solutions $y_1(x)$ and $y_2(x)$, where c_1 and c_2 are constants.

The Wronskian:

Let $y_1(x)$ and $y_2(x)$ be any two solutions of linear differential equation of second order. Then the Wronskian of $y_1(x)$ and $y_2(x)$ is denoted by $W(y_1, y_2)$ or $W(x)$ and is defined as $W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Remark: The Wronskian of three functions $y_1(x)$, $y_2(x)$ and $y_3(x)$ is defined by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

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Theorem: If $y_1(x)$ and $y_2(x)$ are any two solutions of $a_0(x)y''(x)+a_1(x)y'(x)+ a_2(x)y(x) = 0$, then linear combination $c_1y_1(x) + c_2y_2(x) = 0$, where c_1 and c_2 are constants, is also solution of the given equation.

Proof: Consider a given equation $a_0(x)y''(x) + a_1(x)y'(x)+ a_2(x)y(x) = 0$ …… (i)

As $y_1(x)$ and $y_2(x)$ are the solutions of equation (i).

$\therefore a_0(x) y_1''(x) + a_1(x) y_1'(x)+ a_2(x)y_1(x) = 0$ …… (ii)

$\therefore a_0(x) y_2''(x) + a_1(x) y_2'(x)+ a_2(x)y_2(x) = 0$ …… (iii)

Let $u(x) = c_1y_1(x) + c_2y_2(x)$

$\therefore u'(x) = c_1y_1'(x) + c_2y_2'(x)$

$\therefore u''(x) = c_1y_1''(x) + c_2y_2''(x)$

Consider $a_0(x) u''(x) + a_1(x) u'(x)+ a_2(x) u(x)$

$= a_0(x)[c_1y_1''(x) + c_2y_2''(x)]+ a_1(x)[c_1y_1'(x) + c_2y_2'(x)]$
$\quad + a_2(x)[c_1y_1(x) + c_2y_2(x)]$

$= c_1[a_0(x) y_1''(x) + a_1(x) y_1'(x)+ a_2(x)y_1(x)]$
$\quad + c_2[a_0(x) y_2''(x) + a_1(x) y_2'(x)+ a_2(x)y_2(x)]$

$= c_1(0) + c_2(0)$ by (ii) and (iii)

$= 0$

$\therefore c_1y_1(x) + c_2y_2(x)$ is solution of given equation is proved.

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Remark: If $y_1(x), y_2(x), \ldots\ldots y_n(x)$ are solutions of $a_0(x)y''(x)+ a_1(x)y'(x)+ a_2(x)y(x) = 0$, then $c_1y_1(x) + c_2y_2(x) + \ldots\ldots + c_ny_n(x)= 0$ is also solution of the given equation. Where $c_1, c_2, \ldots\ldots c_n$ are constants.

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Theorem: Two solutions $y_1(x)$ and $y_2(x)$ of $a_0(x)y''(x)+ a_1(x)y'(x)+ a_2(x)y(x) = 0$, $a_0(x) \neq 0\ \forall\ x \in (a, b)$, are linearly dependent if and only if their Wronskian is identically zero.

Proof: Suppose two solutions $y_1(x)$ and $y_2(x)$ of $a_0(x)y''(x)+ a_1(x)y'(x)+ a_2(x)y(x) = 0$, $a_0(x) \neq 0\ \forall\ x \in (a, b)$, are linearly dependent ...... (1)

As $y_1(x)$ and $y_2(x)$ are linearly dependent.

$\therefore$ there exists two constants c_1 and c_2 not both zero such that

$c_1y_1(x) + c_2y_2(x) = 0\ \forall\ x \in (a, b)$ ...... (2)

$\therefore c_1y_1'(x) + c_2y_2'(x) = 0\ \forall\ x \in (a, b)$ ...... (3)

As c_1 and c_2 not simultaneously zero.

$$\therefore \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0\ \forall\ x \in (a, b)$$

$$\Rightarrow W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0\ \forall\ x \in (a, b)$$

$\Rightarrow$ Wronskian is zero.

Conversely: Suppose Wronskian is zero.

$$\text{i.e. } \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0\ \forall\ x \in (a, b)$$

Hence for some constants c_1 and c_2 not both zero

$c_1y_1(x) + c_2y_2(x) = 0\ \forall\ x \in (a, b)$

& $c_1y_1'(x) + c_2y_2'(x) = 0\ \forall\ x \in (a, b)$

$\therefore$ solutions $y_1(x)$ and $y_2(x)$ are linearly dependent is proved.

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Theorem: Two solutions $y_1(x)$ and $y_2(x)$ of $a_0(x)y''(x)+ a_1(x)y'(x)+ a_2(x)y(x) = 0$, $a_0(x) \neq 0\ \forall\ x \in (a, b)$, are linearly independent if and only if their Wronskian is non-zero.

Proof: Suppose two solutions $y_1(x)$ and $y_2(x)$ of $a_0(x)y''(x)+ a_1(x)y'(x)+ a_2(x)y(x) = 0$, $a_0(x) \neq 0\ \forall\ x \in (a, b)$, are linearly independent ...... (1)

$\Rightarrow$ Wronskian is non-zero $\because$ if Wronskian is zero, then solutions $y_1(x)$ and $y_2(x)$

are linearly dependent

Conversely : Suppose Wronskian is non-zero.

$\Longrightarrow$ solutions $y_1(x)$ and $y_2(x)$ are linearly independent. $\because$ if solutions $y_1(x)$ and $y_2(x)$ are linearly dependent, then Wronskian is zero,

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Ex.: Find the Wronskian of e^x and xe^x

Solution: Let $y_1 = e^x$ and $y_2 = xe^x$

$\Longrightarrow y_1' = e^x$ and $y_2' = e^x + xe^x$

$\therefore$ The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix}$$

$$= e^{2x}[1+x-x]$$

$\therefore W(x) = e^{2x}$

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Ex.: Find the Wronskian of sinx and cosx

Solution: Let $y_1 = \sin x$ and $y_2 = \cos x$

$\Longrightarrow y_1' = \cos x$ and $y_2' = -\sin x$

$\therefore$ The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} sinx & cosx \\ \cos x & -\sin x \end{vmatrix}$$

$$= -sin^2x - cos^2x$$

$\therefore W(x) = -1$

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Ex.: Find the Wronskian of sinx and sinx - cosx

Solution: Let $y_1 = \sin x$ and $y_2 = \sin x - \cos x$

$\Longrightarrow y_1' = \cos x$ and $y_2' = \cos x + \sin x$

$\therefore$ The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} sinx & sinx - cosx \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= sinxcosx + sin^2x - sinxcosx + cos^2x$$

$\therefore W(x) = 1$

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Ex.: Find the Wronskian of $e^{ax}\cos bx$ and $e^{ax}\sin bx$ $(b \neq 0)$

Solution: Let $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$ $(b \neq 0)$

$\Longrightarrow y_1' = ae^{ax}\cos bx - be^{ax}\sin bx$ and $y_2' = ae^{ax}\sin bx + be^{ax}\cos bx$

$\therefore$ The Wronskian of y_1 and y_2 is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ ae^{ax}\cos bx - be^{ax}\sin bx & ae^{ax}\sin bx + be^{ax}\cos bx \end{vmatrix}$$

$= e^{2ax}\begin{vmatrix} \cos bx & \sin bx \\ a\cos bx - b\sin bx & a\sin bx + b\cos bx \end{vmatrix}$

$= e^{2ax}[a\cos bx\sin bx + b\cos^2 bx - a\sin bx\cos bx + b\sin^2 bx]$

$= e^{2ax}[b\cos^2 bx + b\sin^2 bx]$

$\therefore W(x) = be^{2ax}$

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Ex.: Show that $e^x\cos x$ and $e^x\sin x$ are linearly independent

Proof: Let $y_1 = e^x\cos x$ and $y_2 = e^x\sin x$

$\Rightarrow y_1' = e^x\cos x - e^x\sin x$ and $y_2' = e^x\sin x + e^x\cos x$

$\therefore$ The Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin x & e^x\sin x + e^x\cos x \end{vmatrix}$

$= e^{2x}\begin{vmatrix} \cos x & \sin x \\ \cos x - \sin x & \sin x + \cos x \end{vmatrix}$

$= e^{2x}[\cos x\sin x + \cos^2 x - \sin x\cos x + \sin^2 x]$

$= e^{2x}[\cos^2 x + \sin^2 x]$

$\therefore W(x) = e^{2x} \neq 0$

$\therefore$ Given functions are linearly independent is proved.

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Ex.: Show that e^x and xe^x are linearly independent on the x-axis.

Proof: Let $y_1 = e^x$ and $y_2 = xe^x$

$\Rightarrow y_1' = e^x$ and $y_2' = e^x + xe^x$

$\therefore$ The Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$

$= e^{2x}\begin{vmatrix} 1 & x \\ 1 & 1 + x \end{vmatrix}$

$= e^{2x}[1+x-x]$

$\therefore W(x) = e^{2x} \neq 0$ for $x \neq 0$

$\therefore$ Given functions are linearly independent on x-axis is proved.

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Ex.: Show that the Wronskian of the functions x^2 and $x^2\log x$ is non zero.

Proof: Let $y_1 = x^2$ and $y_2 = x^2\log x$

$\Rightarrow y_1' = 2x$ and $y_2' = 2x\log x + x$

$\therefore$ The Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\log x \\ 2x & 2x\log x + x \end{vmatrix}$

$= x^3 \begin{vmatrix} 1 & \log x \\ 2 & 2\log x + 1 \end{vmatrix}$

$= x^3[2\log x + 1 - 2\log x]$

$\therefore W(x) = x^3 \neq 0$

$\therefore$ Given functions are linearly independent.

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Ex.: Show that sin2x and cos2x are solutions of the differential equation y"+ 4y = 0 and these are linearly independent.

Proof: Let $y_1 = \sin 2x$ and $y_2 = \cos 2x$ …… (1)

$\therefore y_1' = 2\cos 2x$ and $y_2' = -2\sin 2x$

$\therefore y_1'' = -4\sin 2x$ and $y_2'' = -4\cos 2x$

$\therefore y_1'' = -4y_1$ and $y_2'' = -4y_2$ by (1)

$\therefore y_1'' + 4y_1 = 0$ and $y_2'' + 4y_2 = 0$

$\therefore y_1 = \sin 2x$ and $y_2 = \cos 2x$ are the solutions of the differential equation y"+ 4y = 0 is proved.

Now the Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$

$= -2\sin^2 2x - 2\cos^2 2x$

$\therefore W(x) = -2 \neq 0$

$\therefore$ Given solutions are linearly independent is proved.

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Ex.: Show that sin3x and cos3x are linearly independent solutions of the differential equation y"+9y = 0.

Proof: Let $y_1 = \sin 3x$ and $y_2 = \cos 3x$ …… (1)

$\therefore y_1' = 3\cos 3x$ and $y_2' = -3\sin 3x$

$\therefore y_1'' = -9\sin 3x$ and $y_2'' = -9\cos 3x$

$\therefore y_1'' = -9y_1$ and $y_2'' = -9y_2$ by (1)

$\therefore y_1'' + 9y_1 = 0$ and $y_2'' + 9y_2 = 0$

$\therefore y_1 = \sin 3x$ and $y_2 = \cos 3x$ are the solutions of the differential equation y"+ 9y = 0 is proved.

Now the Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix}$

$= -3sin^2 3x - 3cos^2 3x$

$\therefore W(x) = -3 \neq 0$

$\therefore y_1 = \sin 3x$ and $y_2 = \cos 3x$ are linearly independent solutions of the differential equation $y'' + 9y = 0$ is proved

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Ex.: Show that $y_1 = \sin x$ and $y_2 = \sin x - \cos x$ are linearly independent solutions of the differential equation $y'' + y = 0$.

Proof: Let $y_1 = \sin x$ and $y_2 = \sin x - \cos x$ …… (1)

$\therefore y_1' = \cos x$ and $y_2' = \cos x + \sin x$

$\therefore y_1'' = -\sin x$ and $y_2'' = -\sin x + \cos x$

$\therefore y_1'' = -y_1$ and $y_2'' = -y_2$ by (1)

$\therefore y_1'' + y_1 = 0$ and $y_2'' + y_2 = 0$

$\therefore y_1 = \sin x$ and $y_2 = \sin x - \cos x$ are the solutions of the differential equation $y'' + y = 0$.

Now the Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} sinx & sinx - cosx \\ \cos x & \cos x + \sin x \end{vmatrix}$

$= sinxcosx + sin^2x - sinxcosx + cos^2x$

$\therefore W(x) = 1 \neq 0$

$\therefore y_1 = \sin x$ and $y_2 = \sin x - \cos x$ are linearly independent solutions of the differential equation $y'' + y = 0$ is proved

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Ex.: Examine whether e^{2x} and e^{3x} are linearly independent solutions of the differential equation $y'' - 5y' + 6y = 0$ or not?

Solution: Let $y_1 = e^{2x}$ and $y_2 = e^{3x}$ …… (1)

$\therefore y_1' = 2e^{2x}$ and $y_2' = 3e^{3x}$

$\therefore y_1'' = 4e^{2x}$ and $y_2'' = 9e^{3x}$

Consider $y_1'' - 5y_1' + 6y_1 = 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$ and

$y_2'' - 5y_2' + 6y_2 = 9e^{3x} - 15e^{3x} + 6e^{3x} = 0$

$\therefore y_1 = e^{2x}$ and $y_2 = e^{3x}$ are the solutions of the differential equation $y'' - 5y' + 6y = 0$.

Now the Wronskian of y_1 and y_2 is

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

$= 3e^{5x} - 2e^{5x}$

$\therefore$ W(x) = $e^{5x} \neq 0$

$\therefore$ $y_1 = e^{2x}$ and $y_2 = e^{3x}$ are linearly independent solutions of the differential equation $y'' - 5y' + 6y = 0$.

==

Ex.: Show that the functions $1+x$, x^2 and $1+2x$ are linearly independent.

Proof: Let $y_1 = 1+x$, $y_2 = x^2$ and $y_3 = 1+2x$ are the given functions.

$\therefore y_1' = 1, y_2' = 2x$ and $y_3' = 2$

$\therefore y_1'' = 0, y_2'' = 2$ and $y_3'' = 0$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1+x & x^2 & 1+2x \\ 1 & 2x & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$= (1+x)(0-4) - x^2(0-0) + (1+2x)(2-0)$

$= -4-4x+2+4x$

$\therefore$ W(x)= $-2 \neq 0$.

$\therefore$ Given functions are linearly independent.

==

Ex.: Using Wronskian, show that the functions x, x^2, x^3 are linearly independent.

Proof: Let $y_1 = x$, $y_2 = x^2$ and $y_3 = x^3$ are the given functions.

$\therefore y_1' = 1, y_2' = 2x$ and $y_3' = 3x^2$

$\therefore y_1'' = 0, y_2'' = 2$ and $y_3'' = 6x$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$= x(12x^2-6x^2) - x^2(6x-0) + x^3(2-0)$

$= 6x^3-6x^3+2x^3$

$\therefore$ W(x)= $2x^3 \neq 0$.

$\therefore$ Given functions are linearly independent.

==

Ex. Prove that $1, x, x^2$ are linearly independent. Hence form the differential equation whose solutions are $1, x, x^2$.

Proof: Let $y_1 = 1$, $y_2 = x$ and $y_3 = x^2$ are the given functions.

$\therefore y_1' = 0, y_2' = 1$ and $y_3' = 2x$

$\therefore y_1'' = 0, y_2'' = 0$ and $y_3'' = 2$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$= (2-0) - x(0-0)+x^2(0-0)$

$\therefore W(x)= 2 \neq 0.$

$\therefore 1, x, x^2$ are linearly independent solutions.

To find differential equation, let $y = c_1+ c_2x + c_3x^2$ …… (i)

where c_1, c_2, c_3 are constants.

Differentiating equation (i) thrice w.r.t. x, we get,

$\frac{dy}{dx} = c_2 + 2c_3x$

$\frac{d^2y}{dx^2} = 2c_3$

$\frac{d^3y}{dx^3} = 0$ be the required differential equation.

===

Ex. Examine whether the set of functions $1, x^2, x^3$ are linearly independent or not.

Solution: Let $y_1 = 1$, $y_2 = x^2$ and $y_3 = x^3$ are the given functions.

$\therefore y_1' = 0,\ y_2' = 2x$ and $y_3' = 3x^2$

$\therefore y_1'' = 0,\ y_2'' = 2$ and $y_3'' = 6x$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$= (12x^2 - 6x^2) - x^2 (0 - 0) + x^3(0 - 0)$

$\therefore W(x)= 6x^2 \neq 0$

$\therefore$ Given set of functions are linearly independent.

===

Ex.: Examine the functions x^2, e^x, e^{-x} for linear independence.

Solution: Let $y_1 = x^2$, $y_2 = e^x$ and $y_3 = e^{-x}$ are the given functions.

$\therefore y_1' = 2x, y_2' = e^x$ and $y_3' = -e^{-x}$

$\therefore y_1'' = 2, y_2'' = e^x$ and $y_3'' = e^{-x}$

$\therefore$ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x^2 & e^x & e^{-x} \\ 2x & e^x & -e^{-x} \\ 2 & e^x & e^{-x} \end{vmatrix}$$

$= e^{x-x}\begin{vmatrix} x^2 & 1 & 1 \\ 2x & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$

$= x^2(1+1)-(2x+2)+(2x-2)$

∴ $W(x)= 2x^2-4 \neq 0$ if $x^2- 2\neq 0$ i.e. if $x\neq \pm\sqrt{2}$

∴ Given functions are linearly independent if $x\neq \pm\sqrt{2}$ and are linearly dependent if $x= \pm\sqrt{2}$.

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Ex.: Examine whether the set of functions x^2-x+1, x^2-1, $3x^2-x-1$ are linearly dependent or not.

Solution: Let $y_1 = x^2-x+1$, $y_2 = x^2-1$ and $y_3 = 3x^2-x-1$ are the given functions.

∴ $y_1' = 2x - 1, y_2' = 2x$ and $y_3' = 6x-1$

∴ $y_1'' = 2, \quad y_2''= 2$ and $y_3''= 6$

∴ The Wronskian of y_1, y_2 and y_3 is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x^2-x+1 & x^2-1 & 3x^2-x-1 \\ 2x-1 & 2x & 6x-1 \\ 2 & 2 & 6 \end{vmatrix}$$

$= (x^2-x+1)(12x-12x+2)-(x^2-1) (12x-6-12x+2)+(3x^2-x-1)(4x-2-4x)$

$= 2x^2-2x+2+4x^2-4-6x^2+2x+2$

∴ $W(x)= 0$

∴ Given set of functions are linearly dependent.

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Method of Variation of Parameters:

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ …… (i) be a linear differential equation,

where P, Q and R are the functions of x or constants.

Suppose $y = Au + Bv$ …… (ii) be a complementary function (C.F.) of (i).

Where A, B are constants and u, v are functions of x.

As (ii) is C.F. of (i), hence u and v must be the solution of auxiliary equation of (i)

i.e. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ …… (iii)

∴ $\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$ …… (iv)

and $\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0$ …… (v)

In the method of variation of parameter, we assume that $y = Au + Bv$ …… (vi) be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and $u\frac{dA}{dx} + v\frac{dB}{dx} = 0$ …… (vii)

Differentiating equation (vi) w.r.t. x, we get,

$\frac{dy}{dx} = A\frac{du}{dx} + u\frac{dA}{dx} + B\frac{dv}{dx} + v\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = A\frac{du}{dx} + B\frac{dv}{dx}$ …… (viii) using (vii).

Now differentiating equation (viii) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = \frac{dA}{dx}\frac{du}{dx} + A\frac{d^2u}{dx^2} + \frac{dB}{dx}\frac{dv}{dx} + B\frac{d^2v}{dx^2}$ …… (ix)

Using (vi), (viii) and (ix) in (i), we have,

$[\frac{dA}{dx}\frac{du}{dx} + A\frac{d^2u}{dx^2} + \frac{dB}{dx}\frac{dv}{dx} + B\frac{d^2v}{dx^2}] + P[A\frac{du}{dx} + B\frac{dv}{dx}] + Q[Au + Bv] = R$

$\Longrightarrow A[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu] + B[\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv] + [\frac{dA}{dx}\frac{du}{dx} + \frac{dB}{dx}\frac{dv}{dx}] = R$

$\Longrightarrow A(0) + B(0) + [\frac{dA}{dx}\frac{du}{dx} + \frac{dB}{dx}\frac{dv}{dx}] = R$ by (iv) and (v)

$\Longrightarrow \frac{dA}{dx}\frac{du}{dx} + \frac{dB}{dx}\frac{dv}{dx} = R$ …… (x)

Solving (vii) and (x),we get, $\frac{dA}{dx}$ and $\frac{dB}{dx}$.

Integrating $\frac{dA}{dx}$ and $\frac{dB}{dx}$, we get, A and B.

Putting these values of A and B in (vi), we get G.S. of given equation (i).

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Ex.: Solve by method of variation of parameters $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

Solution: Let $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ i.e. $(D^2 + 4)y = 4\tan 2x$ …… (i) be the given equation.

∴ Its A.E. is $D^2 + 4 = 0$ which has roots $D = \pm 2i$.

∴ C.F. is $y = A\cos 2x + B\sin 2x$

By method of variation of parameter assume that $y = A\cos 2x + B\sin 2x$ …… (ii) be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and $\cos 2x\frac{dA}{dx} + \sin 2x\frac{dB}{dx} = 0$ …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -2A\sin 2x + \cos 2x\frac{dA}{dx} + 2B\cos 2x + \sin 2x\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -4A\cos2x - 2\sin2x\frac{dA}{dx} - 4B\sin2x + 2\cos2x\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -4(A\cos2x + B\sin2x) - 2\sin2x\frac{dA}{dx} + 2\cos2x\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -4y - 2\sin2x\frac{dA}{dx} + 2\cos2x\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + 4y = -2\sin2x\frac{dA}{dx} + 2\cos2x\frac{dB}{dx}$

$\therefore -2\sin2x\frac{dA}{dx} + 2\cos2x\frac{dB}{dx} = 4\tan2x$ by (i)

i.e. $\sin2x\frac{dA}{dx} - \cos2x\frac{dB}{dx} = -2\tan2x$....... (v)

To solve (iii) and (v), consider sin2x(iii) - cos2x(v), we get,

$\sin2x\cos2x\frac{dA}{dx} + \sin^2 2x\frac{dB}{dx} - \sin2x\cos2x\frac{dA}{dx} + \cos^2 2x\frac{dB}{dx} = 0 + 2\cos2x\tan2x$

$\therefore \frac{dB}{dx} = 2\sin2x$

Putting value of $\frac{dB}{dx}$ in (iii), we get,

$\cos2x\frac{dA}{dx} + \sin2x(2\sin2x) = 0$

$\therefore \cos2x\frac{dA}{dx} = -2\sin^2 2x \Rightarrow \frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos2x}$

Now $\frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos2x} \Rightarrow A = \int(-\frac{2\sin^2 2x}{\cos2x})dx + c_1 = -2\int(\frac{1-\cos^2 2x}{\cos2x})dx + c_1$

$= -2\int(\sec2x - \cos2x)dx + c_1$

$= -\log(\sec2x + \tan2x) + \sin2x + c_1$

and $\frac{dB}{dx} = 2\sin2x \Rightarrow B = \int 2\sin2x\,dx = -\cos2x + c_2$

Putting these values of A and B in (ii), we get,

$y = [-\log(\sec2x + \tan2x) + \sin2x + c_1]\cos2x + (-\cos2x + c_2)\sin2x$

$\therefore y = c_1\cos2x + c_2\sin2x - \cos2x\log(\sec2x + \tan2x) + \sin2x\cos2x - \cos2x\sin2x$

$\therefore y = c_1\cos2x + c_2\sin2x - \cos2x\log(\sec2x + \tan2x)$

be the required G.S. of given equation.

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Ex.: Solve by method of variation of parameters y'' - 3y' + 2y = 2

Solution: Let y'' - 3y' + 2y = 2 i.e. $(D^2 - 3D + 2)y = 2$ ...... (i) be the given equation.

$\therefore$ Its A.E. is $D^2 - 3D + 2 = 0$ i.e. $(D - 1)(D - 2) = 0$ which has roots D = 1, 2.

$\therefore$ C.F. is $y = Ae^x + Be^{2x}$

By method of variation of parameter assume that $y = Ae^x + Be^{2x}$ ...... (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and $e^x\frac{dA}{dx} + e^{2x}\frac{dB}{dx} = 0$ ...... (iii)

Differentiating equation (ii) w.r.t. x, we get,

y' = $Ae^{x}+ e^{x}\frac{dA}{dx} + 2Be^{2x} + e^{2x}\frac{dB}{dx}$

$\Rightarrow$ y' = $Ae^{x} + 2Be^{2x}$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

y'' = $Ae^{x} + e^{x}\frac{dA}{dx} + 4Be^{2x} + 2e^{2x}\frac{dB}{dx}$

$\therefore$ y'' - 3y' + 2y = 2 gives

$Ae^{x} + e^{x}\frac{dA}{dx} + 4Be^{2x} + 2e^{2x}\frac{dB}{dx} - 3Ae^{x} - 6Be^{2x} + 2Ae^{x} + 2Be^{2x} = 2$

i.e. $e^{x}\frac{dA}{dx} + 2e^{2x}\frac{dB}{dx} = 2$ ……(v)

To solve (iii) and (v), consider (v) - (iii), we get,

$e^{x}\frac{dA}{dx} + 2e^{2x}\frac{dB}{dx} - e^{x}\frac{dA}{dx} - e^{2x}\frac{dB}{dx} = 2 - 0$

$\therefore e^{2x}\frac{dB}{dx} = 2 \Rightarrow \frac{dB}{dx} = 2e^{-2x}$

Putting value of $\frac{dB}{dx}$ in (iii), we get,

$e^{x}\frac{dA}{dx} + e^{2x}(2e^{-2x}) = 0$

$\therefore e^{x}\frac{dA}{dx} = -2 \Rightarrow \frac{dA}{dx} = -2e^{-x}$

Now $\frac{dA}{dx} = -2e^{-x} \Rightarrow A = \int(-2e^{-x})dx + c_1 = 2e^{-x} + c_1$ and

$\frac{dB}{dx} = 2e^{-2x} \Rightarrow B = \int 2e^{-2x}dx = -e^{-2x} + c_2$

Putting these values of A and B in (ii), we get,

$y = [2e^{-x} + c_1]\, e^{x} + (-e^{-2x} + c_2)\, e^{2x}$

$\therefore y = c_1e^{x} + c_2e^{2x} + 2 - 1$

$\therefore y = c_1e^{x} + c_2e^{2x} + 1$ be the required G.S. of given equation.

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Ex.: Using method of variation of parameters solve $\frac{d^2y}{dx^2}$ + y = cosecx

Solution: Let $\frac{d^2y}{dx^2}$ + y = cosecx i.e. $(D^2 + 1)y$ = cosecx …… (i) be the given equation.

$\therefore$ Its A.E. is $D^2 + 1 = 0$ which has roots $D = \pm i$.

$\therefore$ C.F. is y = Acosx + Bsinx

By method of variation of parameter assume that y = Acosx + Bsinx …… (ii) be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied and $\cos x\frac{dA}{dx} + \sin x\frac{dB}{dx} = 0$ …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -Asinx +$ cosx $\frac{dA}{dx} + Bcosx +$ sinx $\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -Asinx + Bcosx$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -Acosx - sinx\frac{dA}{dx} - Bsinx + cosx\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -(Acosx + Bsinx) - sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -y - sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + y = -sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$

$\therefore -sinx\frac{dA}{dx} + cosx\frac{dB}{dx} =$ cosecx …… (v) by (i)

To solve (iii) and (v), consider sinx(iii) + cosx(v), we get,

sinxcosx $\frac{dA}{dx}$ + $\sin^2$x $\frac{dB}{dx}$ - sinxcosx $\frac{dA}{dx}$ + $\cos^2$x $\frac{dB}{dx}$ = 0 + cosxcosecx

$\therefore \frac{dB}{dx}$ = cotx

Putting value of $\frac{dB}{dx}$ in (iii), we get,

cosx $\frac{dA}{dx}$ + sinx(cotx) = 0

$\therefore$ cosx $\frac{dA}{dx} = -$cosx $\Longrightarrow \frac{dA}{dx} = -1$

Now $\frac{dA}{dx} = -1 \Longrightarrow A = \int(-1)dx = -x + c_1$ and

$\frac{dB}{dx}$ = cotx $\Longrightarrow B = \int cotx\, dx$ = logsinx+ c_2

Putting these values of A and B in (ii), we get,

y = $(-x + c_1)$cosx + (logsinx+ c_2)sinx

$\therefore$ y = $c_1$cosx + $c_2$sinx$-x$cosx + sinx(logsinx)

be the required G.S. of given equation.

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Ex.: Using method of variation of parameters solve $\frac{d^2y}{dx^2} + a^2y$ = cosec(ax)

Solution: Let $\frac{d^2y}{dx^2} + a^2y$ = cosec(ax) i.e. $(D^2 + a^2)y$ = cosec(ax) …… (i)

be the given equation.

$\therefore$ Its A.E. is $D^2 + a^2 = 0$ which has roots D = $\pm$ai.

$\therefore$ C.F. is y = Acosax + Bsinax

By method of variation of parameter assume that y = Acosax + Bsinax …… (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied and cosax $\frac{dA}{dx}$ + sinax $\frac{dB}{dx}$ = 0 …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -aAsinax +$ cosax $\frac{dA}{dx} + aBcosax$ + sinax $\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -aAsinax + aBcosax$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -a^2Acosax - asinax\frac{dA}{dx} - a^2Bsinax + acosax\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -a^2(Acosax + Bsinax) - asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -a^2y - asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + a^2y = -asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$

$\therefore -asinax\frac{dA}{dx} + acosax\frac{dB}{dx}$ = cosec(ax) …… (v) by (i)

To solve (iii) and (v), consider asinax(iii)+cosax(v), we get,

asinaxcosax $\frac{dA}{dx}$ + asin^{2}ax $\frac{dB}{dx}$ - asinaxcosax $\frac{dA}{dx}$ + acos^{2}ax $\frac{dB}{dx}$ = 0 + cosaxcosec(ax)

$\therefore$ a $\frac{dB}{dx}$ = cot(ax) $\Longrightarrow \frac{dB}{dx} = \frac{1}{a}$ cot(ax)

Putting value of $\frac{dB}{dx}$ in (iii), we get,

cosax $\frac{dA}{dx}$ + sinax[$\frac{1}{a}$ cot(ax)] = 0

$\therefore$ cosax $\frac{dA}{dx} = -\frac{1}{a}$ cosax $\Longrightarrow \frac{dA}{dx} = -\frac{1}{a}$

Now $\frac{dA}{dx} = -\frac{1}{a} \Longrightarrow A = \int(-\frac{1}{a})dx = -\frac{x}{a} + c_1$ and

$\frac{dB}{dx} = \frac{1}{a}$ cot(ax) $\Longrightarrow B = \int(\frac{1}{a}cotax)dx = \frac{1}{a^2}$ logsinax+ c_2

Putting these values of A and B in (ii), we get,

$y = (-\frac{x}{a} + c_1)$cosax + $(\frac{1}{a^2}$ logsinax+ $c_2)$sinax

$\therefore y = c_1$cosax + $c_2$sinax$- \frac{x}{a}$ cosax + $\frac{1}{a^2}$ sinax(logsinax)

be the required G.S. of given equation.

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Ex.: Using method of variation of parameters solve $\frac{d^2y}{dx^2} + 9y = sec3x$

Solution: Let $\frac{d^2y}{dx^2} + 9y = sec3x$ i.e. $(D^2 + 9)y = sec3x$…… (i)

be the given equation.

∴ Its A.E. is $D^2 + 9 = 0$ which has roots $D = \pm 3i$.

∴ C.F. is y = Acos3x + Bsin3x

By method of variation of parameter assume that y = Acos3x + Bsin3x …… (ii) be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied and $\cos3x\frac{dA}{dx} + \sin3x\frac{dB}{dx} = 0$ …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -3A\sin3x + \cos3x\frac{dA}{dx} + 3B\cos3x + \sin3x\frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -3A\sin3x + 3B\cos3x$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -9A\cos3x - 3\sin3x\frac{dA}{dx} - 9B\sin3x + 3\cos3x\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -9(A\cos3x + B\sin3x) - 3\sin3x\frac{dA}{dx} + 3\cos3x\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -9y - 3\sin3x\frac{dA}{dx} + 3\cos3x\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + 9y = -3\sin3x\frac{dA}{dx} + 3\cos3x\frac{dB}{dx}$

$\therefore -3\sin3x\frac{dA}{dx} + 3\cos3x\frac{dB}{dx} = \sec3x$ …… (v) by (i)

To solve (iii) and (v), consider 3sin3x(iii) + cos3x(v), we get,

$3\sin3x\cos3x\frac{dA}{dx} + 3\sin^23x\frac{dB}{dx} - 3\sin3x\cos3x\frac{dA}{dx} + 3\cos^23x\frac{dB}{dx} = 0 + \cos3x\sec3x$

$\therefore 3\frac{dB}{dx} = 1 \Longrightarrow \frac{dB}{dx} = \frac{1}{3}$

Putting value of $\frac{dB}{dx}$ in (iii), we get,

$\cos3x\frac{dA}{dx} + \sin3x(\frac{1}{3}) = 0$

$\therefore \cos3x\frac{dA}{dx} = -\frac{1}{3}\sin3x \Longrightarrow \frac{dA}{dx} = -\frac{1}{3}\tan3x$

Now $\frac{dA}{dx} = -\frac{1}{3}\tan3x \Longrightarrow A = \int(-\frac{1}{3}\tan3x)dx = \frac{1}{9}\log\cos3x + c_1$ and

$\frac{dB}{dx} = \frac{1}{3} \Longrightarrow B = \int(\frac{1}{3})dx = \frac{x}{3} + c_2$

Putting these values of A and B in (ii), we get,

$y = (\frac{1}{9}\log\cos3x + c_1)\cos3x + (\frac{x}{3} + c_2)\sin3x$

$\therefore y = c_1\cos3x + c_2\sin3x + \frac{1}{9}\cos3x(\log\sin3x) + \frac{x}{3}\sin3x$

be the required G.S. of given equation.

==

Ex.: Using method of variation of parameters solve $\frac{d^2y}{dx^2} + a^2y = \sec(ax)$

Solution: Let $\frac{d^2y}{dx^2} + a^2y = \sec(ax)$ i.e. $(D^2 + a^2)y = \sec(ax)$ …… (i)

be the given equation.

∴ Its A.E. is $D^2 + a^2 = 0$ which has roots $D = \pm ai$.

∴ C.F. is y = Acosax + Bsinax

By method of variation of parameter assume that y = Acosax + Bsinax …… (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and $\cos ax \frac{dA}{dx} + \sin ax \frac{dB}{dx} = 0$ …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

$\frac{dy}{dx} = -aA\sin ax + \cos ax \frac{dA}{dx} + aB\cos ax + \sin ax \frac{dB}{dx}$

$\Rightarrow \frac{dy}{dx} = -aA\sin ax + aB\cos ax$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

$\frac{d^2y}{dx^2} = -a^2A\cos ax - a\sin ax\frac{dA}{dx} - a^2B\sin ax + a\cos ax\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -a^2(A\cos ax + B\sin ax) - a\sin ax\frac{dA}{dx} + a\cos ax\frac{dB}{dx}$

$\therefore \frac{d^2y}{dx^2} = -a^2y - a\sin ax\frac{dA}{dx} + a\cos ax\frac{dB}{dx}$ by (ii)

$\therefore \frac{d^2y}{dx^2} + a^2y = -a\sin ax\frac{dA}{dx} + a\cos ax\frac{dB}{dx}$

$\therefore -a\sin ax\frac{dA}{dx} + a\cos ax\frac{dB}{dx} = \sec(ax)$ …… (v) by (i)

To solve (iii) and (v), consider asinax(iii)+cosax(v), we get,

$a\sin ax\cos ax\frac{dA}{dx} + a\sin^2 ax\frac{dB}{dx} - a\sin ax\cos ax\frac{dA}{dx} + a\cos^2 ax\frac{dB}{dx} = 0 + \cos ax\sec(ax)$

$\therefore a\frac{dB}{dx} = 1 \Longrightarrow \frac{dB}{dx} = \frac{1}{a}$

Putting value of $\frac{dB}{dx}$ in (iii), we get,

$\cos ax\frac{dA}{dx} + \sin ax(\frac{1}{a}) = 0$

$\therefore \cos ax\frac{dA}{dx} = -\frac{1}{a}\sin ax \Longrightarrow \frac{dA}{dx} = -\frac{1}{a}\tan ax$

Now $\frac{dA}{dx} = -\frac{1}{a}\tan ax \Longrightarrow A = \int(-\frac{1}{a}\tan ax)dx = \frac{1}{a^2}\log\cos ax + c_1$ and

$\frac{dB}{dx} = \frac{1}{a} \Longrightarrow B = \int(\frac{1}{a})dx = \frac{x}{a} + c_2$

Putting these values of A and B in (ii), we get,

$y = (\frac{1}{a^2}\log\cos ax + c_1)\cos ax + (\frac{x}{a} + c_2)\sin ax$

$\therefore$ y = $c_1$cosax + $c_2$sinax + $\frac{1}{a^2}$ cosax(logsinax) + $\frac{x}{a}$ sinax

be the required G.S. of given equation.

==

Ex.: Solve by method of variation of parameters y" + y – x = 0

Solution: Let y" + y = x i.e. $(D^2 + 1)y = x$ …… (i)

be the given equation.

$\therefore$ Its A.E. is $D^2 + 1 = 0$ which has roots D = $\pm$i.

$\therefore$ C.F. is y = C_1cosx + C_2sinx

By method of variation of parameter assume that y = Acosx + Bsinx …… (ii)

be the G.S. of the given equation (i).

Where A and B are functions of x so chosen that equation (i) shall be satisfied

and cosx $\frac{dA}{dx}$ + sinx $\frac{dB}{dx}$ = 0 …… (iii)

Differentiating equation (ii) w.r.t. x, we get,

y' = $-Asinx$+ cosx $\frac{dA}{dx}$ + $Bcosx$ + sinx $\frac{dB}{dx}$

$\Rightarrow$ y' = $-Asinx + Bcosx$ …… (iv) using (iii).

Again differentiating equation (iv) w.r.t. x, we get,

y" = $-Acosx - sinx\frac{dA}{dx} - Bsinx + cosx\frac{dB}{dx}$

$\therefore$ y" = $-(Acosx + Bsinx) - sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$

$\therefore$ y" = $-y - sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$ by (ii)

$\therefore$ y" + $y = -sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$

$\therefore$ $-sinx\frac{dA}{dx} + cosx\frac{dB}{dx}$ = x …… (v) by (i)

To solve (iii) and (v), consider sinx(iii) + cosx(v), we get,

sinxcosx $\frac{dA}{dx}$ + $\sin^2$x $\frac{dB}{dx}$ - sinxcosx $\frac{dA}{dx}$ + $\cos^2$x $\frac{dB}{dx}$ = 0 + xcosx

$\therefore$ $\frac{dB}{dx}$ = xcosx

Putting value of $\frac{dB}{dx}$ in (iii), we get,

cosx $\frac{dA}{dx}$ + sinx[$xcosx$] = 0

$\therefore$ cosx $\frac{dA}{dx}$ = – xsinxcosx $\Rightarrow$ $\frac{dA}{dx} = -xsinx$

Now $\frac{dA}{dx} = -xsinx \Rightarrow$ A = $\int(-xsinx)$dx=xcosx-$\int cosx\,dx$+c_1 = xcosx- $sinx$ + $c_1$and

$\frac{dB}{dx}$ = xcosx $\Rightarrow$ B = $\int$ xcosxdx = xsinx-$\int sinx\,dx$ + c_2 = xsinx + cos$x$ + c_2

Putting these values of A and B in (ii), we get,

y = (xcosx- $sinx$+ c_1)cosx + (xsinx + cos$x$ + c_2)sinx

$\therefore$ y = $c_1$cosx + $c_2$sinx + x$\cos^2$x – sinxcosx + x$\sin^2$x + cosxsinx

$\therefore$ y = $c_1$cosx + $c_2$sinx + x be the required G.S. of given equation.

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MULTIPLE CHOICE QUESTIONS [MCQ'S]

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1) $\frac{dy}{dx}$ = f(x, y) with $y(x_0) = y_0$ is called …….

A) initial value problem B) linear equation
C) homogeneous equation D) None of these

2) An initial value problem $\frac{dy}{dx}$ = f(x, y) with $y(x_0) = y_0$ may have ……

A) one solution B) more than one solution
C) no solution. D) all of these.

3) A function f(x, y) defined in a region D in xy-plane is said to satisfy Lipschitz condition
in D if for (x, y_1) and (x, y_2) in D, there exist a positive constant K such that …….

A) $|f(x,y_2) - f(x,y_1)| \leq K|y_2 - y_1|$ B) $|f(x,y_2) - f(x,y_1)| \leq K$
C) $|f(x,y_2) - f(x,y_1)| \geq K|y_2 - y_1|$ D) None of these

4) If a function f(x, y) satisfy Lipschitz condition $|f(x,y_2) - f(x,y_1)| \leq K|y_2 - y_1|$ then K is called ……. for the function f(x, y).

A) constant B) Lipschitz constant C) variable D) None of these

5) Every continuous function …… satisfy Lipschitz condition.

A) may B) must C) may not D) None of these

6) If the function f(x, y) is continuous and bounded for all values of x in a domain D and satisfies Lipschitz's condition $|f(x,y_2) - f(x,y_1)| \leq K|y_2 - y_1|$ for all points in domain D, then initial value problem $\frac{dy}{dx}$ = f(x, y) with $y(x_0) = y_0$ has ……

A) a unique solution. B) no solution.
C) infinite number of solutions D) None of these

7) If S is either a rectangle $|x - x_0| \leq h, |y - y_0| \leq k$ (h, k > 0) or a strip $|x - x_0| \leq h, |y| < \infty$ (h > 0) and f(x, y) is a real valued function defined on S such that $\frac{\partial f}{\partial y}$ exits and continuous on S with $\left|\frac{\partial f}{\partial y}\right| \leq K$ ∀ (x, y) ∈ S for a positive constant K, then f(x, y) satisfies Lipschitz's condition on S with ……constant K.

A) Picard's B) Non Lipschitz's C) Lipschitz's D) None of these

8) A function f(x, y) is said to satisfy Lipschitz condition in a region D in xy plane if there

exist a positive constant K such that $|f(x, y_2) - f(x, y_1)| \leq$.......whenever the points (x, y_1) and (x, y_2) both lie in D.

A) K$|x_1 - x_2|$ B) K$|y_2 - x|$ C) K$|y_2 - y_1|$ D) None of these

9) If S is defined by the rectangle $|x| \leq$ a, $|y| \leq$ b, then the function f(x, y) = xsiny + ycosx satisfies Lipschitz's condition with Lipschitz's constant is ……

A) a B) a - 1 C) a + 1 D) b

10) If S is defined by the rectangle $|x| \leq 1, |y| \leq 1$, then Lipschitz's constant is …… for the function f(x, y) = xy^2.

A) 1 B) 2 C) 3 D) 4

11) If S is defined by the rectangle $|x| \leq$ a, $|y| \leq$ b, then Lipschitz's constant is …… for the function f(x, y) = $x^2 + y^2$.

A) b B) a C) 2b D) 2a

12) Uniqueness and existence is …… for the initial value problem $\frac{dy}{dx} = y^{1/3}$ with y(0) = 0.

A) applicable B) not applicable

C) may or may not applicable D) None of these

13) If $a_0(x)$, $a_1(x)$ and $a_2(x)$ are continuous on an interval (a, b) and $a_0(x) \neq 0\ \forall\ x \in (a, b)$, then an equation $a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$ is called ……

A) a second order linear differential equation

B) a first order linear differential equation

C) a third order linear differential equation

D) None of these

14) The Wronskian of $y_1(x)$ and $y_2(x)$ is denoted by W(y_1, y_2) and is defined as …

A) $\begin{vmatrix} y_1 & y_1' \\ y_2' & y_2 \end{vmatrix}$ B) $\begin{vmatrix} y_1' & y_1' \\ y_2' & y_2 \end{vmatrix}$ C) $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ D) None of these

15) The Wronskian of three functions $y_1(x)$, $y_2(x)$ and $y_3(x)$ is W(x) = ……

A) $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1 & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ B) $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ C) $\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3 \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ D) None of these

16) Two solutions $y_1(x)$ and $y_2(x)$ of $a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$, $a_0(x) \neq 0$ $\forall\ x \in (a, b)$, are linearly dependent if and only if their Wronskian is …….

A) zero B) non zero C) 1 D) None of these

17) Two non zero functions $f_1(x)$ and $f_2(x)$ are linearly dependent iff their Wronskian is…

A) zero B) non zero C) 1 D) None of these

18) Two solutions $y_1(x)$ and $y_2(x)$ of $a_0(x)y''(x)+ a_1(x)y'(x)+ a_2(x)y(x) = 0$, $a_0(x) \neq 0$ $\forall\, x \in (a, b)$, are linearly independent if and only if their Wronskian is …….

A) zero B) 1 C) non zero D) None of these

19) Two non zero functions $f_1(x)$ and $f_2(x)$ of differential equation are linearly independent iff their Wronskian is …

A) non zero B) zero C) non vanishing D) None of these

20) The Wronskian of e^{-x} and e^x is ……

A) 1 B) 2 C) 3 D) None of these

21) The functions e^{-x} and e^x are ……

A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

22) The Wronskian of sinx and cosx is ……

A) -1 B) 0 C) 1 D) None of these

23) The functions sinx and cosx are ……

A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

24) The Wronskian of cosx and sinx is ……

A) -1 B) 0 C) 1 D) None of these

25) The functions cosx and sinx are ……

A) Linearly independent B) Linearly dependent and Linearly independent
C) Linearly dependent D) None of these

26) The Wronskian of sin2x and cos2x is ……

A) -2 B) 0 C) 2 D) None of these

27) The functions sin2x and cos2x are ……

A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

28) The Wronskian of sin3x and cos3x is ……

A) 0 B) 3 C) -3 D) None of these

29) The functions sin3x and cos3x are ……

A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

30) The Wronskian of the functions y_1 = sinx and y_2 = sinx – cosx is ……

A) 0 B) 1 C) $\sin^2 x$ D) $\cos^2 x$

31) The functions y_1 = sinx and y_2 = sinx – cosx are ……

A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

32) The Wronskian of the functions $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ is ……
A) e^{2x} B) e^x C) 1 D) 0

33) The functions $e^x\cos x$ and $e^x\sin x$ are ……
A) Linearly independent B) Linearly dependent and Linearly independent
C) Linearly dependent D) None of these

34) The Wronskian of $e^{2x}\cos 3x$ and $e^{2x}\sin 3x$ is ……
A) $3e^{4x}$ B) 0 C) $3e^{2x}$ D) $2e^{3x}$

35) The functions $e^{2x}\cos 3x$ and $e^{2x}\sin 3x$ are ……
A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

36) The Wronskian of $e^{ax}\cos bx$ and $e^{ax}\sin bx$ $(b \neq 0)$ is ……
A) ae^{2ax} B) 0 C) be^{2ax} D) $2be^{ax}$

37) The functions $e^{ax}\cos bx$ and $e^{ax}\sin bx$ $(b \neq 0)$ are ……
A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

38) The Wronskian of e^{2x} and e^{3x} is ……
A) e^{2x} B) e^{3x} C) e^{5x} D) e^{6x}

39) The functions e^{2x} and e^{3x} are ……
A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

40) The Wronskian of the functions x^2 and $x^2\log x$ is ……
A) $\log x$ B) x^2 C) x^3 D) None of these

41) The functions x^2 and $x^2\log x$ are ……
A) Linearly independent B) Linearly dependent and Linearly independent
C) Linearly dependent D) None of these

42) The Wronskian of the functions $1, x, x^2$ is ……
A) 0 B) 1 C) 2 D) None of these

43) The functions $1, x, x^2$ are ……
A) Linearly dependent B) Linearly dependent and Linearly independent
C) Linearly independent D) None of these

44) The Wronskian of the functions $1, x^2, x^3$ is ……
A) 0 B) $6x^2$ C) $6x^3$ D) None of these

45) The functions $1, x^2, x^3$ are ……

A) Linearly independent B) Linearly dependent and Linearly independent
C) Linearly dependent D) None of these

46) The Wronskian of the functions x, x^2, x^3 is ……
A) 0 B) $2x^3$ C) 2x D) None of these

47) The functions x, x^2, x^3 are ……
A) Linearly dependent B) dependent
C) Linearly independent D) None of these

48) The functions x^2, e^x, e^{-x} are linearly …… if x = $\pm\sqrt{2}$
A) independent B) not dependent
C) dependent D) None of these

49) The functions x^2, e^x, e^{-x} are linearly …… if x $\neq \pm\sqrt{2}$
A) independent B) dependent and independent
C) dependent D) None of these

50) The Wronskian of the functions 1+x, x^2, 1+2x is ……
A) 0 B) -2 C) 2 D) None of these

51) The functions 1+x, x^2, 1+2x are linearly ……
A) independent B) dependent and independent
C) dependent D) None of these

52) The Wronskian of the functions x^2-x+1, x^2-1, $3x^2$ –x-1 is ……
A) 0 B) -2 C) 2 D) None of these

53) The functions x^2-x+1, x^2-1, $3x^2$ –x-1 are linearly ……
A) independent B) dependent and independent
C) dependent D) None of these

54) In a method of variation of parameters, A and B are so chosen such that C.F. y = Au+Bv becomes G.S. of given differential equation is ……
A) $A\frac{du}{dx} + B\frac{dv}{dx} = 0$ B) $u\frac{dA}{dx} + v\frac{dB}{dx} = 0$ C) $u\frac{dA}{dx} + v\frac{dB}{dx} \neq 0$ D) None of these

55) C.F. of $\frac{d^2y}{dx^2} + a^2y = \text{cosec}(ax)$ is y = ……
A) $c_1e^{ax} + c_2e^{-ax}$ B) Acosec(ax)+Bsec(ax)
C) Acos(ax)+Bsin(ax) D) None of these

56) C.F. of y'' + $a^2y = \sec(ax)$ is y = ……
A) $c_1e^{ax} + c_2e^{-ax}$ B) Acosec(ax)+Bsec(ax)
C) Acos(ax)+Bsin(ax) D) None of these

57) C.F. of $\frac{d^2y}{dx^2} + a^2y = \sin(ax)$ is y = ……
A) $c_1e^{ax} + c_2e^{-ax}$ B) Acosec(ax)+Bsec(ax)
C) Acos(ax)+Bsin(ax) D) None of these

58) C.F. of $y'' + 4y = 4\tan 2x$ is y = ……

A) $c_1e^{2x} + c_2e^{-2x}$ B) Acosec2x + Bsec2x

C) Acos2x +Bsin2x D) None of these

59) C.F. of $y'' + 9y = \sec 3x$ is y = ……

A) $c_1e^{3x} + c_2e^{-3x}$ B) Acosec3x + Bsec3x

C) Acos3x + Bsin3x D) None of these

60) C.F. of $y'' + y - x = 0$ is y = ……

A) Acosx + Bsinx B) Acosecx + Bsecx C) $c_1e^{x} + c_2e^{-x}$ D) None of these

61) C.F. of $y'' - y = e^{2x}$ is y = ……

A) Acosx + Bsinx B) $c_1 + c_2e^{x}$ C) $c_1e^{x} + c_2e^{-x}$ D) None of these

62) C.F. of $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ is y = ……

A) Acosx + Bsinx B) $c_1e^{x} + c_2e^{-x}$ C) $c_1e^{x} + c_2$ D) None of these

63) C.F. of $y'' - 2y' = e^{x}\sin x$ is y = ……

A) Acos2x + Bsin2x B) $c_1 + c_2e^{2x}$ C) $c_1e^{x} + c_2e^{-x}$ D) None of these

64) C.F. of $y'' - 3y' + 2y = 2$ is y = ……

A) $c_1e^{x} + c_2e^{2x}$ B) $c_1 + c_2e^{2x}$ C) Acos2x + Bsin2x D) None of these

65) C.F. of $y'' + k^2y = \cos kx$ is y = ……

A) Acoskx + Bsinkx B) $c_1 + c_2e^{kx}$ C) $c_1e^{kx} + c_2e^{-kx}$ D) None of these

UNIT-2: SIMULTANEOUS DIFFERENTIAL EQUATIONS

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Simultaneous Linear Differential Equation of First Order: The general form of a set of simultaneous linear differential equation of first order of three variables x, y, z is $P_1dx + Q_1dy + R_1dz = 0$ and $P_2dx + Q_2dy + R_2dz = 0$

where P_1, Q_1, R_1 and P_2, Q_2, R_2 are functions of x, y, z.

Simultaneous Differential Equation: If P, Q, R are the functions of x, y, z, then differential equation of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is called simultaneous differential equation of first order.

Methods of Solving Simultaneous Differential Equation:

Rule-I(A) Method of Combinations:

By taking any two pairs of the three ratios of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ in which third variable is absent or cancelled. Integrating and taking product of these solutions, we get G.S. of given equation.

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Ex.: Solve $\frac{dx}{zy} = \frac{dy}{zx} = \frac{dz}{xy}$

Solution: Let $\frac{dx}{zy} = \frac{dy}{zx} = \frac{dz}{xy}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{zy} = \frac{dy}{zx} \Rightarrow \frac{dx}{y} = \frac{dy}{x} \Rightarrow xdx = ydy \Rightarrow 2xdx - 2ydy = 0$

Integrating, we get,

$x^2 - y^2 = c_1$ i.e. $x^2 - y^2 - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{zy} = \frac{dz}{xy} \Rightarrow \frac{dx}{z} = \frac{dz}{x} \Rightarrow xdx = zdz \Rightarrow 2xdx - 2zdz = 0$

Integrating, we get,

$x^2 - z^2 = c_2$ i.e. $x^2 - z^2 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^2 - y^2 - c_1)(x^2 - z^2 - c_2) = 0.$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{z^2} = \frac{ydy}{xz^2} = \frac{dz}{xy}$

Solution: Let $\frac{dx}{z^2} = \frac{ydy}{xz^2} = \frac{dz}{xy}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{z^2} = \frac{ydy}{xz^2} \Longrightarrow xdx = ydy \Longrightarrow 2xdx - 2ydy = 0$

Integrating, we get,

$x^2 - y^2 = c_1$ i.e. $x^2 - y^2 - c_1 = 0$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{ydy}{xz^2} = \frac{dz}{xy} \Longrightarrow y^2dy = z^2dz \Longrightarrow 3y^2dy - 3z^2dz = 0$

Integrating, we get,

$y^3 - z^3 = c_2$ i.e. $y^3 - z^3 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^2 - y^2 - c_1)(y^3 - z^3 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{xdx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2}$

Solution: Let $\frac{xdx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2}$…… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{xdx}{y^2z} = \frac{dy}{zx} \Longrightarrow x^2dx = y^2dy \Longrightarrow 3x^2dx - 3y^2dy = 0$

Integrating, we get,

$x^3 - y^3 = c_1$ i.e. $x^3 - y^3 - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{xdx}{y^2z} = \frac{dz}{y^2} \Longrightarrow xdx = zdz \Longrightarrow 2xdx - 2zdz = 0$

Integrating, we get,

$x^2 - z^2 = c_2$ i.e. $x^2 - z^2 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^3 - y^3 - c_1)(x^2 - z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{0} = \frac{dy}{-z} = \frac{dz}{y}$ (Oct.2019)

Solution: Let $\frac{dx}{0} = \frac{dy}{-z} = \frac{dz}{y}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{0} = \frac{dy}{-z} \Longrightarrow dx = 0$

Integrating, we get,

$x = c_1$ i.e. $x - c_1 = 0$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{dy}{-z} = \frac{dz}{y} \Longrightarrow ydy = -zdz \Longrightarrow 2ydy + 2zdz = 0$

Integrating, we get,

$y^2 + z^2 = c_2$ i.e. $y^2 + z^2 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x - c_1)(y^2 + z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$

Solution: Let $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{z} = \frac{dy}{0} \Longrightarrow dy = 0$

Integrating, we get,

$y = c_1$ i.e. $y - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{z} = \frac{dz}{-x} \Longrightarrow xdx = -zdz \Longrightarrow 2xdx + 2zdz = 0$

Integrating, we get,

$x^2 + z^2 = c_2$ i.e. $x^2 + z^2 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(y - c_1)(x^2 + z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{-x} = \frac{dy}{0} = \frac{dz}{z}$ (Oct.2019)

Solution: Let $\frac{dx}{-x} = \frac{dy}{0} = \frac{dz}{z}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{-x} = \frac{dy}{0} \Longrightarrow dy = 0$

Integrating, we get,

$y = c_1$ i.e. $y - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{-x} = \frac{dz}{z} \Longrightarrow \frac{dx}{x} = -\frac{dz}{z} \Longrightarrow \frac{dx}{x} + \frac{dz}{z} = 0$

Integrating, we get,

$\log x + \log z = \log c_2$ i.e. $xz = c_2$ i.e. $xz - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(y - c_1)(xz - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{x^2 z} = \frac{dy}{0} = \frac{dz}{-x^2}$

Solution: Let $\frac{dx}{x^2 z} = \frac{dy}{0} = \frac{dz}{-x^2}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{x^2 z} = \frac{dy}{0} \Longrightarrow dy = 0$

Integrating, we get,

$y = c_1$ i.e. $y - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{x^2 z} = \frac{dz}{-x^2} \Longrightarrow dx = -zdz \Longrightarrow 2dx + 2zdz = 0$

Integrating, we get,

$2x + z^2 = c_2$ i.e. $2x + z^2 - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(y - c_1)(2x + z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$

Solution: Let $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{y} = \frac{dy}{-x} \Longrightarrow$ xdx = -ydy $\Longrightarrow$ 2xdx + 2ydy = 0

Integrating, we get,

$x^2 + y^2 = c_1$ i.e. $x^2 + y^2 - c_1 = 0$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{y} = \frac{dz}{0} \Longrightarrow dz = 0$

Integrating, we get,

$z = c_2$ i.e. $z - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^2 + y^2 - c_1)(z - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Solution: Let $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{x} = \frac{dy}{y}$

Integrating, we get,

$\log x = \log y + \log c_1$ i.e. $x = c_1 y$ i.e. $x - c_1 y = 0$…… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{x} = \frac{dz}{z}$

Integrating, we get,

$\log x = \log z + \log c_2$ i.e. $x = c_2 z$ i.e. $x - c_2 z = 0$ …… (iii)

∴ By (ii) and (iii),

$(x - c_1 y)(x - c_2 z) = 0$

be the required general solution of given equation.

==

Ex.: Solve $dx = dy = dz$

Solution: Let $dx = dy = dz$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$dx = dy$

Integrating, we get,

$x = y + c_1$ i.e. $x – y - c_1 = 0$…… (ii)

Now taking first and third ratios of (i), we have

$dx = dz$

Integrating, we get,

$x = z + c_2$ i.e. $x – z - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x – y - c_1)(x – z - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $dx = dy = cosecx\ dz$

Solution: Let $dx = dy = cosecx\ dz$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$dx = dy$

Integrating, we get,

$x = y + c_1$ i.e. $x – y - c_1 = 0$…… (ii)

Now taking first and third ratios of (i), we have

$dx = cosecx\ dz \Longrightarrow dz = \sin x\ dx \Longrightarrow dz - \sin x\ dx = 0$

Integrating, we get,

$z + \cos x = c_2$ i.e. $z + \cos x - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x – y - c_1)(z + \cos x - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $dx = dy = tanx\ dz$ (Oct.2019)

Solution: Let $dx = dy = tanx\ dz$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$dx = dy$

Integrating, we get,

x = y + c_1 i.e. x – y - c_1 = 0…… (ii)

Now taking first and third ratios of (i), we have

$dx = tanx\ dz$ ⟹ dz = cotx dx

Integrating, we get,

z = logsinx + c_2 i.e. z - logsinx - c_2 = 0 …… (iii)

∴ By (ii) and (iii),

(x – y - c_1)(z - logsinx - c_2) = 0

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$

Solution: Let $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$ …(i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{tanx} = \frac{dy}{tany}$ ⟹ cotxdx = cotydy

Integrating, we get,

logsinx = logsiny + logc_1

i.e. sinx = $c_1$siny i.e. sinx - $c_1$siny = 0…… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{tanx} = \frac{dz}{tanz}$ ⟹ cotxdx = cotzdz

Integrating, we get,

logsinx = logsinz + logc_2

i.e. sinx = $c_2$sinz i.e. sinx – $c_2$sinz = 0 …… (iii)

∴ By (ii) and (iii),

(sinx - $c_1$siny)(sinx – $c_2$sinz) = 0

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{cotx} = \frac{dy}{coty} = \frac{dz}{cotz}$

Solution: Let $\frac{dx}{cotx} = \frac{dy}{coty} = \frac{dz}{cotz}$ …(i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{cotx} = \frac{dy}{coty} \Longrightarrow$ tanxdx = tanydy

Integrating, we get,

logsecx = logsecy + $\log c_1$

i.e. secx = $c_1$secy i.e. secx - $c_1$secy = 0…… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{cotx} = \frac{dz}{cotz} \Longrightarrow$ tanxdx = tanzdz

Integrating, we get,

logsecx = logsecz + $\log c_2$

i.e. secx = $c_2$secz i.e. secx – $c_2$secz = 0 …… (iii)

∴ By (ii) and (iii),

(secx - $c_1$secy)(secx – $c_2$secz) = 0

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$

Solution: Let $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$ …… (i)

be the given simultaneous differential equation.

Taking first two ratios of (i), we have

$\frac{dx}{y^2} = \frac{dy}{x^2} \Longrightarrow x^2dx = y^2dy \Longrightarrow 3x^2dx - 3y^2dy = 0$

Integrating, we get,

$x^3 - y^3 = c_1$ i.e. $x^3 - y^3 - c_1 = 0$…… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{y^2} = \frac{dz}{x^2y^2z^2} \Longrightarrow x^2dx = z^{-2}dz \Longrightarrow 3x^2dx - 3z^{-2}dz = 0$

Integrating, we get,

$x^3 + 3z^{-1} = c_2$ i.e. $x^3 + \frac{3}{z} - c_2 = 0$ …… (iii)

∴ By (ii) and (iii),

$(x^3 - y^3 - c_1)(x^3 + \frac{3}{z} - c_2) = 0$

be the required general solution of given equation.

==

Rule-I(B) Method of Combinations:

By taking one pair of ratios of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ in which third variable is absent or cancelled, solving it we get one solution and using this solution we eliminate third variable from another pair of ratios and solve it which contain two constants c_1 and c_2 . In this solution put the value of first constant c_1, we get G.S. of given simultaneous differential equation.

==

Ex.: Solve $\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$

Solution: Let $\frac{dx}{x+z} = \frac{dy}{y} = \frac{dz}{z+y^2}$ …… (i)

be the given simultaneous differential equation.

Taking second and third ratios of (i) in which third variable x is absent, we have

$\frac{dy}{y} = \frac{dz}{z+y^2} \Rightarrow zdy + y^2dy = ydz \Rightarrow y^2dy = ydz - zdy$

$\Rightarrow dy = \frac{ydz-zdy}{y^2} \Rightarrow dy = d(\frac{z}{y}) \Rightarrow d(\frac{z}{y}) = dy$

Integrating, we get,

$\frac{z}{y} = y + c_1$ i.e. $z = y^2 + c_1y$ …… (ii)

Now taking first and second ratios of (i), we have

$\frac{dx}{x+z} = \frac{dy}{y} \Rightarrow \frac{dx}{x+y^2+c_1y} = \frac{dy}{y}$ by (ii)

$\Rightarrow ydx = xdy + y^2dy + c_1ydy$

$\Rightarrow ydx - xdy = y^2dy + c_1ydy \Rightarrow \frac{ydx-xdy}{y^2} = dy + \frac{c_1}{y}dy$

$\Rightarrow d(\frac{x}{y}) = dy + \frac{c_1}{y}dy$

Integrating, we get,

$\frac{x}{y} = y + c_1 logy + c_2$

i.e. $x = y^2 + c_1 ylogy + c_2y$

i.e. $x = y^2 + (z - y^2)logy + c_2y$ by (ii)

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$ (Oct.2019)

Solution: Let $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{xy} = \frac{dy}{y^2} \Longrightarrow \frac{dx}{x} = \frac{dy}{y}$

Integrating, we get,

$logx = logy + logc_1$ i.e. $x = c_1 y$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{dy}{y^2} = \frac{dz}{zxy-2x^2} \Longrightarrow \frac{dy}{y^2} = \frac{dz}{c_1 zy^2-2c_1{}^2y^2}$ by (ii)

$\Longrightarrow dy = \frac{dz}{c_1(z-2c_1)}$

Integrating, we get,

$y = \frac{1}{c_1}\log(z-2c_1) + c_2$

i.e. $y = \frac{y}{x}\log(z-2\frac{x}{y}) + c_2$ by (ii)

i.e. $xy = y\log(\frac{yz-2x}{y}) + c_2 x$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz-zx^2}$

Solution: Let $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz-zx^2}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{xy} = \frac{dy}{y^2} \Longrightarrow \frac{dx}{x} = \frac{dy}{y}$

Integrating, we get,

$logx = logy + logc_1$ i.e. $x = c_1 y$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{dy}{y^2} = \frac{dz}{xyz-zx^2} \Longrightarrow \frac{dy}{y^2} = \frac{dz}{c_1y^2z-zc_1{}^2y^2}$ by (ii)

$\Longrightarrow dy = \frac{dz}{(c_1-c_1{}^2)z}$

Integrating, we get,

$y = \frac{1}{(c_1-c_1{}^2)}\log z + c_2$

i.e. $y = \frac{1}{\left[\frac{x}{y}-\left(\frac{x}{y}\right)^2\right]}$ logz + c_2 by (ii)

i.e. $y = \frac{y^2}{(xy-x^2)}$ logz + c_2

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$

Solution: Let $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{1} = \frac{dy}{3} \Longrightarrow$ dy = 3dx

Integrating, we get,

y = 3x + c_1i.e. y – 3x = c_1…… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)} \Longrightarrow dx = \frac{dz}{5z+tanc_1}$ by (ii)

Integrating, we get,

$x = \frac{1}{5}\log(5z + tanc_1)$ + c_2

i.e. $5x = \log[5z + \tan(y - 3x)] + 5c_2$ by (ii)

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$

Solution: Let $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{y} = \frac{dy}{x} \Longrightarrow$ xdx = ydy $\Longrightarrow$ 2xdx - 2ydy = 0

Integrating, we get,

$x^2 - y^2 = c_1$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{y} = \frac{dz}{xyz^2(x^2-y^2)} \Longrightarrow xdx = \frac{dz}{c_1 z^2}$ by (ii)

Integrating, we get,

$\frac{x^2}{2} = -\frac{1}{c_1 z} + c_2$

i.e. $\frac{x^2}{2} = -\frac{1}{z(x^2 - y^2)} + c_2$ by (ii)

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$

Solution: Let $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{z} = \frac{dy}{-z} \Longrightarrow dx = -dy \Longrightarrow dx + dy = 0$

Integrating, we get,

$x + y = c_1$ …… (ii)

Now taking first and third ratios of (i), we have

$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2} \Longrightarrow 2dx = \frac{2zdz}{z^2+{c_1}^2}$ by (ii)

Integrating, we get,

$2x = \log(z^2 + {c_1}^2) + c_2$

i.e. $2x = \log[z^2 + (x+y)^2] + c_2$ by (ii)

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{x+y} = \frac{dy}{x+y} = \frac{dz}{-x-y-2z}$

Solution: Let $\frac{dx}{x+y} = \frac{dy}{x+y} = \frac{dz}{-x-y-2z}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i) in which third variable z is absent, we have

$\frac{dx}{x+y} = \frac{dy}{x+y} \Longrightarrow dx = dy$

Integrating, we get,

$x = y + c_1$ …… (ii)

Now taking second and third ratios of (i), we have

$\frac{dy}{x+y} = \frac{dz}{-x-y-2z} \Rightarrow \frac{dy}{y + c_1 + y} = \frac{dz}{-y - c_1 - y - 2z}$ by (ii)

$\Rightarrow \frac{dy}{2y + c_1} = \frac{dz}{-2y - c_1 - 2z}$

$\Rightarrow -2ydy - c_1 dy - 2zdy = 2ydz + c_1 dz$

$\Rightarrow 2ydy + c_1 dy + 2zdy + 2ydz + c_1 dz = 0$

$\Rightarrow dy^2 + c_1 d(y+z) + 2d(yz) = 0$

Integrating, we get,

$y^2 + c_1(y+z) + 2yz = c_2$

i.e. $y^2 + (x-y)(y+z) + 2yz = c_2$ by (ii)

be the required general solution of given equation.

==

Rule-II: Method of Multipliers:

Let $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ … (i) be the given simultaneous differential equation.

If possible there exists a multipliers l, m, n which are functions of x or constants such that $lP+mQ+nR = 0$, then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx+mdy+ndz}{lP+mQ+nR}$

Now $lP+mQ+nR = 0 \Rightarrow ldx + mdy + ndz = 0$

Integrating we get a solution say

$\emptyset(x, y, z) = c_1$ i.e. $\emptyset(x, y, z) - c_1 = 0$ … (ii)

Again choose multipliers l_1, m_1, n_1 which are functions of x or constants such that $l_1P + m_1Q + n_1R = 0$ then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l_1dx+m_1dy+n_1dz}{l_1P+m_1Q+n_1R}$

and $l_1P + m_1Q + n_1R = 0 \Rightarrow l_1dx + m_1dy + n_1dz = 0$

Integrating we get a solution say

$\psi(x, y, z) = c_2$ i.e. $\psi(x, y, z) - c_2 = 0$ … (iii)

By (ii) and (iii), the G.S. of (i) is

$[\emptyset(x, y, z) - c_1][\psi(x, y, z) - c] = 0$

==

Ex.: Solve $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$ (Oct.2019)

Solution: Let $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers 1, 1, 1, we get,

Each Ratio of (i) = $\frac{dx+dy+dz}{z-y+x-z+y-x} = \frac{dx+dy+dz}{0}$

$\Rightarrow dx + dy + dz = 0$

Integrating, we get,

$x + y + z = c_1$

i.e. $x + y + z - c_1 = 0$ …… (ii)

Again by taking multipliers x, y, z, we get,

Each Ratio of (i) = $\frac{xdx+ydy+zdz}{xz-xy+yx-yz+zy-zx} = \frac{xdx+ydy+zdz}{0}$

$\Rightarrow xdx + ydy + zdz = 0$

$\Rightarrow 2xdx + 2ydy + 2zdz = 0$

Integrating, we get,

$x^2 + y^2 + z^2 = c_2$

i.e. $x^2 + y^2 + z^2 - c_2 = 0$ …… (iii)

By (ii) and (iii),

$(x + y + z - c_1)(x^2 + y^2 + z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

Solution: Let $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers 1, 1, 1, we get,

Each Ratio of (i) = $\frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{xy-xz+yz-yx+zx-zy} = \frac{dx+dy+dz}{0}$

$\Rightarrow dx + dy + dz = 0$

Integrating, we get,

$x + y + z = c_1$

i.e. $x + y + z - c_1 = 0$ …… (ii)

Again by taking multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get,

Each Ratio of (i) = $\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get,

$\log x + \log y + \log z = \log c_2$

i.e. $xyz = c_2$

i.e. $xyz - c_2 = 0$ …… (iii)

By (ii) and (iii),

$(x + y + z - c_1)(xyz - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$

Solution: Let $\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers x, y, z, we get,

Each Ratio of (i) = $\frac{xdx+ydy+zdz}{x^2y^2-x^2z^2-y^2z^2-y^2x^2+z^2x^2+z^2y^2} = \frac{xdx+ydy+zdz}{0}$

$\Rightarrow xdx + ydy + zdz = 0$

$\Rightarrow 2xdx + 2ydy + 2zdz = 0$

Integrating, we get,

$x^2 + y^2 + z^2 = c_1$

i.e. $x^2 + y^2 + z^2 - c_1 = 0$ …… (ii)

Again by taking multipliers $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$, we get,

Each Ratio of (i) = $\frac{\frac{1}{x}dx-\frac{1}{y}dy-\frac{1}{z}dz}{y^2-z^2+z^2+x^2-x^2-y^2} = \frac{\frac{1}{x}dx-\frac{1}{y}dy-\frac{1}{z}dz}{0}$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0 \Rightarrow \frac{1}{x}dx = \frac{1}{y}dy + \frac{1}{z}dz$

Integrating, we get,

$\log x = \log y + \log z + \log c_2$

i.e. $x = c_2yz$

i.e. $x - c_2yz = 0$ …… (iii)

By (ii) and (iii),

$(x^2 + y^2 + z^2 - c_1)(x - c_2yz) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$

Solution: Let $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers x, y, -1, we get,

Each Ratio of (i) = $\frac{xdx+ydy-dz}{x^2y^2+x^2z-y^2x^2-y^2z-zx^2+zy^2} = \frac{xdx+ydy-dz}{0}$

$\Rightarrow xdx + ydy - dz = 0$

$\Rightarrow 2xdx + 2ydy - 2dz = 0$

Integrating, we get,

$x^2 + y^2 - 2z = c_1$

i.e. $x^2 + y^2 - 2z - c_1 = 0$ …… (ii)

Again by taking multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get,

Each Ratio of (i) = $\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get,

$\log x + \log y + \log z = \log c_2$

i.e. $xyz = c_2$

i.e. $xyz - c_2 = 0$ …… (iii)

By (ii) and (iii),

$(x^2 + y^2 - 2z - c_1)(xyz - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{yzdx}{y-z} = \frac{zxdy}{z-x} = \frac{xydz}{x-y}$

Solution: Let $\frac{yzdx}{y-z} = \frac{zxdy}{z-x} = \frac{xydz}{x-y}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers 1, 1, 1, we get,

Each Ratio of (i) = $\frac{yzdx+zxdy+xydz}{y-z+z-x+x-y} = \frac{d(xyz)}{0}$

$\Rightarrow d(xyz) = 0$

Integrating, we get,

xyz = c_1

i.e. xyz - c_1 = 0 …… (ii)

Again by taking multipliers x, y, z, we get,

Each Ratio of (i) = $\frac{xyzdx+xyzdy+xyzdz}{xy-xz+yz-yx+zx-zy} = \frac{xyzd(x+y+z)}{0}$

$\Rightarrow xyzd(x+y+z) = 0$

$\Rightarrow d(x+y+z) = 0$

Integrating, we get,

x + y + z = c_2

i.e. x + y + z - c_2 = 0 …(iii)

By (ii) and (iii),

(xyz - c_1)(x + y + z - c_2) = 0

be the required general solution of given equation.

==

Ex.: Solve $\frac{adx}{bc(y-z)} = \frac{bdy}{ca(z-x)} = \frac{cdz}{ab(x-y)}$

Solution: Let $\frac{adx}{bc(y-z)} = \frac{bdy}{ca(z-x)} = \frac{cdz}{ab(x-y)}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers, a, b, c, we get,

Each Ratio of (i) = $\frac{a^2dx+b^2dy+c^2dz}{abc(y-z+z-x+x-y)} = \frac{a^2dx+b^2dy+c^2dz}{0}$

$\Rightarrow a^2dx + b^2dy + c^2dz = 0$

Integrating, we get,

$a^2x + b^2y + c^2z = c_1$

i.e. $a^2x + b^2y + c^2z - c_1 = 0$ …… (ii)

Again by taking multipliers ax, by, cz, we get,

Each Ratio of (i) = $\frac{a^2xdx+b^2ydy+c^2zdz}{abc(xy-xz+yz-yx+zx-zy)} = \frac{a^2xdx+b^2ydy+c^2zdz}{0}$

$\Rightarrow a^2xdx + b^2ydy + c^2zdz = 0$

$\Rightarrow a^2 2xdx + b^2 2ydy + c^2 2zdz = 0$

Integrating, we get,

$a^2x^2 + b^2y^2 + c^2z^2 = c_2$

i.e. $a^2x^2 + b^2y^2 + c^2z^2 - c_2 = 0$ …(iii)

By (ii) and (iii),

$(a^2x + b^2y + c^2z - c_1)(a^2x^2 + b^2y^2 + c^2z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{adx}{yz(b-c)} = \frac{bdy}{zx(c-a)} = \frac{cdz}{xy(a-b)}$

Solution: Let $\frac{adx}{yz(b-c)} = \frac{bdy}{zx(c-a)} = \frac{cdz}{xy(a-b)}$ …… (i)

be the given simultaneous differential equation.

Taking multipliers x, y, z, we get,

Each Ratio of (i) = = $\frac{axdx+bydy+czdz}{xyz(b-c+c-a+a-b)} = \frac{axdx+bydy+czdz}{0}$

$\Rightarrow axdx + bydy + czdz = 0$

$\Rightarrow 2axdx + 2bydy + 2czdz = 0$

Integrating, we get,

$ax^2 + by^2 + cz^2 = c_1$

i.e. $ax^2 + by^2 + cz^2 - c_1 = 0$ …… (ii)

Again by taking multipliers ax, by, cz, we get,

Each Ratio of (i) = = $\frac{a^2xdx+b^2ydy+c^2zdz}{xyz(ab-ac+bc-ba+ca-cb)} = \frac{a^2xdx+b^2ydy+c^2zdz}{0}$

$\Rightarrow a^2xdx + b^2ydy + c^2zdz = 0$

$\Rightarrow a^2 2xdx + b^2 2ydy + c^2 2zdz = 0$

Integrating, we get,

$a^2x^2 + b^2y^2 + c^2z^2 = c_2$

i.e. $a^2x^2 + b^2y^2 + c^2z^2 - c_2 = 0$ …(iii)

By (ii) and (iii),

$(ax^2 + by^2 + cz^2 - c_1)(a^2x^2 + b^2y^2 + c^2z^2 - c_2) = 0$

be the required general solution of given equation.

==

Rule-III: Properties of Ratios:

Let $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ … (i) be the given simultaneous differential equation.

If there does not exists a multipliers l, m, n such that lP+mQ+nR = 0,

then choose the multipliers P_1, Q_1, R_1 and P_2, Q_2, R_2

such that $d(P_1P + Q_1Q + R_1R) = P_1dx + Q_1dy + R_1dz$

and $d(P_2P + Q_2Q + R_2R) = P_2dx + Q_2dy + R_2dz$, then we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{P_1\mathrm{dx}+Q_1\mathrm{dy}+R_1\mathrm{dz}}{P_1\mathrm{P}+Q_1\mathrm{Q}+R_1\mathrm{R}} = \frac{P_2\mathrm{dx}+Q_2\mathrm{dy}+R_2\mathrm{dz}}{P_2\mathrm{P}+Q_2\mathrm{Q}+R_2\mathrm{R}}$$

Taking any two pairs of suitable ratios, we get solutions say

$\emptyset(x, y, z) - c_1 = 0$ … (ii) and $\psi(x, y, z) - c_2 = 0$ … (iii)

By (ii) and (iii), the G.S. of (i) is

$[\emptyset(x, y, z) - c_1][\psi(x, y, z) - c_2] = 0$

==

Ex.: Solve $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

Solution: Let $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ …… (i)

be the given simultaneous differential equation.

Taking second and third ratios of (i), we have

$$\frac{dy}{2xy} = \frac{dz}{2xz} \Longrightarrow \frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get,

$\log y = \log z + \log c_1$

i.e. $y = c_1 z$

i.e. $y - c_1 z = 0$…… (ii)

Now by taking multipliers x, y, z, we get,

Each Ratio of (i) = $\frac{\mathrm{xdx+ydy+zdz}}{x^3-xy^2-xz^2+2xy^2+2xz^2} = \frac{\mathrm{xdx+ydy+zdz}}{x^3+xy^2+xz^2} = \frac{\mathrm{xdx+ydy+zdz}}{x(x^2+y^2+z^2)}$

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{\mathrm{xdx+ydy+zdz}}{x(x^2+y^2+z^2)} \text{ …… (iii)}$$

Taking second and fourth ratios of (iii), we have,

$$\frac{dy}{2xy} = \frac{\mathrm{xdx+ydy+zdz}}{x(x^2+y^2+z^2)}$$

$$\Longrightarrow \frac{dy}{y} = \frac{\mathrm{2xdx+2ydy+2zdz}}{(x^2+y^2+z^2)}$$

$$\Longrightarrow \frac{dy}{y} = \frac{\mathrm{d}(x^2+y^2+z^2)}{(x^2+y^2+z^2)}$$

Integrating, we get,

$\log y = \log(x^2 + y^2 + z^2) + \log c_2$

i.e. $y = c_2 (x^2 + y^2 + z^2)$

i.e. $y - c_2 (x^2 + y^2 + z^2) = 0$…… (iv)

By (ii) and (iv),

$(y - c_1 z)[y - c_2 (x^2 + y^2 + z^2)] = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$

Solution: Let $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ …… (i)

be the given simultaneous differential equation.

Taking first and second ratios of (i), we have

$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} \Longrightarrow \frac{dx}{(x+y)} = \frac{dy}{(x-y)}$

$\Longrightarrow xdx - ydx = xdy + ydy$

$\Longrightarrow xdx - ydx - xdy - ydy = 0$

$\Longrightarrow 2xdx - 2ydx - 2xdy - 2ydy = 0$

$\Longrightarrow d(x^2 - 2xy - y^2) = 0$

Integrating, we get,

$x^2 - 2xy - y^2 = c_1$

i.e. $x^2 - 2xy - y^2 - c_1 = 0$…… (ii)

Now by taking multipliers x, -y, -z, we get,

Each Ratio of (i) = $\frac{xdx-ydy-zdz}{x^2z+xyz-xyz+y^2z-zx^2-zy^2} = \frac{xdx-ydy-zdz}{0}$

$xdx - ydy - zdz = 0$

$\Longrightarrow 2xdx - 2ydy - 2zdz = 0$

$\Longrightarrow d(x^2 - y^2 - z^2) = 0$

Integrating, we get,

$x^2 - y^2 - z^2 = c_2$

i.e. $x^2 - y^2 - z^2 - c_2 = 0$…… (iii)

By (ii) and (iii),

$(x^2 - 2xy - y^2 - c_1)(x^2 - y^2 - z^2 - c_2) = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$

Solution: Let $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ …… (i)

be the given simultaneous differential equation.

By taking multipliers 1, 1, 1, we get,

Each Ratio of (i) = $\frac{dx+dy+dz}{y+z+z+x+x+y}$

i.e. Each Ratio of (i) = $\frac{dx+dy+dz}{2x+2y+2z}$

i.e. Each Ratio of (i) = $\frac{d(x+y+z)}{2(x+y+z)}$

Again by taking multipliers 1, -1, 0 and 0, 1, -1 we get,

Each Ratio of (i) = $\frac{dx-dy+0}{y+z-z-x+0} = \frac{0+dy-dz}{0+z+x-x-y}$

i.e. Each Ratio of (i) = $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

i.e. Each Ratio of (i) = $\frac{d(x+y+z)}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dx-dz}{z-x}$

Consider $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

$\Longrightarrow \frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$

Integrating, we get,

$\log(x-y) = \log(y-z) + \log c_2$

i.e. $(x-y) = c_1(y-z)$

i.e. $(x-y) - c_1(y-z) = 0$ …… (ii)

Again consider $\frac{d(x+y+z)}{2(x+y+z)} = \frac{dx-dy}{y-x}$

$\Longrightarrow \frac{d(x+y+z)}{(x+y+z)} = -2\frac{d(x-y)}{(x-y)}$

$\Longrightarrow \frac{d(x+y+z)}{(x+y+z)} + 2\frac{d(x-y)}{(x-y)} = 0$

Integrating, we get,

$\log(x+y+z) + 2\log(x-y) = \log c_1$

i.e. $(x+y+z)(x-y)^2 = c_2$

i.e. $(x+y+z)(x-y)^2 - c_2 = 0$…… (iii)

By (ii) and (iii),

$[(x-y) - c_1(y-z)][(x+y+z)(x-y)^2 - c_2] = 0$

be the required general solution of given equation.

==

Ex.: Solve $\frac{dx}{y^2+yz+z^2} = \frac{dy}{z^2+zx+x^2} = \frac{dz}{x^2+xy+y^2}$

Solution: Let $\frac{dx}{y^2+yz+z^2} = \frac{dy}{z^2+zx+x^2} = \frac{dz}{x^2+xy+y^2}$ …… (i)

be the given simultaneous differential equation.

By taking multipliers as -1, 1, 0; 0, -1, 1 and -1, 0, 1, we have,

Each Ratio of (i) = $\frac{dy-dx}{z^2+zx+x^2-y^2-yz-z^2} = \frac{dz-dy}{x^2+xy+y^2-z^2-zx-x^2} = \frac{dz-dx}{x^2+xy+y^2-y^2-yz-z^2}$

i.e. Each Ratio of (i) = $\frac{dy-dx}{zx+x^2-y^2-yz} = \frac{dz-dy}{xy+y^2-z^2-zx} = \frac{dz-dx}{x^2+xy-yz-z^2}$

i.e. Each Ratio of (i) = $\frac{dy-dx}{(x-y)(x+y)+z(x-y)} = \frac{dz-dy}{x(y-z)+(y-z)(y+z)} = \frac{dz-dx}{(x-z)(x+z)+y(x-z)}$

i.e. Each Ratio of (i) = $\frac{dy-dx}{(x-y)(x+y+z)} = \frac{dz-dy}{(y-z)(x+y+z)} = \frac{dz-dx}{(x-z)(x+y+z)}$

Consider $\frac{dy-dx}{(x-y)(x+y+z)} = \frac{dz-dy}{(y-z)(x+y+z)}$

$\Rightarrow \frac{d(y-x)}{(y-x)} = \frac{d(z-y)}{(z-y)}$

Integrating, we get,

$\log(y-x) = \log(z-y) + \log c_1$

i.e. $(y-x) = c_1(z-y)$

i.e. $(y-x) - c_1(z-y) = 0$…… (ii)

Again consider $\frac{dy-dx}{(x-y)(x+y+z)} = \frac{dz-dx}{(x-z)(x+y+z)}$

$\Rightarrow \frac{d(y-x)}{(y-x)} = \frac{d(z-x)}{(z-x)}$

Integrating, we get,

$\log(y-x) = \log(z-x) + \log c_2$

i.e. $(y-x) = c_2(z-x)$

i.e. $(y-x) - c_2(z-x) = 0$…… (iii)

By (ii) and (iii),

$[(y-x) - c_1(z-y)][(y-x) - c_2(z-x)] = 0$

be the required general solution of given equation.

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MULTIPLE CHOICE QUESTIONS [MCQ'S]

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1) If P, Q, R are the functions of x, y, z, then differential equation of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is called …… differential equation of first order.

[A]linear [B] simultaneous [C] homogeneous [D] None of these

2) Solution of $\frac{dx}{0} = \frac{dy}{-z} = \frac{dz}{y}$ is……

[A] $(x-c_1)(y^2+z^2-c_2) = 0$ [B] $(y-c_1)(x^2+z^2-c_2) = 0$

[C] $(z-c_1)(x^2+y^2-c_2) = 0$ [D] $(x-c_1)(y^2-z^2-c_2) = 0$

3) Solution of $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$ is……

[A] $(x-c_1)(y^2+z^2-c_2) = 0$ [B] $(y-c_1)(x^2+z^2-c_2) = 0$

[C] $(z-c_1)(x^2+y^2-c_2) = 0$ [D] $(x-c_1)(y^2-z^2-c_2) = 0$

4) Solution of $\frac{dx}{-x} = \frac{dy}{0} = \frac{dz}{z}$ is……

[A] $(x-c_1)(y+z-c_2) = 0$ [B] $(y-c_1)(x+z-c_2) = 0$

[C] $(y-c_1)(xz-c_2) = 0$ [D] $(x-c_1)(yz-c_2) = 0$

5) Solution of $\frac{dx}{x^2z} = \frac{dy}{0} = \frac{dz}{-x^2}$ is……

[A] $(x-c_1)(2y+z^2-c_2) = 0$ [B] $(y-c_1)(2x-z^2-c_2) = 0$

[C] $(z-c_1)(x^2+y^2-c_2) = 0$ [D] $(y-c_1)(2x+z^2-c_2) = 0$

6) Solution of $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{0}$ is……

[A] $(x^2+y^2-c_1)(z-c_2) = 0$ [B] $(x^2-y^2-c_1)(z-c_2) = 0$

[C] $(x+y-c_1)(z-c_2) = 0$ [D] $(x^3+y^3-c_1)(z-c_2) = 0$

7) Taking first and second ratios of simultaneous D.E. $dx = dy = dz$, the solution is……

[A] $(x - y)(y + z) = c$ [B] $y - z = c$

[C] $x - y = c$ [D] $(x + 2y)(y + z) = c$

8) Taking second and third ratios of simultaneous D.E. $dx = dy = dz$, the solution is……

[A] $(x - y)(y + z) = c$ [B] $y - z = c$

[C] $x - y = c$ [D] $(x + 2y)(y + z) = c$

9) Taking first and third ratios of simultaneous D.E. $dx = dy = dz$, the solution is……

[A] $x - z = c$ [B] $y - z = c$

[C] $x - y = c$ [D] $x + z = c$

10) Solution of $dx = dy = dz$ is……

[A] $(x - y - c_1)(x - z - c_2) = 0$ [B] $(x + y - c_1)(x - z - c_2) = 0$

[C] $(x - y - c_1)(x + z - c_2) = 0$ [D] $(x + y - c_1)(x + z - c_2) = 0$

11) Taking first and second ratios of simultaneous D.E. $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$, the solution is……

[A] $x = cy$ [B] $x+y = c$ [C] $x - y = c$ [D] $xy = c$

12) Taking second and third ratios of simultaneous D.E. $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$, the solution is……

[A] $yz = c$ [B] $y + z = c$ [C] $y - z = c$ [D] $y = cz$

13) Taking first and third ratios of simultaneous D.E. $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$, the solution is……

[A] $xz = c$ [B] $x = cz$ [C] $x - z = c$ [D] $x + z = c$

14) Solution of $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ is……

[A] $(x - c_1y)(x - c_2z) = 0$ [B] $(x + c_1y)(x - c_2z) = 0$

[C] $(x - c_1y)(x + c_2z) = 0$ [D] $(x + c_1y)(x + c_2z) = 0$

15) Taking first and second ratios of simultaneous D.E. $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$,

the solution is……

[A] $x^2 = y^2$ [B] $x^2 - y^2 = c$ [C] $x^2 - 3y^2 = c$ [D] $4x^2 = 5y^2$

16) Taking first and third ratios of simultaneous D.E. $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$,

the solution is……

[A] $x^2 - z^2 = c$ [B] $x^2 + z^2 = 0$ [C] $x^2 - 3z^2 = 0$ [D] $x^2 = 5z^2 + 2$

17) Taking second and third ratios of simultaneous D.E. $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$,

the solution is……

[A] $y^2 - z^2 - c = 0$ [B] $x^2 - z^2 = 0$ [C] $x^2 - 3z^2 = c$ [D] None of these

18) Solution of $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ is……

[A] $(x^2 + y^2 - c_1)(x^2 - z^2 - c_2) = 0$ [B] $(x^2 - y^2 - c_1)(x^2 + z^2 - c_2) = 0$

[C] $(x^2 - y^2 - c_1)(x^2 - z^2 - c_2) = 0$ [D] $(x^2 + y^2 - c_1)(x^2 + z^2 - c_2) = 0$

19) Taking first and second ratios of simultaneous differential equation $\frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$,

the solution is ……

[A] $x^3 + 2y^3 = c$ [B] $x^3 - y^3 = c$ [C] $x^3 + 4y^3 = c$ [D] $2x^3 + 3y^3 = c$

20) Taking first and third ratios of simultaneous D.E. $\frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$,

the solution is ……

[A] $x^2 + z^2 = c$ [B] $x^2 - z^2 = c$ [C] $x^2 + 3y^2 = c$ [D] $4x^2 + 5y^2 = c$

21) Solution of $\frac{xdx}{y^2z} = \frac{dy}{xz} = \frac{dz}{y^2}$ is ……

[A] $(x^3 - y^3 - c_1)(x^2 - z^2 - c_2) = 0$ [B] $(x^3 + y^3 - c_1)(x^2 - z^2 - c_2) = 0$

[C] $(x^3 - y^3 - c_1)(x^2 + z^2 - c_2) = 0$ [D] $(x^3 + y^3 - c_1)(x^2 + z^2 - c_2) = 0$

22) Taking first and second ratio of simultaneous differential equation

$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{5z+\tan(y-2x)}$, the solution is ……

[A] $xy = c$ [B] $x^2 + z^2 = c$ [C] $x = 2y + c$ [D] $y = 2x + c$

23) Taking first and second ratio of simultaneous differential equation

$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$, the solution is ……

[A] $y-3x-c_1 = 0$ [B] $y+3x-c_1 = 0$ [C] $x-3y-c_1 = 0$ [D] $x+3y-c_1 = 0$

24) Taking first and second ratios of simultaneous D.E. $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$,

the solution is……

[A] sinx – csiny = 0 [B] sinx – csinz = 0

[C] sinx + csiny = 0 [D] siny – csinz = 0

25) Taking first and third ratios of simultaneous D.E. $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$,

the solution is……

[A] sinx – csiny = 0 [B] sinx – csinz = 0

[C] sinx + csiny = 0 [D] siny – csinz = 0

26) Taking second and third ratios of simultaneous D.E. $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$,

the solution is……

[A] sinx – csiny = 0 [B] sinx – csinz = 0

[C] sinx + csiny = 0 [D] siny – csinz = 0

27) Solution of $\frac{dx}{tanx} = \frac{dy}{tany} = \frac{dz}{tanz}$ is……

[A] (sinx + csiny)(sinx – csinz) = 0 [B] (sinx – csiny)(sinx + csinz) = 0

[C] (sinx – csiny)(sinx – csinz) = 0 [D] (sinx + csiny)(sinx + csinz) = 0

28) Taking first and second ratios of simultaneous D.E. $\frac{dx}{cotx} = \frac{dy}{coty} = \frac{dz}{cotz}$,

the solution is……

[A] secx – csecy = 0 [B] secx – csecz = 0

[C] secy – csecz = 0 [D] secx + csecy = 0

29) Taking first and third ratios of simultaneous D.E. $\frac{dx}{cotx} = \frac{dy}{coty} = \frac{dz}{cotz}$,

the solution is……

[A] secx – csecy = 0 [B] secx – csecz = 0

[C] secy – csecz = 0 [D] secx + csecy = 0

30) Taking second and third ratios of simultaneous D.E. $\frac{dx}{cotx} = \frac{dy}{coty} = \frac{dz}{cotz}$,

the solution is……

[A] secx – csecy = 0 [B] secx – csecz = 0

[C] secy – csecz = 0 [D] secx + csecy = 0

31) Solution of $\frac{dx}{cotx} = \frac{dy}{coty} = \frac{dz}{cotz}$ is……

[A] (secx – csecy)(secx – csecz) = 0 [B] (secx + csecy)(secx – csecz) = 0

[C] (secx – csecy)(secx + csecz) = 0 [D] (secx + csecy)(secx + csecz) = 0

32) Solution of $dx = dy = cosecx dz$ is……

[A] $(x + y - c_1)(z + \cos x - c_2) = 0$ [B] $(x - y - c_1)(z + \cos x - c_2) = 0$
[C] $(x - y - c_1)(z - \cos x - c_2) = 0$ [D] $(x + y - c_1)(z - \cos x - c_2) = 0$

33) Taking first and second ratios of simultaneous D.E. $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$,

the solution is……

[A] $x^2 - y^2 = c$ [B] $x^2+y^2 = c$ [C] $x = cy$ [D] $x - y = c$

34) Taking first and second ratios of simultaneous D.E. $\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$,

the solution is……

[A] $x^2 - y^2 = c$ [B] $x^2+y^2 = c$ [C] $x = cy$ [D] $x + y = c$

35) Taking first and second ratios of simultaneous D.E. $\frac{dx}{x+y} = \frac{dy}{x+y} = \frac{dz}{-x-y-2z}$,

the solution is……

[A] $x - y = c$ [B] $x+y = c$ [C] $x = cy$ [D] $xy = c$

36) Taking first and second ratios of simultaneous D.E. $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$,

the solution is……

[A] $xy = c$ [B] $x^2+z^2 = c$ [C] $x - y = c$ [D] $x/y = c$

37) Taking first and second ratios of simultaneous D.E. $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$,

the solution is……

[A] $x^2 + y^2 = c$ [B] $x^3 - y^3 = c$ [C] $x^3 + y^3 = c$ [D] $x^2 - y^2 = c$

38) Taking second and third ratios of simultaneous D.E. $\frac{dx}{x^2+2y^2} = \frac{dy}{-xy} = \frac{dz}{xz}$,

the solution is……

[A] $yz = c$ [B] $xz = c$ [C] $xy = c$ [D] $y - z = c$

39) Taking second and third ratios of simultaneous D.E. $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$,

the solution is……

[A] $yz = c$ [B] $xz = c$ [C] $y = cz$ [D] $y - z = c$

40) Taking first and second ratios of simultaneous D.E. $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-a\sqrt{x^2+y^2+z^2}}$,

the solution is……

[A] $x+y = c$ [B] $xy = c$ [C] $x = cy$ [D] $x - y = c$

41) The solution of simultaneous differential equation $\frac{dx}{a} = \frac{dy}{a} = dz$ is……

[A] $(x - ay - c_1)(y + z - c_2) = 0$ [B] $(x + ay - c_1)(y - z - c_2) = 0$
[C] $(x - y - c_1)(y - az - c_2) = 0$ [D] $(x + ay - c_1)(y + az - c_2) = 0$

42) The solution set of $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ is……

[A] $xy = C_1$, $yz = C_2$ [B] $x = C_1y$, $y = C_2z$
[C] $x = C_1+z$, $y = C_2z$ [D] $y = C_1z$, $y = C_2+x$

43) The solution set of $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z}$ is……

[A] $x^2 - y^2 = C_1$ and $x + y = C_2z$ [B] $x^2 + y^2 = C_1$ and $x - y = C_2z$
[C] $x^2 + z^2 = C_1$, $x + y + z = C_2z$ [D] None of these

44) The solution set of $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$ is……

[A] $x^2 - y^2 = C_1$ and $x^2 - z^2 = C_2$ [B] $x^2 + y^2 = C_1$ and $x^2 - z^2 = C_2$
[C] $x + y + z = C_1$ and $x^2 + z^2 = C_2$ [D] None of these

45) The solution set of $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$ is……

[A] $x^2 + y^2 = C_1$ and $y = C_2$ [B] $y = C_1$ and $x^2 + z^2 = C_2$
[C] $x^2 + z^2 = C_1$ and $z = C_2$ [D] None of these

46) The solution set of $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$ is……

[A] $y-x = C_1(z-y)$ and $(x-y)^2(x+y+z)= C_2$
[B] $x+y = C_1(y+z)$ and $(x-y)^2(x+y+z)= C_2$
[C] $y-z = C_1(x-y)$ and $(x+y)^2(x+y-z)= C_2$
[D] $x-y = C_1(y-z)$ and $(x+y+z)= C_2(x-y)^2$

47) If $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{A}{lP+mQ+nR}$, then A = ……

[A] $ldx + mdy + ndz$ [B] $mdx + ldy + ndz$
[C] $ldx - mdy + ndz$ [D] $ldx + mdy - ndz$

48) If $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{xdx+ydy+zdz}{A}$, then A = ……

[A] $xP + yQ + zR$ [B] $xP - yQ + zR$
[C] $xP + yQ - zR$ [D] $yP - xQ + zR$

49) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$ are …

[A] 1, 1, 0 and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ [B] 1, 1, 1 and x, y, z
[C] 1, 0, 1 and x, y, -z [D] 1, 1, 0 and x, -y, -z

50) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$ are …

[A] 1, 1, 1 and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ [B] 0, 1, 1 and x, y, z
[C] 1, 0, 1 and x, y, -z [D] 1, 1, 0 and x, -y, -z

51) Set of multipliers used to solve simultaneous D.E. $\frac{yzdx}{y-z} = \frac{zxdy}{z-x} = \frac{xydz}{x-y}$ are …

[A] 1, 0, 1 and $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ [B] 1, 1, 1 and x, y, z
[C] 1, 0, 1 and x, y, -z [D] 1, 1, 0 and x, -y, -z

52) Set of multipliers used to solve simultaneous D.E. $\frac{adx}{bc(y-z)} = \frac{bdy}{ca(z-x)} = \frac{cdz}{ab(x-y)}$ are …

[A] 1, 1, 1 and $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ [B] a, -b, -c and x, y, z

[C] a, b, c and ax, by, cz [D] a, b, -c and x, -y, -z

53) Set of multipliers used to solve simultaneous D.E. $\frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)zx} = \frac{cdz}{(a-b)xy}$ are …

[A] 1, 1, 1 and $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ [B] x, y, z and ax, by, cz

[C] a, b, c and ax, -by, cz [D] a, b, -c and x, -y, -z

54) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)}$ are …

[A] 1, 1, 1 and x, -y, z [B] 1, 0, 1 and x, y, -z

[C] -1, 0, 1 and x, -y, -z [D] x, y, z and $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$

55) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ are …

[A] x, y, -z and x, -y, -z [B] -x, y, z and x, -y, -z

[C] y, x, -z and x, -y, -z [D] y, x, z and x, y, -z

56) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$ are …

[A] $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ and x, y, -1 [B] x, y, z and 1, y, z

[C] 1, 1, 1 and x, y, z [D] None of these

57) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{x(2y^4-z^4)} = \frac{dy}{y(z^4-2x^4)} = \frac{dz}{z(x^4-y^4)}$ are …

[A] $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$ and x^3, y^3, z^3 [B] x^2, y^2, z^2 and 1, y, z

[C] 1, 1, 1 and x^4, y^4, z^4 [D] None of these

58) Set of multipliers used to solve simultaneous D.E. $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$ are …

[A] x, y, -z and 1, 0, 0 [B] -x, y, z and l, -m, -n

[C] y, x, -z and 1, 1, 1 [D] x, y, z and l, m, n

59) If we use multipliers 1, 1, 0 to solve simultaneous D.E. $\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$, then each ratio = ……

[A] $\frac{dx+dz}{2+x+z}$ [B] $\frac{dx+dy}{1+x+y}$ [C] $\frac{dx+dy}{2+y}$ [D] $\frac{dx+dy}{2+x+y}$

60) If we use multipliers a, b, 1 to solve simultaneous D.E. $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{bx-ay}$, then the solution is ……

[A] $ax+by = c_1$ [B] $x+y+z = c_1$ [C] $ax-y+z = c_1$ [D] $ax+by+z = c_1$

UNIT-3: TOTAL DIFFERENTIAL OR PFAFFIAN DIFFERENTIAL EQUATIONS

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Pfaffian Differential Equation of First Order: The differential equation of the form $u_1dx_1 + u_2dx_2 + \ldots\ldots + u_ndx_n = 0$ is called Pfaffian differential equation or total differential equation in n independent variables $x_1, x_2, \ldots\ldots, x_n$.

Pfaffian Differential Equation: If P, Q, R are the functions of x, y, z, then differential equation of the form Pdx + Qdy + Rdz = 0 is called Pfaffian differential equation or total differential equation.

Exact Differential Equation: A Pfaffian differential equation Pdx + Qdy + Rdz = 0 is said to be exact if there exists a function u(x, y, z) such that Pdx + Qdy + Rdz = du.

Integrable Differential Equation: A Pfaffian differential equation Pdx + Qdy + Rdz = 0 is said to be integrable if it is either exact or can be made exact.

Note: Every exact differential equation is integrable. But every integrable differential equation may not be exact.

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The Necessary Condition: If the Pfaffian differential equation Pdx + Qdy + Rdz = 0 is integrable, then $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$

Proof: Let the differential equation Pdx + Qdy + Rdz = 0 …… (i)

where P, Q, R are the functions of x, y, z be integrable say its integral is

u(x, y, z) = c …… (ii)

∴ equation (i) is either exact or can be made exact.

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \qquad \ldots\ldots \text{(iii)}$$

As (ii) is an integral of (i), we have,

$$\frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \lambda \Longrightarrow \lambda P = \frac{\partial u}{\partial x}, \lambda Q = \frac{\partial u}{\partial y}, \lambda R = \frac{\partial u}{\partial z} \qquad \ldots\ldots \text{(iv)}$$

From the first two equations of (iv), we get,

$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q)$$

i.e. $\lambda\frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$

$\therefore \lambda(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y}$ …… (v)

Similarly, $\lambda(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z}$ …… (vi)

$\lambda(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x}$ …… (vii)

Consider (v) ×R + (vi) × P + (vii) × Q, we get,

$P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = 0$

==

Sufficient Condition for Integrability: The Pfaffian differential equation

Pdx + Qdy + Rdz = 0 is integrable if $P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = 0$.

Condition for Exactness: The Pfaffian differential equation Pdx + Qdy + Rdz = 0 is exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$.

Method of Solution by Inspection: The Pfaffian differential equation Pdx + Qdy + Rdz = 0 can be solved by arranging the terms or dividing by suitable function of x, y, z. The modified equation may contain several parts which are exact differentials. The following list will help to re-write the given equation in differential form:

i) ydx + xdy = d(xy)

ii) yzdx + xzdy + xydz = d(xyz)

iii) $2(xdx + ydy) = d(x^2+y^2)$

iv) $2(xdx + ydy + zdz) = d(x^2+y^2+z^2)$

v) $y^2dx + 2xydy = d(xy^2)$

vi) $\frac{df(x,y,z)}{f(x,y,z)} = d[\log f(x,y,z)]$

vii) $\frac{xdy-ydx}{x^2} = d(\frac{y}{x})$

viii) $\frac{ydx-xdy}{y^2} = d(\frac{x}{y})$

ix) $\frac{xdy+ydx}{xy} = d(\log xy)$

x) $\frac{xdy-ydx}{xy} = d(\log\frac{y}{x})$

xi) $\frac{xdy-ydx}{x^2+y^2} = d(\tan^{-1}\frac{y}{x})$

xii) $\frac{xdy+ydx}{x^2+y^2} = d[\frac{1}{2}\log(x^2+y^2)]$

==

Ex. Show that the given equation (yz+2x) dx + (zx – 2z) dy + (xy – 2y) dz = 0 is exact. (Oct. 2019)

Proof: Let (yz+2x) dx + (zx – 2z) dy + (xy – 2y) dz = 0 be the given equation, comparing it with Pdx + Qdy + Rdz = 0, we get,

P = yz+2x, Q = zx – 2z and R = xy – 2y

$\therefore \frac{\partial P}{\partial y} = z, \frac{\partial P}{\partial z} = y, \frac{\partial Q}{\partial x} = z, \frac{\partial Q}{\partial z} = x - 2, \frac{\partial R}{\partial x} = y$ and $\frac{\partial R}{\partial y} = x - 2$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

Hence the given equation is exact is proved.

==

Ex. Show that the given equation $(x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz = 0$ is exact.

Proof: Let $(x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = x^2 - yz$, $Q = y^2 - zx$ and $R = z^2 - xy$

$\therefore \frac{\partial P}{\partial y} = -z, \frac{\partial P}{\partial z} = -y, \frac{\partial Q}{\partial x} = -z, \frac{\partial Q}{\partial z} = -x, \frac{\partial R}{\partial x} = -y$ and $\frac{\partial R}{\partial y} = -x$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

Hence the given equation is exact is proved.

==

Ex. Show that the given equation $(yz - x^3)\,dx + (zx - y^3)\,dy + (xy - z^3)\,dz = 0$ is exact.

Proof: Let $(yz - x^3)\,dx + (zx - y^3)\,dy + (xy - z^3)\,dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = yz - x^3$, $Q = zx - y^3$ and $R = xy - z^3$

$\therefore \frac{\partial P}{\partial y} = z, \frac{\partial P}{\partial z} = y, \frac{\partial Q}{\partial x} = z, \frac{\partial Q}{\partial z} = x, \frac{\partial R}{\partial x} = y$ and $\frac{\partial R}{\partial y} = x$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

Hence the given equation is exact is proved.

==

Ex. Show that the given equation $(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ is exact.

Proof: Let $(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = 2x + y^2 + 2xz$, $Q = 2xy$ and $R = x^2$

$\therefore \frac{\partial P}{\partial y} = 2y, \frac{\partial P}{\partial z} = 2x, \frac{\partial Q}{\partial x} = 2y, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 2x$ and $\frac{\partial R}{\partial y} = 0$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

$\therefore$ the given equation is exact is proved.

==

Ex. Show that the equation $(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ is integrable.

Proof: Let $(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = 2x + y^2 + 2xz$, $Q = 2xy$ and $R = x^2$

$\therefore \frac{\partial P}{\partial y} = 2y, \frac{\partial P}{\partial z} = 2x, \frac{\partial Q}{\partial x} = 2y, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 2x$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (2x + y^2 + 2xz)(0 - 0) + 2xy(2x - 2x) + x^2(2y - 2y)$

$= 0$

Hence the given equation is integrable is proved.

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Ex. Show that the equation $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$ is integrable. Is it exact? Verify.

Proof: Let $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = yz^2(x^2 - yz) = x^2yz^2 - y^2z^3$, $Q = zx^2(y^2 - xz) = x^2zy^2 - x^3z^2$ and

$R = xy^2(z^2 - xy) = xy^2z^2 - x^2y^3$

$\therefore \frac{\partial P}{\partial y} = x^2z^2 - 2yz^3, \frac{\partial P}{\partial z} = 2x^2yz - 3y^2z^2, \frac{\partial Q}{\partial x} = 2xzy^2 - 3x^2z^2, \frac{\partial Q}{\partial z} = x^2y^2 - 2x^3z,$

$\frac{\partial R}{\partial x} = y^2z^2 - 2xy^3$ and $\frac{\partial R}{\partial y} = 2xyz^2 - 3x^2y^2$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (x^2yz^2 - y^2z^3)(x^2y^2 - 2x^3z - 2xyz^2 + 3x^2y^2) + (x^2zy^2 - x^3z^2)(y^2z^2 - 2xy^3 - 2x^2yz$
$+ 3y^2z^2) + (xy^2z^2 - x^2y^3)(x^2z^2 - 2yz^3 - 2xzy^2 + 3x^2z^2)$

$= (x^2yz^2 - y^2z^3)(4x^2y^2 - 2x^3z - 2xyz^2) + (x^2zy^2 - x^3z^2)(4y^2z^2 - 2xy^3 - 2x^2yz)$
$+ (xy^2z^2 - x^2y^3)(4x^2z^2 - 2yz^3 - 2xzy^2)$

$= (x^2yz^2 - y^2z^3)(4x^2y^2 - 2x^3z - 2xyz^2) + (x^2zy^2 - x^3z^2)(4y^2z^2 - 2xy^3 - 2x^2yz)$
$+ (xy^2z^2 - x^2y^3)(4x^2z^2 - 2yz^3 - 2xzy^2)$

$= 4x^4y^3z^2 - 4x^2y^4z^3 - 2x^5yz^3 + 2x^3y^2z^4 - 2x^3y^2z^4 + 2xy^3z^5 + 4x^2y^4z^3 - 4x^3y^2z^4 - 2x^3y^5z$
$+ 2x^4y^3z^2 - 2x^4y^3z^2 + 2x^5yz^3 + 4x^3y^2z^4 - 4x^4y^3z^2 - 2xy^3z^5 + 2x^2y^4z^3 - 2x^2y^4z^3 + 2x^3y^5z$

$= 0$

Hence the given equation is integrable is proved.

But it is not exact $\because \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} \neq \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} \neq \frac{\partial P}{\partial z}$

Ex. Solve (y + z) dx + dy + dz = 0.

Proof: Let (y + z) dx + dy + dz = 0 be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

P = y + z, Q = 1 and R = 1

$\therefore \frac{\partial P}{\partial y} = 1, \frac{\partial P}{\partial z} = 1, \frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (y + z)(0 - 0) + (0 - 1) + (1 - 0)$

$= 0 - 1 + 1$

$= 0$

∴ The given equation is integrable.

Divide the given equation by (y + z), we get,

$dx + \frac{dy+dz}{y+z} = 0$

Integrating, we get,

$x + \log(y + z) = c$

be the solution of given equation.

==

Ex. Solve $xdy - ydx - 2x^2zdz = 0$.

Proof: Let $xdy - ydx - 2x^2zdz = 0$ be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

$P = -y$, $Q = x$ and $R = -2x^2z$

$\therefore \frac{\partial P}{\partial y} = -1, \frac{\partial P}{\partial z} = 0, \frac{\partial Q}{\partial x} = 1, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = -4xz$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (-y)(0 - 0) + x(-4xz - 0) - 2x^2z(-1-1)$

$= 0 - 4x^2z + 4x^2z$

$= 0$

∴ The given equation is integrable.

Divide the given equation by x^2, we get,

$\frac{xdy-ydx}{x^2} - 2zdz = 0$

i. e. $d(\frac{y}{x}) - d(z^2) = 0$

Integrating, we get,

$\frac{y}{x} - z^2 = c$

$\therefore y - xz^2 = cx$

be the solution of given equation.

==

Ex. Solve $zydx = zxdy + y^2dz$.

Proof: Let $zydx = zxdy + y^2dz$

i.e. $zydx - zxdy - y^2dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = zy$, $Q = -zx$ and $R = -y^2$

$\therefore \frac{\partial P}{\partial y} = z, \frac{\partial P}{\partial z} = y, \frac{\partial Q}{\partial x} = -z, \frac{\partial Q}{\partial z} = -x, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = -2y$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (zy)(-x + 2y) - zx(0 - y) - y^2(z + z)$

$= -xyz + 2y^2z + xyz - 2y^2z$

$= 0$

$\therefore$ The given equation is integrable.

Divide the given equation by y^2z, we get,

$\frac{ydx - xdy}{y^2} - \frac{dz}{z} = 0$

i. e. $d(\frac{x}{y}) - \frac{dz}{z} = 0$

Integrating, we get,

$\frac{x}{y} - \log z = c$

$\therefore x - y\log z = cy$

be the solution of given equation.

==

Ex. Solve $xz^2dx - zdy + ydz = 0$.

Proof: Let $xz^2dx - zdy + ydz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = xz^2$, $Q = -z$ and $R = y$

$\therefore \frac{\partial P}{\partial y} = 0, \frac{\partial P}{\partial z} = 2xz, \frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial z} = -1, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = 1$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (xz^2)(-1 - 1) - z(0 - 2xz) + y(0 - 0)$

$= -2xz^2 + 2xz^2 + 0$

$= 0$

$\therefore$ The given equation is integrable.

Divide the given equation by z^2, we get,

$xdx - \frac{zdy - ydz}{z^2} = 0$

i.e. $\frac{1}{2}d(x^2) - d(\frac{y}{z}) = 0$

i.e. $d(x^2) - 2d(\frac{y}{z}) = 0$

Integrating, we get,

$x^2 - 2(\frac{y}{z}) = c$

$\therefore x^2z - 2y = cz$

be the solution of given equation.

==

Ex. Solve $(x-y)dx - xdy + zdz = 0$.

Proof: Let $(x-y)dx - xdy + zdz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = x-y$, $Q = -x$ and $R = z$

$\therefore \frac{\partial P}{\partial y} = -1, \frac{\partial P}{\partial z} = 0, \frac{\partial Q}{\partial x} = -1, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (x-y)(0 - 0) - x(0 - 0) + z(-1 + 1)$

$= 0$

$\therefore$ The given equation is integrable.

Rearrange the terms of given equation as:

$xdx - ydx - xdy + zdz = 0$

i.e. $2xdx - 2(ydx + xdy) + 2zdz = 0$

i.e. $d(x^2) - 2d(xy) + d(z^2) = 0$

Integrating, we get,

$x^2 - 2xy + z^2 = c$

be the solution of given equation.

==

Ex. Solve $(a - z)(ydx + xdy) + xydz = 0$.

Proof: Let $(a - z)(ydx + xdy) + xydz = 0$

i.e. $(a - z)ydx + (a - z)xdy + xydz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = (a - z)y$, $Q = (a - z)x$ and $R = xy$

$\therefore \frac{\partial P}{\partial y} = a - z, \frac{\partial P}{\partial z} = -y, \frac{\partial Q}{\partial x} = a - z, \frac{\partial Q}{\partial z} = -x, \frac{\partial R}{\partial x} = y$ and $\frac{\partial R}{\partial y} = x$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (a - z)y(-x - x) + (a - z)x(y + y) + xy(a - z - a + z)$

$= -2(a - z)xy + 2(a - z)xy + 0$

$= 0$

$\therefore$ The given equation is integrable.

Divide the given equation by $xy(a - z)$, we get,

$\frac{ydx+xdy}{xy} + \frac{dz}{a-z} = 0$

i.e. $\frac{d(xy)}{xy} - \frac{d(z-a)}{z-a} = 0$

Integrating, we get,

$\log xy - \log (z-a) = \log c$

i.e. $\frac{xy}{z-a} = c$

$\therefore xy = c(z-a)$

be the solution of given equation.

==

Ex. Solve $(x^2 - yz) dx + (y^2 - zx) dy + (z^2 - xy) dz = 0$.

Proof: Let $(x^2 - yz) dx + (y^2 - zx) dy + (z^2 - xy) dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = x^2 - yz$, $Q = y^2 - zx$ and $R = z^2 - xy$

$\therefore \frac{\partial P}{\partial y} = -z, \frac{\partial P}{\partial z} = -y, \frac{\partial Q}{\partial x} = -z, \frac{\partial Q}{\partial z} = -x, \frac{\partial R}{\partial x} = -y$ and $\frac{\partial R}{\partial y} = -x$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

∴ The given equation exacts and hence integrable.

Now we rearrange the terms as:

$(x^2 dx + y^2 dy + z^2 dz) - (yzdx + zxdy + xydz) = 0$

∴ $(3x^2 dx + 3y^2 dy + 3z^2 dz) - 3(yzdx + zxdy + xydz) = 0$

∴ $d(x^3 + y^3 + z^3) - 3d(xyz) = 0$

Integrating, we get,

$x^3 + y^3 + z^3 - 3xyz = c$

be the solution of given equation.

==

Ex. Solve $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$. (Oct. 2019)

Proof: Let $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = y^2 + z^2 - x^2$, $Q = -2xy$ and $R = -2xz$

∴ $\frac{\partial P}{\partial y} = 2y$, $\frac{\partial P}{\partial z} = 2z$, $\frac{\partial Q}{\partial x} = -2y$, $\frac{\partial Q}{\partial z} = 0$, $\frac{\partial R}{\partial x} = -2z$ and $\frac{\partial R}{\partial y} = 0$

∴ $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (y^2 + z^2 - x^2)(0 - 0) - 2xy(-2z - 2z) - 2xz(2y+2y)$

$= 0 + 8xyz - 8xyz$

$= 0$

∴ The given equation integrable.

Now we rearrange the terms as:

$(x^2 + y^2 + z^2)dx - 2x^2dx - 2xydy - 2xzdz = 0$

i.e. $(x^2 + y^2 + z^2)dx - x(2xdx + 2ydy + 2zdz) = 0$

i.e. $(x^2 + y^2 + z^2)dx - xd(x^2 + y^2 + z^2) = 0$

Dividing by $x(x^2 + y^2 + z^2)$, we get,

∴ $\frac{dx}{x} - \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)} = 0$

i.e. $\frac{dx}{x} = \frac{d(x^2+y^2+z^2)}{(x^2+y^2+z^2)}$

Integrating, we get,

$\log x = \log(x^2 + y^2 + z^2) + \log c$

∴ $x = c(x^2 + y^2 + z^2)$

be the solution of given equation.

==

Ex. Solve 2yzdx + zxdy – xy(1+z)dz = 0.

Proof: Let 2yzdx + zxdy – xy(1+z)dz = 0 be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

P = 2yz, Q = zx and R = − xy(1+z)

$\therefore \frac{\partial P}{\partial y} = 2z, \frac{\partial P}{\partial z} = 2y, \frac{\partial Q}{\partial x} = z, \frac{\partial Q}{\partial z} = x, \frac{\partial R}{\partial x} = -y(1+z)$ and $\frac{\partial R}{\partial y} = -x(1+z)$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

= (2yz)[x +x(1+z)] + zx[-y(1+z) – 2y) - xy(1+z)(2z-z)

= (2yz)(2x +xz) + zx(-yz – 3y) - xyz(1+z)

$= 4xyz+2xyz^2 - xyz^2 - 3xyz - xyz - xyz^2$

= 0

∴ The given equation integrable.

Divide the given equation by xyz, we get,

$\frac{2dx}{x} + \frac{dy}{y} - (\frac{1}{z} + 1)dz = 0$

Integrating, we get,

2logx + logy – logz – z = logc

i.e. $\log x^2 + \log y - \log z - \log e^z = \log c$

i.e. $\log(\frac{x^2y}{ze^z}) = \log c$

$\therefore \frac{x^2y}{ze^z} = c$

i.e. $x^2y = cze^z$

be the solution of given equation.

==

Ex. Solve $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$.

Proof: Let $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$ be the given equation,

comparing it with Pdx + Qdy + Rdz = 0, we get,

$P = 2x^2 + 2xy + 2xz^2 + 1$, Q = 1 and R = 2z

$\therefore \frac{\partial P}{\partial y} = 2x, \frac{\partial P}{\partial z} = 4xz, \frac{\partial Q}{\partial x} = 0, \frac{\partial Q}{\partial z} = 0, \frac{\partial R}{\partial x} = 0$ and $\frac{\partial R}{\partial y} = 0$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= (2x^2 + 2xy + xz^2 + 1)(0 - 0) + (0 - 4xz) + 2z(2x - 0)$

$= 0 - 4xz + 4xz$

$= 0$

$\therefore$ The given equation integrable.

Rearrange the given terms as:

$2x(x + y + z^2)dx + dx + dy + 2zdz = 0$

Divide the given equation by $(x+y+z^2)$, we get,

$2xdx + \frac{dx+dy+2zdz}{x+y+z^2} = 0$

i.e. $d(x^2) + \frac{d(x+y+z^2)}{x+y+z^2} = 0$

Integrating, we get,

$x^2 + \log(x + y + z^2) = c$

be the solution of given equation.

==

Ex. Solve $\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0$

Proof: Let $\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\frac{y}{x}dz = 0$ be the given equation,

comparing it with $Pdx + Qdy + Rdz = 0$, we get,

$P = \frac{yz}{x^2+y^2}$, $Q = -\frac{xz}{x^2+y^2}$ and $R = -\tan^{-1}\frac{y}{x}$

$\therefore \frac{\partial P}{\partial y} = z\,\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{z(x^2-y^2)}{(x^2+y^2)^2}$, $\frac{\partial P}{\partial z} = \frac{y}{x^2+y^2}$,

$\frac{\partial Q}{\partial x} = -z\,\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{z(x^2-y^2)}{(x^2+y^2)^2}$, $\frac{\partial Q}{\partial z} = -\frac{x}{x^2+y^2}$,

$\frac{\partial R}{\partial x} = -\frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{y}{x^2+y^2}$ and $\frac{\partial R}{\partial y} = -\frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{-x}{x^2+y^2}$

$\therefore P(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}) + Q(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}) + R(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})$

$= \frac{yz}{x^2+y^2}[-\frac{x}{x^2+y^2} + \frac{x}{x^2+y^2}] - \frac{xz}{x^2+y^2}[\frac{y}{x^2+y^2} - \frac{y}{x^2+y^2}]$

$- \tan^{-1}(\frac{y}{x})[\frac{z(x^2-y^2)}{(x^2+y^2)^2} - \frac{z(x^2-y^2)}{(x^2+y^2)^2}]$

$= 0$

∴ The given equation is integrable.

Rearrange the given equation as:

$z[\frac{ydx-xdy}{x^2+y^2}] - \tan^{-1}\frac{y}{x}\,dz = 0$

i.e. $z[\frac{xdy-ydx}{x^2+y^2}] + \tan^{-1}\frac{y}{x}\,dz = 0$

i.e. $\frac{1}{\tan^{-1}\frac{y}{x}}[\frac{xdy-ydx}{x^2+y^2}] + \frac{dz}{z} = 0$

i.e. $\frac{d(\tan^{-1}\frac{y}{x})}{\tan^{-1}\frac{y}{x}} + \frac{dz}{z} = 0$

Integrating, we get,

$\log\tan^{-1}\frac{y}{x} + \log z = \log c$

$\therefore z\tan^{-1}\frac{y}{x} = c$

be the solution of given equation.

==

Homogeneous Equation: If P, Q, R, are homogeneous functions of same degree of variables x, y, z, then the Pfaffian differential equation Pdx + Qdy + Rdz = 0 is called homogeneous equation.

Method of Solving Homogeneous Equation: If Pdx + Qdy + Rdz = 0 is homogeneous equation, then find Px + Qy + Rz.

Case-i) If $\rho = Px + Qy + Rz \neq 0$, then

step-1) Find an I.F. $\frac{1}{\rho}$ of given homogeneous equation.

2) Multiply given equation by $\frac{1}{\rho}$.

3) Find $d(\rho)$.

4) Express given equation in the form $\frac{d(\rho)}{\rho} \pm \dots$

5) By integrating we get, the solution.

Case-ii) If $\rho = Px + Qy + Rz = 0$, then

step-1) Verify given homogeneous equation is integrable.

2) Put x = zu and y = zv, hence dx = udz + zdu and dy = vdz + zdv into the equation.

3) case-a) If coefficient of dz is zero, then we get equation in two variables u and v, regrouping and integrating we get solution.

4) case-b) If coefficient of dz is not zero, then we will be able to separate the equation into form $\frac{f(u,v)du+g(u,v)dv}{f(u,v)} + \frac{dz}{z} = 0$

5) Take ρ = f(u, v) and find d(ρ).

6) Express given equation in the form $\frac{d(\rho)}{\rho} \pm \ldots$ and rearrange the terms.

7) By integrating we get, the solution.

===

Remark: If Px + Qy + Rz ≠ 0, then the homogeneous equation Pdx + Qdy + Rdz = 0 is always integrable. But if Px + Qy + Rz = 0, then it may or may not be integrable.

===

Ex. Solve z(z - y)dx + z(z + x)dy + x(x + y)dz = 0

Proof: Let z(z - y)dx + z(z + x)dy + x(x + y)dz = 0 be the given homogeneous equation, with P = z(z - y) , Q = z(z + x) and R = x(x + y)

$\therefore \rho$ = Px + Qy + Rz = xz(z - y) + yz(z + x) + xz(x + y)

= $xz^2 - xyz + yz^2 + xyz + x^2z + xyz$

= $xz^2 + yz^2 + x^2z + xyz$

= $z(xz + yz + x^2 + xy)$

= z(x + y)(z + x) ≠ 0

∴ The given equation is integrable.

Divide the given equation by z(x + y)(z + x), we get,

$$\frac{z(z-y)}{z(x+y)(z+x)}\,dx + \frac{z(z+x)}{z(x+y)(z+x)}\,dy + \frac{x(x+y)}{z(x+y)(z+x)}\,dz = 0$$

$$i.e.\ \frac{(z-y)}{(x+y)(z+x)}\,dx + \frac{1}{(x+y)}\,dy + \frac{x}{z(z+x)}\,dz = 0$$

$$i.e.\ \frac{[(z+x)-(x+y)]}{(x+y)(z+x)}\,dx + \frac{1}{(x+y)}\,dy + \frac{[(z+x)-z]}{z(z+x)}\,dz = 0$$

$$i.e.\ \frac{1}{(x+y)}\,dx - \frac{1}{(z+x)}\,dx + \frac{1}{(x+y)}\,dy + \frac{1}{z}\,dz - \frac{1}{(z+x)}\,dz = 0$$

$$i.e.\ \frac{dx+dy}{(x+y)} - \frac{dx+dz}{(z+x)} + \frac{dz}{z} = 0$$

$$i.e.\ \frac{d(x+y)}{(x+y)} + \frac{dz}{z} = \frac{d(x+z)}{(x+z)}$$

Integrating, we get,

log (x + y) + log z = log (x + z) + log c

$\therefore$ (x + y)z = c(x + z)

be the solution of given equation.

==

Ex. Solve $y(y + z)dx + x(x - z)dy + x(x + y)dz = 0$

Proof: Let $y(y + z)dx + x(x - z)dy + x(x + y)dz = 0$ be the given homogeneous equation,

with $P = y(y + z)$, $Q = x(x - z)$ and $R = x(x + y)$

$\therefore \rho = Px + Qy + Rz = xy(y + z) + yx(x - z) + zx(x + y)$

$= xy^2 + xyz + x^2y - xyz + x^2z + xyz$

$= xy^2 + x^2y + x^2z + xyz$

$= x(y^2 + xy + xz + yz)$

$= x(x + y)(y + z) \neq 0$

$\therefore$ The given equation is integrable.

Divide the given equation by x(x + y)(y + z), we get,

$$\frac{y(y+z)}{x(x+y)(y+z)} dx + \frac{x(x-z)}{x(x+y)(y+z)} dy + \frac{x(x+y)}{x(x+y)(y+z)} dz = 0$$

$$i.e. \frac{y}{x(x+y)} dx + \frac{(x-z)}{(x+y)(y+z)} dy + \frac{1}{(y+z)} dz = 0$$

$$i.e. \frac{[(x+y)-x]}{x(x+y)} dx + \frac{[(x+y)-(y+z)]}{(x+y)(y+z)} dy + \frac{1}{(y+z)} dz = 0$$

$$i.e. \frac{1}{x} dx - \frac{1}{(x+y)} dx + \frac{1}{(y+z)} dy - \frac{1}{(x+y)} dy + \frac{1}{(y+z)} dz = 0$$

$$i.e. \frac{dx}{x} - \frac{dx+dy}{(x+y)} + \frac{dy+dz}{(y+z)} = 0$$

$$i.e. \frac{dx}{x} + \frac{d(y+z)}{(y+z)} = \frac{d(x+y)}{(x+y)}$$

Integrating, we get,

log x + log (y + z) = log (x + y) + logc

$\therefore$ x(y + z) = c(x + y)

be the solution of given equation.

==

Ex. Solve $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0$

Proof: Let $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0$

be the given homogeneous equation,

with $P = y^2 + yz$, $Q = z^2 + zx$ and $R = y^2 - xy$

$\therefore \rho = Px + Qy + Rz = x(y^2 + yz) + y(z^2 + zx) + z(y^2 - xy)$

$= xy^2 + xyz + yz^2 + xyz + y^2z - xyz$

$= xy^2 + yz^2 + y^2z + xyz$

$= y(xy + z^2 + yz + xz)$

$= y(x + z)(y + z) \neq 0$

$\therefore$ The given equation is integrable.

Divide the given equation by y(x+z)(y+z), we get,

$$\frac{y(y+z)}{y(x+z)(y+z)}\,dx + \frac{z(x+z)}{y(x+z)(y+z)}\,dy + \frac{y(y-x)}{y(x+z)(y+z)}\,dz = 0$$

$$i.e.\ \frac{1}{(x+z)}\,dx + \frac{z}{y(y+z)}\,dy + \frac{y-x}{(x+z)(y+z)}\,dz = 0$$

$$i.e.\ \frac{1}{(x+z)}\,dx + \frac{[(y+z)-y]}{y(y+z)}\,dy + \frac{[(y+z)-(x+z)]}{(x+z)(y+z)}\,dz = 0$$

$$i.e.\ \frac{1}{(x+z)}\,dx + \frac{1}{y}\,dy - \frac{1}{(y+z)}\,dy + \frac{1}{(x+z)}\,dz - \frac{1}{(y+z)}\,dz = 0$$

$$i.e.\ \frac{dx+dz}{(x+z)} + \frac{1}{y}\,dy - \frac{dy+dz}{(y+z)} = 0$$

$$i.e.\ \frac{d(x+z)}{(x+z)} + \frac{dy}{y} = \frac{d(y+z)}{(y+z)}$$

Integrating, we get,

$\log(x + z) + \log y = \log(y + z) + \log c$

$\therefore (x + z)y = c(y + z)$

be the solution of given equation.

==

Ex. Solve $(yz + z^2)dx - xzdy + xydz = 0$

Proof: Let $(yz + z^2)dx - xzdy + xydz = 0$ be the given homogeneous equation,

with $P = yz + z^2$, $Q = -xz$ and $R = xy$

$\therefore \rho = Px + Qy + Rz = x(yz + z^2) + y(-xz) + z(xy)$

$= xyz + xz^2 - xyz + xyz$

$= xyz + xz^2$

$= xz(y + z) \neq 0$

$\therefore$ The given equation is integrable.

Divide the given equation by xz(y + z), we get,

$\frac{z(y+z)}{xz(y+z)}$ dx - $\frac{xz}{xz(y+z)}$ dy + $\frac{xy}{xz(y+z)}$ dz = 0

$i.e.\ \frac{1}{x}$ dx - $\frac{1}{(y+z)}$ dy + $\frac{y}{z(y+z)}$ dz = 0

$i.e.\ \frac{1}{x}$ dx - $\frac{1}{(y+z)}$ dy + $\frac{[(y+z)-z]}{z(y+z)}$ dz = 0

$i.e.\ \frac{1}{x}$ dx - $\frac{1}{(y+z)}$ dy + $\frac{1}{z}$ dz - $\frac{1}{(y+z)}$ dz = 0

$i.e..\ \frac{1}{x}$ dx + $\frac{1}{z}$ dz - $\frac{dy+dz}{(y+z)}$ = 0

$i.e.\quad \frac{dx}{x} + \frac{dz}{z} = \frac{d(y+z)}{(y+z)}$

Integrating, we get,

log x + log z = log (y + z) + logc

∴ xz = c(y + z)

be the solution of given equation.

==

Ex. Solve $z^2dx + (z^2 - 2yz)dy + (2y^2 – yz – xz)dz = 0$

Proof: Let $z^2dx + (z^2 - 2yz)dy + (2y^2 – yz – xz)dz = 0$

be the given homogeneous equation,

with $P = z^2$, $Q = z^2 - 2yz$ and $R = 2y^2 – yz – xz$ and is integrable.

∴ $\rho = Px + Qy + Rz = xz^2 + y(z^2 - 2yz) + z(2y^2 – yz – xz)$

$= xz^2 + yz^2 - 2y^2z + 2zy^2 - yz^2 - xz^2$

$= 0$

∴ To solve the given equation put x = zu and y = zv,

∴ dx = udz + zdu and dy = vdz + zdv

∴ the given equation becomes

$z^2(udz + zdu) + (z^2 - 2vz^2)(vdz + zdv) + (2z^2v^2 – z^2v – uz^2)dz = 0$

i.e. $z^3du + z^3(1 - 2v)dv) + (z^2u + z^2v - 2z^2v^2 + 2z^2v^2 – z^2v – uz^2)dz = 0$

i.e. $z^3du + z^3(1 - 2v)dv) + 0dz = 0$

i.e. $du + (1 - 2v)dv = 0$

Integrating, we get,

$u + v – v^2 = c$

$\therefore \frac{x}{z} + \frac{y}{z} - \frac{y^2}{z^2} = c$

i.e. $(x + y)z - y^2 = cz^2$

be the solution of given equation.

==

Ex. Solve yzdx + 2zxdy – 3xydz = 0

Proof: Let yzdx + 2zxdy – 3xydz = 0 be the given homogeneous equation,

with P = yz, Q = 2zx and R = – 3xy and is integrable.

$\therefore \rho = Px + Qy + Rz = xyz + 2yzx - 3zxy = 0$

∴ To solve the given equation put x = zu and y = zv,

∴ dx = udz + zdu and dy = vdz + zdv

∴ the given equation becomes

$vz^2(udz + zdu) + (2z^2u)(vdz + zdv) - (3z^2uv)dz = 0$

i.e. $vz^3du + 2uz^3dv + (uvz^2 + 2z^2uv - 3z^2uv)dz = 0$

i.e. $vz^3du + 2uz^3dv + 0dz = 0$

i.e. $vdu + 2udv = 0$

i.e. $\frac{du}{u} + 2\frac{dv}{v} = 0$

Integrating, we get,

$\log u + 2\log v = \log c$

$\therefore uv^2 = c$

i.e. $\frac{x}{z}\left(\frac{y^2}{z^2}\right) = c$

i.e. $xy^2 = cz^3$

be the solution of given equation.

==

Ex. Solve $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$

Proof: Let $yz^2(x^2 - yz)\,dx + zx^2(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$

be the given homogeneous equation, which is integrable with

$P = yz^2(x^2 - yz)$, $Q = zx^2(y^2 - xz)$ and $R = xy^2(z^2 - xy)$

$\therefore Px + Qy + Rz = xyz^2(x^2 - yz) + yzx^2(y^2 - xz) + zxy^2(z^2 - xy)$

$= xyz\,(x^2z - yz^2 + xy^2 - x^2z + yz^2 - xy^2)$

$= 0$

∴ To solve the given equation put $x = zu$ and $y = zv$,

∴ $dx = udz + zdu$ and $dy = vdz + zdv$

∴ the given equation becomes

$vz^3(u^2z^2 - vz^2)(udz + zdu) + u^2z^3(v^2z^2 - uz^2)(vdz + zdv) + uv^2z^3(z^2 - uvz^2)dz = 0$

i.e. $z^5[(u^2v - v^2)(udz + zdu) + (u^2v^2 - u^3)(vdz + zdv) + (uv^2 - u^2v^3)dz] = 0$

i.e. $(u^2v - v^2)zdu + (u^2v^2 - u^3)zdv + (u^3v - uv^2 + u^2v^3 - u^3v + uv^2 - u^2v^3)dz = 0$

i.e. $(u^2 - v)vzdu + (v^2 - u)u^2zdv = 0$

i.e. $u^2vdu - v^2du + u^2v^2dv - u^3dv = 0$

i.e. $u^2(vdu - udv) + u^2v^2dv - v^2du = 0$

Dividing by u^2v^2, we get,

i.e. $\frac{vdu - udv}{v^2} + dv - \frac{du}{u^2} = 0$

i.e. $d(\frac{u}{v}) + dv + d(\frac{1}{u}) = 0$

Integrating, we get,

$\frac{u}{v} + v + \frac{1}{u} = c$

∴ $u^2 + uv^2 + v = cuv$

i.e. $(\frac{x^2}{z^2}) + \frac{x}{z}(\frac{y^2}{z^2}) + \frac{y}{z} = c(\frac{x}{z})(\frac{y}{z})$

i.e. $x^2z + xy^2 + yz^2 = cxyz$

be the solution of given equation.

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MULTIPLE CHOICE QUESTIONS [MCQ'S)]

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1) The differential equation of the form $u_1dx_1 + u_2dx_2 + \ldots\ldots + u_ndx_n = 0$ is called …… differential equation in n independent variables $x_1, x_2, \ldots\ldots, x_n$.

A) Pfaffian B) Linear C) Homogeneous D) None of these

2) Pfaffian differential equation is also called …… differential equation.

A) linear B) total C) homogeneous D) None of these

3) If P, Q, R, are functions of x, y, z, then the differential equation $Pdx + Qdy + Rdz = 0$ is called …… differential equation.

A) simultaneous B) Pfaffian C) linear D) Non Linear

4) If there exists a function u(x, y, z), such that Pdx + Qdy + Rdz = du, then the Pfaffian differential equation Pdx + Qdy + Rdz = 0 is said to be ……

. A) exact B) not exact

C) may or may not be exact D) None of these

5) Pfaffian differential equation Pdx + Qdy + Rdz = 0 is said to be ……, if the equation is exact or can be made exact.

A) not integrable B) integrable C) linear D) None of these

6) Statement 'Every exact differential equation is integrable.' is ……

A) true B) false

C) may be true or false D) None of these

7) Every exact differential equation is ……

A) not integrable B) integrable

C) may or may not integrable D) None of these

8) Statement 'Every integrable differential equation is exact' is ……

A) true B) false

C) may be true or false D) None of these

9) An integrable differential equation ……

A) is exact B) is not exact

C) may or may not exact D) None of these

10) If the Pfaffian differential equation Pdx + Qdy + Rdz = 0 satisfies the conditions $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ and $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$, then given equation is……

A) exact B) not exact

C) may or may not exact D) None of these

11) If the Pfaffian differential equation Pdx + Qdy + Rdz = 0 is exact, then it satisfies the conditions……

A) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ B) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ C) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ D) All above

12) If the differential equation Pdx + Qdy + Rdz = 0 is exact, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ & $\frac{\partial R}{\partial x} =$……

A) $\frac{\partial P}{\partial z}$ B) $\frac{\partial P}{\partial y}$ C) $\frac{\partial z}{\partial P}$ D) $\frac{\partial y}{\partial P}$

13) The differential equation (yz + 2x) dx + (zx – 2z) dy + (xy – 2y) dz = 0 is ……

A) exact B) not exact

C) may or may not exact D) None of these

14) The differential equation $(x^2 - yz)\,dx + (y^2 - zx)\,dy + (z^2 - xy)\,dz = 0$ is……

A) exact B) not exact

C) may or may not exact D) None of these

15) The differential equation $(yz - x^3)\,dx + (zx - y^3)\,dy + (xy - z^3)\,dz = 0$ is……

A) exact B) not exact

C) may or may not exact D) None of these

16) The differential equation $(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ is……

A) exact B) not exact

C) may or may not exact D) None of these

17) The differential equation $(y + z)dx + (z + x)dy + (x + y)dz = 0$ is……

A) exact B) not exact

C) may or may not exact D) None of these

18) The differential equation $(y + z)\,dx + dy + dz = 0$ is……

A) exact B) not exact

C) may or may not exact D) None of these

19) For the differential equation $xdx + ydy + zdz = 0$. Which of the following is true?

A) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ B) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ C) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ D) All above

20) For the differential equation $(y + z)dx + (z + x)dy + (x + y)dz = 0$.
Which of the following is true?

A) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ B) $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ C) $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ D) All above

21) The differential equation $(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2\,dz = 0$ is……

A) not integrable B) integrable

C) may or may not integrable D) None of these

22) The differential equation $(y + z)\,dx + dy + dz = 0$ is……

A) integrable B) not integrable

C) may or may not integrable D) None of these

23) If the differential equation $Pdx + Qdy + Rdz = 0$ is $(a - z)(ydx + xdy) + xydz = 0$,
then P = ……

A) a - z B) (a – z)y C) (a – z)x D) xy

24) If the differential equation $Pdx + Qdy + Rdz = 0$ is $(a - z)(ydx + xdy) + xydz = 0$,
then Q = ……

A) a - z B) (a – z)y C) (a – z)x D) xy

25) If the differential equation $Pdx + Qdy + Rdz = 0$ is $zydx = zxdy + y^2dz$,
then Q = ……

A) zx B) – zx C) zy D) y^2

26) If the differential equation Pdx + Qdy + Rdz = 0 is $zydx = zxdy + y^2dz$, then R = ……

A) zy B) – zx C) y^2 D) $-y^2$

27) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then the value of P is ……

A) $2x^2y$ B) $3xy^2$ C) z D) x^3

28) If the differential equation Pdx + Qdy + Rdz = 0 is $(yz + 2x)\,dx + (zx - 2z)\,dy + (xy - 2y)\,dz = 0$, then Q = ……

A) yz + 2x B) zx – 2z C) xy – 2y D) None of these

29) If the differential equation Pdx + Qdy + Rdz = 0 is $(yz - x^3)\,dx + (zx - y^3)\,dy + (xy - z^3)\,dz = 0$, then R = ……

A) $yz - x^3$ B) $zx - y^3$ C) $xy - z^3$ D) 0

30) If the differential equation Pdx + Qdy + Rdz = 0 is $(yz + xyz)\,dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0$, then Q = ……

A) yz + xyz B) zx + xyz C) xy + xyz D) xy + xyz + yz

31) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then $\frac{\partial P}{\partial y}$ = ……

A) $2x^2$ B) $3xy^2$ C) z D) x^3

32) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then $\frac{\partial P}{\partial z}$ = ……

A) $2x^2$ B) $3xy^2$ C) z D) 0

33) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then $\frac{\partial Q}{\partial x}$ = ……

A) $2x^2$ B) $3y^2$ C) z D) x^3

34) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then $\frac{\partial Q}{\partial z}$ = ……

A) $2x^2$ B) $3y^2$ C) z D) 0

35) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then $\frac{\partial R}{\partial x}$ = ……

A) $2x^2$ B) $3y^2$ C) z D) 0

36) If the differential equation Pdx + Qdy + Rdz = 0 is $2x^2ydx + 3xy^2dy + zdz = 0$, then $\frac{\partial R}{\partial y}$ = ……

A) $2x^2$ B) $3y^2$ C) z D) 0

37) If the differential equation Pdx + Qdy + Rdz = 0 is (a – z)(ydx + xdy) + xydz = 0, then $\frac{\partial P}{\partial y}$ = ……

A) a - z B) (a – z)y C) (a – z)x D) x

38) If the differential equation Pdx + Qdy + Rdz = 0 is (a – z)(ydx + xdy) + xydz = 0, then $\frac{\partial P}{\partial z}$ = ……

A) a - z B) (a – z)y C) – y D) y

39) If the differential equation Pdx + Qdy + Rdz = 0 is (a – z)(ydx + xdy) + xydz = 0, then $\frac{\partial Q}{\partial x}$ = ……

A) a - z B) (a – z)x C) x D) –x

40) If the differential equation Pdx + Qdy + Rdz = 0 is (a – z)(ydx + xdy) + xydz = 0, then $\frac{\partial Q}{\partial z}$ = ……

A) a - z B) (a – z)x C) x D) –x

41) If the differential equation Pdx + Qdy + Rdz = 0 is zydx = zxdy + y^2dz, then $\frac{\partial R}{\partial y}$ = ……

A) z B) 0 C) – 2y D) 2y

42) If the differential equation Pdx + Qdy + Rdz = 0 is zydx = zxdy + y^2dz, then $\frac{\partial Q}{\partial x}$ = ……

A) –z B) z C) – 2y D) 2y

43) If the differential equation Pdx + Qdy + Rdz = 0 is zydx = zxdy + y^2dz, then $\frac{\partial Q}{\partial z}$ = ……

A) x B) –x C) – 2y D) 2y

44) To solve the equation (y + z)dx + dy + dz = 0, we divide the equation by ……

A) z B) y C) y + z D) xyz

45) To solve the equation xdy – ydx – $2x^2$zdz = 0, we divide the equation by ……

A) x^2z B) x^2 C) xy D) xyz

46) To solve the equation zydx = zxdy + y^2dz, we divide the equation by ……

A) y^2z B) zy C) zx D) xyz

47) To solve the equation xz^2dx - zdy + ydz = 0, we divide the equation by ……

A) xz^2 B) x C) z^2 D) xyz

48) To solve the equation $(y^2 + z^2 – x^2)$dx – 2xydy – 2xzdz = 0, we rearrange the terms

by adding and subtracting ……

A) x^2dx B) y^2dx C) z^2dx D) xyz

49) Solution of equation ydx + xdy = 0 is ……

A) xy = c B) yz = c C) zx = c D) xyz = c

50) Solution of equation yzdx + xzdy + xydz = 0 is ……

A) xy = c B) yz = c C) zx = c D) xyz = c

51) Solution of equation (y + z)dx + dy + dz = 0 is ……

A) x + log(y + z) = c B) log x + y + z = c

C) y + z = c D) x + y + z = c

52) If P, Q, R, are homogeneous functions of x, y, z, of same degree, then the Pfaffian differential equation Pdx + Qdy + Rdz = 0 is called …… differential equation

A) simultaneous B) homogeneous C) linear D) Non Linear

53) The Pfaffian differential equation (x-y)dx - xdy + zdz = 0 is …… equation.

A) homogeneous B) non homogeneous

C) may or may not homogeneous D) None of these

54) The Pfaffian differential equation 2(y + z)dx – (x + z)dy + (2y – x + z)dz = 0 is ……

A) homogeneous equation B) non homogeneous equation

C) simultaneous equation D) None of these

55) The differential equation $(y^2 + z^2 – x^2)dx – 2xydy – 2xzdz = 0$ is …… equation.

A) not homogeneous B) homogeneous

C) may or may not homogeneous D) None of these

56) The differential equation $yz^2(x^2 - yz)\, dx + zx^2(y^2 – xz)\, dy + xy^2(z^2-xy)\, dz = 0$ is …… equation.

A) simultaneous B) homogeneous C) linear D) Non homogeneous

57) If Pdx + Qdy + Rdz = 0 is homogeneous differential equation with $\rho = Px + Qy + Rz \neq 0$, then it is always integrable since it has an I.F.= …

A) ρ B) $\frac{1}{\rho}$ C) e^{ρ} D) None of these

58) For homogeneous differential equation Pdx + Qdy + Rdz = 0, if $\rho = Px + Qy + Rz \neq 0$, then the differential equation is always ……

A) integrable B) not integrable

C) may or may not integrable D) None of these

59) For homogeneous differential equation Pdx + Qdy + Rdz = 0 if $\rho = Px + Qy + Rz = 0$, then the differential equation is ……

A) integrable B) not integrable

C) may or may not integrable D) None of these

60) For homogeneous differential equation Pdx + Qdy + Rdz = 0,

if $\rho = Px + Qy + Rz = 0$, then to solve this equation we put ……

A) $x = zu$ and $y = zv$ B) $u = xz$ and $v = yz$

C) $z = xu$ and $z = yv$ D) None of these

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UNIT-4: DIFFERENCE EQUATIONS

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Shift Operator: Shift operator E is defined as Ef(x) = f(x+h).

Note: $E^2f(x) = E[Ef(x)] = Ef(x+h) = f(x+2h)$, Similarly $E^3f(x) = f(x+3h)$ and so on,

In general $E^nf(x) = f(x+nh)$, where n is any real number.

Forward difference Operator: Forward difference operator Δ is defined as

$\Delta f(x) = f(x+h) - f(x)$.

Note: i) $\Delta f(x) = f(x+h) - f(x)$ is called first forward difference of f(x) and

$\Delta^n f(x)$ is called n^{th} forward difference of f(x).

ii) $\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E - 1)f(x)$

$\therefore \Delta = E - 1$ i.e $E = \Delta + 1$

be the relation between shift operator and forward difference operator.

Difference Equations: A relation of the form $F\left[x, y, \frac{\Delta y}{\Delta x}, \frac{\Delta^2 y}{\Delta x^2}, \ldots\ldots, \frac{\Delta^n y}{\Delta x^n}\right] = 0$ is called a difference equation.

Note: i) If y = f(x), then $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{h} = \frac{Ef(x)-f(x)}{h} = \frac{(E-1)f(x)}{h}$,

$\frac{\Delta^2 y}{\Delta x^2} = \frac{f(x+2h)-2f(x+h)+f(x)}{h^2} = \frac{E^2f(x)-2Ef(x)+f(x)}{h^2} = \frac{(E-1)^2f(x)}{h^2}$,

and so on, ingeneral, $\frac{\Delta^n y}{\Delta x^n} = \frac{(E-1)^n f(x)}{h^n}$. Where h is the interval of differencing.

ii) If y = f(x), then a relation of the form

$\varphi\,[x, f(x), f(x+h), f(x+2h), \ldots\ldots, f(x+nh)] = 0$ is called a difference equation.

Order of a difference equation: The difference between the largest and smallest arguments for the function involved divided by h is called order of a difference equation.

e.g. Order of a difference equation

$\varphi\,[x, f(x), f(x+h), f(x+2h), \ldots\ldots, f(x+nh)] = 0$ is $\frac{(x+nh)-(x)}{h} = n$.

Solution of a difference equation: Any function which satisfies the given difference equation is called solution of a difference equation.

Subscript Notation: y = f(x) is written in subscript form as $y_x = f(x)$ and $y_{x+n} = f(x+nh)$.

e.g. i) The difference equation $f(x+2h) - 5f(x+h) + 6f(x) = 0$ is written in subscript form as $y_{x+2} - 5y_{x+1} + 6y_x = 0$.

ii) A difference equation $\varphi[\,x, f(x), f(x+h), f(x+2h), \ldots\ldots, f(x+nh)] = 0$

is written in subscript form as φ [x, y_x, y_{x+1}, y_{x+2}, ……, y_{x+n}] = 0.

Note: Difference equation φ[x, y_x, y_{x+1}, y_{x+2}, ……, y_{x+n}] = 0 is also expressed as φ[x, y_x, Ey_x, E^2y_x, ……, E^ny_x] = 0.

Linear Difference Equation: An equation of the form

$a_0(x)E^ny_x+a_1(x)E^{n-1}y_x + a_2(x)E^{n-2}y_x+ \ldots\ldots +a_n(x)y_x = R(x)$ i.e. $\Phi(E)y_x = R(x)$ …(1)

where $\Phi(E) = a_0(x)E^n + a_1(x)E^{n-1} + a_2(x)E^{n-2} + \ldots\ldots + a_n(x)$, $a_0(x) \neq 0$ and $a_i(x)$ (i = 0, 1, 2, ……) are constants, then the equation (1) is called a linear difference equation with constant coefficients.

Non-Linear Difference Equation: If a difference equation is not of the form $\Phi(E)y_x = R(x)$, then the equation it is called a non-linear difference equation.

e. g. i) $(E^3 - 6E^2 + 12)\, y_x = 0$ is a linear difference equation with constant coefficients.

ii) $(x E^2 - xE + 4)\, y_x = 4x + 1$ is a linear difference equation with variable coefficients.

iii) $y_x^2 + y_x y_{x+1}$=10x is a non-linear difference equation.

Formulation of Difference Equation: From the general solution of a difference equation which contain k arbitrary constants, to find a difference equation, we operate Δ, k times on this G.S. and eliminate these arbitrary constants.

Note: In this unit we take interval difference h =1

i.e. Ef(x) = f(x+1) & Δf(x) = f(x+1) – f(x)

==

Ex. Given $f(x) = c.3^x + x.3^{x-1}$, find the corresponding difference equation.

Solution: Given solution $f(x) = c.3^x + x.3^{x-1}$, contain only one arbitrary constant, so we operate Δ once on this f(x), we get,

$\Delta f(x) = f(x+1) - f(x) = c.3^{x+1} + (x+1).3^x - c.3^x - x.3^{x-1}$

$= 3c.3^x + 3x.3^{x-1}+3.3^{x-1} - c.3^x - x.3^{x-1}$

$= 2c.3^x + (2x+3)3^{x-1}$

$= 2[f(x)- x.3^{x-1}] + (2x+3)3^{x-1}$ from given equation $c.3^x = f(x)- x.3^{x-1}$

$\therefore f(x+1) - f(x) = 2f(x) - 2x.3^{x-1}+ 2x.3^{x-1}+3^x$

i. e. $f(x+1) - 3f(x) = 3^x$ be the required difference equation.

==

Ex. Given $u_x = c_1 2^x + c_2 3^x + \frac{1}{2}$, find the corresponding difference equation.

Solution: Given solution $u_x = c_1 2^x + c_2 3^x + \frac{1}{2}$ contain two arbitrary constants, so we operate Δ twice on this u_x, we get,

$\Delta u_x = u_{x+1} - u_x = c_1 2^{x+1} + c_2 3^{x+1} + \frac{1}{2} - c_1 2^x - c_2 3^x - \frac{1}{2}$

$= 2c_1 2^x + 3c_2 3^x - c_1 2^x - c_2 3^x$

$= c_1 2^x + 2c_2 3^x$ …… (i)

$\Delta^2 u_x = c_1 2^{x+1} + 2c_2 3^{x+1} - c_1 2^x - 2c_2 3^x$

$= 2c_1 2^x + 6c_2 3^x - c_1 2^x - 2c_2 3^x$

$= c_1 2^x + 4c_2 3^x$ …… (ii)

Now equation (ii) – (i) gives,

$\Delta^2 u_x - \Delta u_x = c_1 2^x + 4c_2 3^x - c_1 2^x - 2c_2 3^x = 2c_2 3^x$

From (i), we get,

$\Delta u_x = c_1 2^x + \Delta^2 u_x - \Delta u_x$ i.e. $c_1 2^x = 2\Delta u_x - \Delta^2 u_x$

Hence from given equation, we have,

$u_x = 2\Delta u_x - \Delta^2 u_x + \frac{1}{2}(\Delta^2 u_x - \Delta u_x) + \frac{1}{2}$

$= \frac{3}{2}\Delta u_x - \frac{1}{2}\Delta^2 u_x + \frac{1}{2}$

$= \frac{3}{2}(u_{x+1} - u_x) - \frac{1}{2}(u_{x+2} - 2u_{x+1} + u_x) + \frac{1}{2}$

$\therefore 2u_x = 3u_{x+1} - 3u_x - u_{x+2} + 2u_{x+1} - u_x + 1$

$\therefore u_{x+2} - 5u_{x+1} + 6u_x = 1$ be the required difference equation.

==

Ex. Form the difference equation corresponding to the family of curves $y = ax^2 + bx - 3$,

Solution: Given family of curves $y_x = ax^2 + bx - 3$ contain two arbitrary constants,

so we operate Δ twice on this y_x, we get,

$\Delta y_x = y_{x+1} - y_x = a(x+1)^2 + b(x+1) - 3 - ax^2 - bx + 3$

$= 2ax + a + b$ …… (i)

$\Delta^2 y_x = 2a(x+1) + a + b - 2ax - a - b$

$= 2a$

$\therefore a = \frac{1}{2}\Delta^2 y_x$

Putting in (i), we get,

$\Delta y_x = x\Delta^2 y_x + \frac{1}{2}\Delta^2 y_x + b$

$\therefore b = \Delta y_x - x\Delta^2 y_x - \frac{1}{2}\Delta^2 y_x$

Hence from given equation, we have,

$y_x = x^2 \frac{1}{2} \Delta^2 y_x + x(\Delta y_x - x\Delta^2 y_x - \frac{1}{2} \Delta^2 y_x) - 3$

$\therefore 2y_x = x^2 \Delta^2 y_x + 2x\Delta y_x - 2x^2 \Delta^2 y_x - x\Delta^2 y_x - 6$

$\therefore 2y_x = -(x^2 + x) \Delta^2 y_x + 2x\Delta y_x - 6$

$\therefore 2y_x = -(x^2 + x)(y_{x+2} - 2y_{x+1} + y_x) + 2x(y_{x+1} - y_x) - 6$

$\therefore 2y_x = -(x^2 + x) y_{x+2} + 2(x^2 + x) y_{x+1} - (x^2 + x) y_x + 2xy_{x+1} - 2xy_x - 6$

$\therefore 2y_x = -(x^2 + x) y_{x+2} + 2(x^2 + 2x) y_{x+1} - (x^2 + 3x) y_x - 6$

$\therefore (x^2 + x) y_{x+2} - 2(x^2 + 2x) y_{x+1} + (x^2 + 3x + 2) y_x + 6 = 0$

be the required difference equation.

==

Ex. Form the difference equation given that $y_n = A3^n + B5^n$, where A and B are arbitrary constants.

Solution: Given equation $y_n = A3^n + B5^n$ …… (i)

$\therefore y_{n+1} = A3^{n+1} + B5^{n+1} = 3A3^n + 5B5^n$ …… (ii)

& $y_{n+2} = A3^{n+2} + B5^{n+2} = 9A3^n + 25B5^n$ …… (iii)

Eliminating A and B from equations (i), (ii), (iii), we get,

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & 5 \\ y_{n+2} & 9 & 25 \end{vmatrix} = 0$$

i.e. $y_n(-25y_{n+1} + 5y_{n+2} + 9y_{n+1} - 3y_{n+2} = 0$

i.e. $2y_{n+2} - 16y_{n+1} + 30y_n = 0$

i.e. $y_{n+2} - 8y_{n+1} + 15y_n = 0$

be the required difference equation.

==

Ex. Form the difference equation corresponding to the following general solution:

a) $y = c_1x^2 + c_2x + c_3$ b) $y = (c_1 + c_2n)(-2)^n$

Solution: a) Given solution $y_x = c_1x^2 + c_2x + c_3$ …… (1)

contain three arbitrary constants c_1, c_2 and c_3, so we operate Δ thrice on this y_x, we get

$\Delta y_x = y_{x+1} - y_x = c_1(x+1)^2 + c_2(x+1) + c_3 - c_1x^2 - c_2x - c_3$

$= 2c_1x + c_1 + c_2$ …… (2)

$\Delta^2 y_x = [2c_1(x+1) + c_1 + c_2] - [2c_1x + c_1 + c_2]$

$= 2c_1$ …… (3)

& $\Delta^3 y_x = 2c_1 - 2c_1$

$\therefore$ $(E - 1)^3 y_x = 0$

$\therefore$ $(E^3 - 3E^2 + 3E - 1)y_x = 0$

$\therefore$ $y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x = 0$ be the required difference equation.

b) Given solution $y_n = (c_1 + c_2 n)(-2)^n$ i.e. $y_n = c_1(-2)^n + c_2 n(-2)^n$ …… (1)

contain two arbitrary constants c_1 and c_2.

$\therefore$ $y_{n+1} = c_1(-2)^{n+1} + c_2(n+1)(-2)^{n+1} = -2c_1(-2)^n - 2c_2(n+1)(-2)^n$ …… (ii)

& $y_{n+2} = c_1(-2)^{n+2} + c_2(n+2)(-2)^{n+2} = 4c_1(-2)^n + 4c_2(n+2)(-2)^n$ …… (iii)

Eliminating c_1 and c_2 from equations (i), (ii), (iii), we get,

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -2 & -2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

i.e. $y_n[-8n-16+8n+8] - y_{n+1}[4n+8-4n] + y_{n+2}[-2n-2+2n] = 0$

i.e. $-2y_{n+2} - 8y_{n+1} - 8y_n = 0$

i.e. $y_{n+2} + 4y_{n+1} + 4y_n = 0$

be the required difference equation.

==

Ex. Find the order of the difference equation $y_{x+2} - 7y_x = 5$

Solution: Given difference equation is $y_{x+2} - 7y_x = 5$

Here difference between the highest subscript and lowest subscript = x+2 – x = 2

$\therefore$ order of given difference equation is 2.

==

Ex. Find the order of the difference equation $y_{x+4} - 5y_{x+2} + 6y_x = 0$.

Solution: Given difference equation is $y_{x+4} - 5y_{x+2} + 6y_x = 0$.

Here difference between the highest subscript and lowest subscript = x+4 – x = 4

$\therefore$ order of given difference equation is 4.

==

Ex. Find the order of the difference equation $\Delta^3 y_x + 2\Delta y_x + y_x = x + 3$.

Solution: Given difference equation is $\Delta^3 y_x + 2\Delta y_x + y_x = x + 3$

i.e. $(E-1)^3 y_x + 2(E-1)y_x + y_x = x + 3$

i.e. $(E^3 - 3E^2 + 3E - 1)y_x + (2E-2)y_x + y_x = x + 3$

i.e. $y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x + 2y_{x+1} - 2y_x + y_x = x + 3$

Here difference between the highest subscript and lowest subscript = x+3 – x = 3

∴ order of given difference equation is 3.

==

Ex. Show that $y_x = \frac{x(x-1)}{2}$ is a solution of the difference equation $y_{x+1} - y_x = x$.

Proof: We have $y_x = \frac{x(x-1)}{2}$

∴ $y_{x+1} = \frac{(x+1)x}{2}$

Consider

LHS = $y_{x+1} - y_x$

$= \frac{(x+1)x}{2} - \frac{x(x-1)}{2}$

$= \frac{x}{2}[x+1-x+1]$

$= x$

= RHS

∴ $y_x = \frac{x(x-1)}{2}$ is a solution of the given difference equation is proved.

==

Ex. Show that $y_x = 1 - \frac{2}{x}$, x = 1, 2, 3, …. is a solution of the first order difference equation $(x+1)y_{x+1} + xy_x = 2x - 3$, x = 1, 2, 3, …

Proof: We have $y_x = 1 - \frac{2}{x}$, x = 1, 2, 3, ….

∴ $y_{x+1} = 1 - \frac{2}{x+1}$

Consider

LHS = $(x+1)y_{x+1} + xy_x$

$= (x+1)\left(1 - \frac{2}{x+1}\right) + x\left(1 - \frac{2}{x}\right)$

$= x + 1 - 2 + x - 2$

$= 2x - 3$

= RHS

∴ $y_x = 1 - \frac{2}{x}$, x = 1, 2, 3, …. is a solution of the given difference equation is proved.

==

Ex. Show that $y_x = c_1 + c_2\, 2^x - x$ is a solution of the difference equation

$y_{x+2} - 3y_{x+1} + 2y_x = 1$

Proof: We have $y_x = c_1 + c_2 2^x - x$

$\therefore y_{x+1} = c_1 + c_2 2^{x+1} - (x+1) = c_1 + 2c_2 2^x - x - 1$

& $y_{x+2} = c_1 + c_2 2^{x+2} - (x+2) = c_1 + 4c_2 2^x - x - 2$

Consider

LHS $= y_{x+2} - 3y_{x+1} + 2y_x$

$= c_1 + 4c_2 2^x - x - 2 - 3[c_1 + 2c_2 2^x - x - 1] + 2[c_1 + c_2 2^x - x]$

$= c_1 + 4c_2 2^x - x - 2 - 3c_1 - 6c_2 2^x + 3x + 3 + 2c_1 + 2c_2 2^x - 2x$

$= 1$

= RHS

$\therefore y_x = c_1 + c_2 2^x - x$ is a solution of the given difference equation is proved.

==

Second Order Homogenous Difference Equations: If $a_2 \neq 0$, then $y_{x+2} + a_1 y_{x+1} + a_2 y_x = 0$ is called second order homogenous difference equation.

General Homogenous Difference Equations:

If $a_n \neq 0$, then $y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \ldots\ldots + a_{n-1} y_{x+1} + a_n y_x = 0$ is called n^{th} order homogenous difference equation.

Auxiliary Equations: When $y_x = m^x$, the auxiliary equation of general n^{th} order homogenous difference equation is $m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_{n-1} m + a_n = 0$

Remark: i) If m_1 and m_2 are distinct real roots of an auxiliary equation $m^2 + a_1 m + a_2 = 0$ of given second order homogenous difference equation, then the solution is $y_x = c_1 m_1^x + c_2 m_2^x$.

ii) If m_1 and m_2 are equal real roots of an auxiliary equation $m^2 + a_1 m + a_2 = 0$ of given second order homogenous difference equation, then the solution is $y_x = (c_1 + c_2 x) m_1^x$.

iii) If $m = \alpha \pm i\beta$ are the complex roots of an auxiliary equation $m^2 + a_1 m + a_2 = 0$ of given second order homogenous difference equation, then the solution is $y_x = \rho^x (c_1 \cos x\theta + c_2 \sin x\theta)$ where $\rho = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\frac{\beta}{\alpha})$ and c_1, c_2 are constants.

iv) If $m_1, m_2, \ldots, m_n$ are distinct real roots of an auxiliary equation of given n^{th} order homogenous difference equation, then the solution is

$y_x = c_1 m_1^x + c_2 m_2^x + \ldots\ldots + c_n m_n^x$

v) If $m_1, m_2, \ldots, m_k$ are equal real roots of an auxiliary equation of given n^{th}

order homogenous difference equation repeated k times, then the solution is

$y_x = (c_1 + c_2 x + \cdots + c_k x^{k-1}) m_1^x$.

Ex. Solve the difference equation $y_{x+3} - 3y_{x+2} - 10y_{x+1} + 24y_x = 0$.

Solution: Given difference equation is $y_{x+3} - 3y_{x+2} - 10y_{x+1} + 24y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$m^3 - 3m^2 - 10m + 24 = 0$

$(m - 2)(m + 3)(m - 4) = 0$

$\therefore$ m = 2, -3, 4 are the roots of an A.E.

Thus, the G.S. is

$y_x = C_1 2^x + C_2(-5)^x + C_3 4^x$.

Ex. Solve the difference equation $y_{x+2} - 7y_{x+1} + 12y_x = 0$.

Solution: Given difference equation is $y_{x+2} - 7y_{x+1} + 12y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$m^2 - 7m + 12 = 0$

$(m - 3)(m - 4) = 0$

$\therefore$ m = 3, 4 are the roots of an A.E.

Thus, the G.S. is

$y_x = C_1 3^x + C_2 4^x$

Ex. Solve the difference equation $y_{x+4} - 4y_{x+3} + 6y_{x+2} - 4y_{x+1} + y_x = 0$.

Solution: Given difference equation is $y_{x+4} - 4y_{x+3} + 6y_{x+2} - 4y_{x+1} + y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$m^4 - 4m^3 + 6m^2 - 4m + 1 = 0$

$(m - 1)^4 = 0$

$\therefore$ m = 1, 1, 1, 1 are the roots of an A.E.

Thus, the G.S. is

$y_x = (C_1 + C_2 x + C_3 x^2 + C_4 x^3).1^x = C_1 + C_2 x + C_3 x^2 + C_4 x^3$

Ex. Solve the difference equation $y_{x+4} - 8y_{x+3} + 18y_{x+2} - 27\, y_x = 0$.

Solution: Given difference equation is $y_{x+4} - 8y_{x+3} + 18y_{x+2} - 27\, y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$m^4 - 8m^3 + 18m^2 - 27 = 0$

$(m+1)(m-3)^3 = 0$ $\therefore$ m = -1, 3, 3, 3 are the roots of an A.E.

Thus, the G.S. is

$y_x = C_1(-1)^x + (C_2 + C_3x + C_4x^2).3^x$

Ex. Solve the difference equation $y_{x+3} + y_{x+2} - 8y_{x+1} - 12y_x = 0$.

Solution: Given difference equation is $y_{x+3} + y_{x+2} - 8y_{x+1} - 12y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$m^3 + m^2 - 8m - 12 = 0$

$(m - 3)(m^2 + 4m + 4) = 0$

$(m - 3)(m + 2)^2 = 0$

$\therefore$ m = 3, -2, -2 are the roots of an A.E.

Thus, the G.S. is

$y_x = C_1 3^x + (C_2 + C_3x).(-2)^x$

==

Ex. Solve the difference equation $2y_{x+2} - 5y_{x+1} + 2y_x = 0$. Also find the particular solution satisfying the initial conditions $y_0 = 0$ and $y_1 = 1$.

Solution: Given difference equation is $2y_{x+2} - 5y_{x+1} + 2y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$2m^2 - 5m + 2 = 0$

$(2m - 1)(m - 2) = 0$

$\therefore m = \frac{1}{2}, 2$ are the roots of an A.E.

Thus, the G.S. is

$y_x = C_1(\frac{1}{2})^x + C_2 2^x$

When x = 0 and x = 1, we get,

$y_0 = C_1 + C_2 = 0$ and $y_1 = \frac{1}{2}C_1 + 2C_2 = 1$

Solving these, we get $C_1 = -\frac{2}{3}$ and $C_2 = \frac{2}{3}$

$\therefore$ The particular solution is $y_x = -\frac{2}{3}(\frac{1}{2})^x + \frac{2}{3}2^x$

==

Ex. Solve the difference equation $9y_{x+2} - 6y_{x+1} + y_x = 0$. Also find the particular solution satisfying the initial conditions $y_0 = 0$ and $y_1 = 1$.

Solution: Given difference equation is $9y_{x+2} - 6y_{x+1} + y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$9m^2 - 6m + 1 = 0$

$(3m - 1)^2 = 0$

$\therefore m = \frac{1}{3}, \frac{1}{3}$ are the roots of an A.E.

Thus, the G.S. is

$y_x = (C_1 + C_2x)\left(\frac{1}{3}\right)^x$

When x = 0 and x = 1, we get,

$y_0 = C_1 = 0$ and $y_1 = \frac{1}{3}(C_1 + C_2) = 1$

Solving these, we get $C_1 = 0$ and $C_2 = 3$

$\therefore$ The particular solution is $y_x = 2\left(\frac{1}{3}\right)^x x$

==

Ex. Solve the difference equation $y_{x+2} + y_x = 0$ with $y_0 = 0$ and $y_1 = 1$.

Solution: Given difference equation is $y_{x+2} + y_x = 0$.

When we take $y_x = m^x$, the A.E.is

$m^2 + 1 = 0$

$\therefore m = \pm i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ are the roots of an A.E. with $\rho = 1$ and $\theta = \frac{\pi}{2}$

Thus, the G.S. is

$y_x = C_1\cos\frac{\pi}{2}x + C_2\sin\frac{\pi}{2}x$

When x = 0 and x = 1, we get,

$y_0 = C_1 = 0$ and $y_1 = C_2 = 1$

$\therefore$ The particular solution is $y_x = \sin\frac{\pi}{2}x$

==

Ex. Solve $y_{x+1} - 2y_x\cos\alpha + y_{x-1} = 0$.

Solution: Given difference equation is $y_{x+1} - 2y_x\cos\alpha + y_{x-1} = 0$.

When we take $y_{x-1} = m^x$, the A.E.is

$m^2 - 2m\cos\alpha + 1 = 0$

$\therefore m = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} = \cos\alpha \pm i\sin\alpha$ are the roots of an A.E.

with $\rho = 1$ and $\theta = \alpha$

Thus, the G.S. is

$y_x = C_1\cos\alpha x + C_2\sin\alpha x$

==

Ex. Solve the difference equation $3y_{x+2} – 6y_{x+1} + 4y_x = 0$. Also find the particular solution satisfying the initial conditions $y_0 = 0$ and $y_1 = 1$.

Solution: Given difference equation is $3y_{x+2} – 6y_{x+1} + 4y_x = 0$.

When we take $y_x = m^x$, the A.E. is

$3m^2 - 6m + 4 = 0$

$\therefore m = \frac{6\pm\sqrt{36-48}}{6} = 1 \pm \frac{1}{\sqrt{3}}i = \frac{2}{\sqrt{3}}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ are the roots of an A.E.

with $\rho = \frac{2}{\sqrt{3}}$ and $\theta = \frac{\pi}{6}$

Thus, the G.S. is

$y_x = (\frac{2}{\sqrt{3}})^x (C_1\cos\frac{\pi}{6}x + C_2\sin\frac{\pi}{6}x)$

==

Fibonacci Sequence: A sequence of type 0, 1, 1, 2, 3, 5, 8, ……. is called Fibonacci sequence. which is formulated in difference equation form as

$y_{x+1} = y_x + y_{x-1}$ i.e. $y_{x+2} - y_{x+1} - y_x = 0$ i.e. $(E^2 - E – 1)y_x = 0$ with $y_0 = 0$ and $y_1 = 1$ is called Fibonacci difference equation.

Formulation of Fibonacci difference equation:

Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ……. is formulated in difference equation form as $y_{x+1} = y_x + y_{x-1}$ i.e. $y_{x+2} - y_{x+1} - y_x = 0$ with $y_0 = 0$ and $y_1 = 1$

Method of solving Fibonacci difference equation:

Let. $y_{x+2} - y_{x+1} - y_x = 0$ i.e. $(E^2 - E – 1)y_x = 0$ with $y_0 = 0$ and $y_1 = 1$ be the Fibonacci difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 – m – 1 = 0$

$\therefore m = \frac{1\pm\sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ are the roots of an A.E.

$\therefore$ The G. S. of the given Fibonacci difference equation is

$y_x = c_1(\frac{1}{2} + \frac{\sqrt{5}}{2})^x + c_2(\frac{1}{2} - \frac{\sqrt{5}}{2})^x$

i.e. $y_x = \frac{1}{2^x}[c_1(1 + \sqrt{5})^x + c_2(1 - \sqrt{5})^x]$

Now $y_0 = 0$ and $y_1 = 1$ gives

$0 = c_1 + c_2$ …… (i) and

$1 = \frac{1}{2}[c_1(1+\sqrt{5}) + c_2(1-\sqrt{5})]$

$= \frac{1}{2}[c_1 + c_1\sqrt{5} + c_2 - c_2\sqrt{5}]$

$1 = \frac{\sqrt{5}}{2}[c_1 - c_2]$

i.e. $c_1 - c_2 = \frac{2}{\sqrt{5}}$ …… (ii)

Adding equation (i) and (ii), we get,

$2c_1 = \frac{2}{\sqrt{5}}$ i.e. $c_1 = \frac{1}{\sqrt{5}}$

Putting in (i), we get, $c_2 = -\frac{1}{\sqrt{5}}$

∴ Required particular solution of Fibonacci difference equation is

$y_x = \frac{1}{2^x}[\frac{1}{\sqrt{5}}(1+\sqrt{5})^x - \frac{1}{\sqrt{5}}(1-\sqrt{5})^x]$

i.e. $y_x = \frac{1}{\sqrt{5}}[(1+\sqrt{5})^x - (1-\sqrt{5})^x].2^{-x}$

===

Theorem: If $y_x^{(1)}, y_x^{(2)}, \ldots, y_x^{(n)}$ are any n solutions of n^{th} order homogeneous linear difference equation with constant coefficients $(a_0E^n + a_1E^{n-1} + \ldots + a_n)y_x = 0$,
i.e. $a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots + a_{n-1}y_{x+1} + a_ny_x = 0$,
then combination $\lambda_1 y_x^{(1)} + \lambda_2 y_x^{(2)} + \ldots + \lambda_n y_x^{(n)}$ is also solution,
where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary constants.

Proof: As $y_x^{(1)}, y_x^{(2)}, \ldots, y_x^{(n)}$ are any n solutions of the given homogeneous linear difference equation $a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots + a_{n-1}y_{x+1} + a_ny_x = 0$

∴ $a_0y_{x+n}^{(1)} + a_1y_{x+n-1}^{(1)} + a_2y_{x+n-2}^{(1)} + \ldots + a_{n-1}y_{x+1}^{(1)} + a_ny_x^{(1)} = 0$ …… (1)

$a_0y_{x+n}^{(2)} + a_1y_{x+n-1}^{(2)} + a_2y_{x+n-2}^{(2)} + \ldots + a_{n-1}y_{x+1}^{(2)} + a_ny_x^{(2)} = 0$ …… (2)

… … … … …

… … … … …

… … … … …

$a_0y_{x+n}^{(n)} + a_1y_{x+n-1}^{(n)} + a_2y_{x+n-2}^{(n)} + \ldots + a_{n-1}y_{x+1}^{(n)} + a_ny_x^{(n)} = 0$ …… (n)

Multiplying equation (1) by λ_1, (2) by λ_2, …., (n) by λ_n and adding we get

$\lambda_1[a_0y_{x+n}^{(1)} + a_1y_{x+n-1}^{(1)} + a_2y_{x+n-2}^{(1)} + \ldots + a_{n-1}y_{x+1}^{(1)} + a_ny_x^{(1)}]$

$+ \lambda_2[a_0y_{x+n}^{(2)} + a_1y_{x+n-1}^{(2)} + a_2y_{x+n-2}^{(2)} + \ldots + a_{n-1}y_{x+1}^{(2)} + a_ny_x^{(2)}] + \ldots$

… … … … …

$+[a_0y_{x+n}^{(n)} + a_1y_{x+n-1}^{(n)} + a_2y_{x+n-2}^{(n)} + \ldots + a_{n-1}y_{x+1}^{(n)} + a_ny_x^{(n)}] = 0$

i.e. $a_0[\lambda_1 y_{x+n}^{(1)} + \lambda_2 y_{x+n}^{(2)} + \ldots + \lambda_n y_{x+n}^{(n)}] + a_1[\lambda_1 y_{x+n-1}^{(1)} + \lambda_2 y_{x+n-1}^{(2)} + \ldots + \lambda_n y_{x+n-1}^{(n)}]$

$+ a_2[\lambda_1 y_{x+n-2}^{(1)} + \lambda_2 y_{x+n-2}^{(2)} + \ldots + \lambda_n y_{x+n-2}^{(n)}] + \ldots\ldots$

$+ a_n[\lambda_1 y_x^{(1)} + \lambda_2 y_x^{(2)} + \ldots + \lambda_n y_x^{(n)}] = 0$

$\therefore \lambda_1 y_x^{(1)} + \lambda_2 y_x^{(2)} + \ldots + \lambda_n y_x^{(n)}$ is solution of given difference equation is proved.

==

Theorem: If Y is a solutions of n^{th} order homogeneous linear difference equation with constant coefficients $a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots\ldots + a_ny_x = 0$, and Y^* is a solutions of non-homogeneous linear difference equation with constant coefficients $a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots\ldots + a_ny_{kx} = R_x$ then $Y+Y^*$ is a solution of $a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots\ldots + a_ny_x = R_x$

Proof: As Y is a solutions of the given homogeneous linear difference equation

$a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots\ldots + a_ny_x = 0$

$\therefore a_0Y_{x+n} + a_1Y_{x+n-1} + a_2Y_{x+n-2} + \ldots\ldots + a_nY_x = 0 \ldots\ldots (1)$

Also Y^* is a solutions of the non-homogeneous linear difference equation

$a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots\ldots + a_ny_x = R_x$

$\therefore a_0Y_{x+n}^* + a_1Y_{x+n-1}^* + a_2Y_{x+n-2}^* + \ldots\ldots + a_nY_x^* = R_x \ldots\ldots (2)$

Adding equation (1) & (2), we get

$a_0Y_{x+n} + a_1Y_{x+n-1} + a_2Y_{x+n-2} + \ldots\ldots + a_nY_x$

$+a_0Y_{x+n}^* + a_1Y_{x+n-1}^* + a_2Y_{x+n-2}^* + \ldots\ldots + a_nY_x^* = R_x$

$a_0(Y_{x+n}+Y_{x+n}^*) + a_1(Y_{x+n-1}+Y_{x+n-1}^*) + a_2(Y_{x+n-2}+Y_{x+n-2}^*) + \ldots\ldots + a_n(Y_x+Y_x^*) = R_x$

Hence $Y+Y^*$ is a solution of $a_0y_{x+n} + a_1y_{x+n-1} + a_2y_{x+n-2} + \ldots\ldots + a_ny_x = R_x$ is proved.

==

Non-Homogenous Linear Difference Equations: If $a_2 \neq 0$, then $y_{x+2} + a_1y_{x+1} + a_2y_x = f(x)$ is called second order homogenous linear difference equation.

General Non-Homogenous Difference Equations:

If $a_0 \neq 0$, then $(a_0E^n + a_1E^{n-1} + a_2E^{n-2} + \ldots\ldots + a_n)y_x = f(x)$ i.e. $\Phi(E)y_x = f(x)$ is called n^{th} order non-homogenous linear difference equation.

Remark: i) If $\Phi(a) \neq 0$, then particular solution of non-homogenous linear difference equation $\Phi(E)y_x = a^x$ is $\frac{1}{\Phi(E)}a^x = \frac{a^x}{\Phi(a)}$

ii) If $\Phi(a) = 0$ i.e. $\Phi(E) = (E-a)^n\psi(E)$ with $\psi(a) \neq 0$, then particular solution of non-homogenous linear difference equation $\Phi(E)y_x = a^x$ is

$\frac{1}{\Phi(E)}a^x = \frac{1}{(E-a)^n\psi(E)}a^x = \frac{x(x-1)(x-2)\ldots(x-n+1)a^{x-n}}{n!\psi(a)}$

iii) If non-homogenous linear difference equation is of type $\Phi(E)y_x = f(x)$, where f(x) is polynomial in x of degree r, then its particular solution is

$\frac{1}{\Phi(E)}f(x) = \frac{1}{\Phi(1+\Delta)} f(x)$

We expand $\frac{1}{\Phi(1+\Delta)}$ in ascending powers of Δ and operate on f(x).

iv) If non-homogenous linear difference equation is of type $\Phi(E)y_x = a^x f(x)$, then its particular solution is $\frac{1}{\Phi(E)}a^x f(x) = a^x\frac{1}{\Phi(aE)} f(x)$

v) If non-homogenous linear difference equation is of type $\Phi(E)y_x = \cos ax$, then its particular solution is $\frac{1}{\Phi(E)}\cos ax$ = Real part of $\frac{1}{\Phi(E)} e^{iax}$

vi) If non-homogenous linear difference equation is of type $\Phi(E)y_x = \sin ax$, then its particular solution is $\frac{1}{\Phi(E)}\sin ax$ = Imaginary part of $\frac{1}{\Phi(E)} e^{iax}$

==

Ex. Solve the following difference equations:

a) $y_{x+1} - 3y_x = 1$ b) $y_{x+1} - 3y_x = 0,\ y_0 = 2$

Solution: a) Let $y_{x+1} - 3y_x = 1$ i.e. $(E - 3)\, y_x = 1$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E.is

$m - 3 = 0$

∴ m = 3 is the roots of an A. E.

∴ The G. S. of reduced homogeneous difference equation is

$y_x = c\, 3^x$

Now particular solution of given non-homogeneous equation is

P.S. $= \frac{1}{(E-3)}1$

$= \frac{1}{(E-3)}1^x$

$= \frac{1}{(1-3)}$

$= -\frac{1}{2}$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = c\,3^x - \frac{1}{2}$

b) Let $y_{x+1} - 3y_x = 0$ i.e. $(E - 3)\,y_x = 0$

be the given homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E.is

$m - 3 = 0$

∴ m = 3 is the roots of an A. E.

∴ The G. S. of given homogeneous difference equation is

$y_x = c3^x$

Now $y_0 = 2$ gives $c3^0 = 2$ i.e. $2 = c$

Hence particular solution of given equation is

$y_x = 2.3^x$

==

Ex. Solve the following equation $y_{x+2} - 3y_{x+1} + 2y_x = 1$.

Solution: Let $y_{x+2} - 3y_{x+1} + 2y_x = 1$

i.e. $(E^2 - 3E + 2)y_x = 1$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 3m + 2 = 0$

$(m - 1)\,(m - 2) = 0$

∴ m = 1, 2 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 + C_2 2^x$

Now particular solution of given non-homogeneous equation is

$$\text{P.S.} = \frac{1}{(E^2-3E+2)}1$$

$$= \frac{1}{(E-1)(E-2)}1^x$$

$$= \frac{x}{(1-2)}$$

$$= -x$$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 + C_2 2^x - x$

==

Ex. Solve $y_{x+2} - 3y_{x+1} + 2y_x = a^x$, where a is some constant

Solution: Let $y_{x+2} - 3y_{x+1} + 2y_x = a^x$

i.e. $(E^2 - 3E + 2)y_x = a^x$, where a is some constant

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 3m + 2 = 0$

$(m - 1)(m - 2) = 0$

$\therefore$ m = 1, 2 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 + C_2 2^x$

Now particular solution of given non-homogeneous equation is

$$\text{P.S.} = \frac{1}{(E^2 - 3E + 2)} a^x$$

$$= \frac{1}{(E-1)(E-2)} a^x$$

$$= \frac{a^x}{(a-1)(a-2)} \text{ when } a \neq 1 \text{ and } a \neq 2$$

If a = 1, then $\text{P.S.} = \frac{1}{(E^2-3E+2)}1^x = \frac{1}{(E-1)(E-2)}1^x = \frac{x1^{x-1}}{1!(1-2)} = -x$

If a = 2, then $\text{P.S.} = \frac{1}{(E^2-3E+2)}2^x = \frac{1}{(E-1)(E-2)}2^x = \frac{x2^{x-1}}{1!(2-1)} = x2^{x-1}$

Hence complete solution of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 + C_2 2^x + \frac{a^x}{(a-1)(a-2)}$ when $a \neq 1$ and $a \neq 2$

$y_x = C_1 + C_2 2^x - x$ when $a = 1$

$y_x = C_1 + C_2 2^x + x2^{x-1}$ when $a = 2$

==

Ex. Solve $y_{x+2} - 4y_{x+1} + 4y_x = 3^x + 2^x + 4$.

Solution: Let $y_{x+2} - 4y_{x+1} + 4y_x = 3^x + 2^x + 4$.

i.e. $(E^2 - 4E + 4)y_x = 3^x + 2^x + 4$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 4m + 4 = 0$

$(m - 2)^2 = 0$

$\therefore$ m = 2, 2 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = (C_1 + C_2x)\, 2^x$

Now particular solution of given non-homogeneous equation is

P.S. $= \frac{1}{(E^2-4E+4)}(3^x + 2^x + 4)$

$= \frac{1}{(E-2)^2}(3^x + 2^x + 4.1^x)$

$= \frac{1}{(E-2)^2}(3^x) + \frac{1}{(E-2)^2}(2^x) + \frac{1}{(E-2)^2}(4.1^x)$

$= \frac{3^x}{(3-2)^2} + \frac{x(x-1)2^{x-2}}{2!} + \frac{4.1^x}{(1-2)^2}$

` $= 3^x + x(x-1)2^{x-3} + 4$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = (C_1 + C_2x)\, 2^x + 3^x + x(x-1)2^{x-3} + 4$

===

Ex. Solve $y_{x+2} - 4y_{x+1} + 3y_x = 3^x + 1$.

Solution: Let $y_{x+2} - 4y_{x+1} + 3y_x = 3^x + 1$.

i.e. $(E^2 - 4E + 3)y_x = 3^x + 1$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 4m + 3 = 0$

$(m - 1)(m - 3) = 0$

$\therefore$ m = 1, 3 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 + C_2 3^x$

Now particular solution of given non-homogeneous equation is

P.S. $= \frac{1}{(E^2-4E+3)}(3^x + 1)$

$= \frac{1}{(E-1)(E-3)}(3^x + 1^x)$

$= \frac{1}{(E-1)(E-3)}(3^x) + \frac{1}{(E-1)(E-3)}(1^x)$

$= \frac{x3^{x-1}}{1!(3-1)} + \frac{x1^{x-1}}{1!(1-3)}$

` $= \frac{x3^{x-1}}{2} - \frac{x}{2}$

$= \frac{1}{2}x(3^{x-1}-1)$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 + C_2 3^x + \frac{1}{2}x(3^{x-1}-1)$

===

Ex. Solve $y_{x+2} - 4y_{x+1} + 4y_x = 3x + 2^x$

Solution: Let $y_{x+2} - 4y_{x+1} + 4y_x = 3x + 2^x$

i.e. $(E^2 - 4E + 4)y_x = 3x + 2^x$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E.is

$m^2 - 4m + 4 = 0$

$(m - 2)^2 = 0$

$\therefore$ m = 2, 2 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = (C_1 + C_2x)\,2^x$

Now particular solution of given non-homogeneous equation is

$$\text{P.S.} = \frac{1}{(E^2-4E+4)}(3x + 2^x)$$

$$= \frac{1}{(E-2)^2}(3x + 2^x)$$

$$= \frac{1}{(1+\Delta-2)^2}(3x) + \frac{1}{(E-2)^2}(2^x)$$

$$= \frac{3}{(\Delta-1)^2}x + \frac{x(x-1)2^{x-2}}{2!}$$

` $$= 3(1-\Delta)^{-2}x + \frac{x(x-1)2^{x-2}}{2}$$

$$= 3(1+2\Delta+\cdots)x + x(x-1)2^{x-3}$$

$$= 3(x+2(x+1-x)+\cdots) + x(x-1)2^{x-3}$$

$$= 3x + 6 + x(x-1)2^{x-3}$$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = (C_1 + C_2x)\,2^x + 3x + 6 + x(x-1)2^{x-3}$

===

Ex. Solve $u_{x+2} - 5u_{x+1} + 6u_x = 36$

Solution: Let $u_{x+2} - 5u_{x+1} + 6u_x = 36$

i.e. $(E^2 - 5E + 6)u_x = 36$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

∴ m = 2, 3 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 2^x + C_2 3^x$

Now particular solution of given non-homogeneous equation is

$$\text{P.S.} = \frac{1}{(E^2 - 5E + 6)}(36)$$

$$= \frac{36}{(E-2)(E-3)}(1^x)$$

$$= \frac{36}{(1-2)(1-3)}$$

$$= 18$$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 2^x + C_2 3^x + 18$

==

Ex. Solve $y_{x+2} - 5y_{x+1} + 6y_x = 2$. Also find the solution satisfying the initial conditions $y_0 = 1$ and $y_1 = -1$

Solution: Let $y_{x+2} - 5y_{x+1} + 6y_x = 2$

i.e. $(E^2 - 5E + 6)y_x = 2$

be the given non- homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

∴ m = 2, 3 are the roots of an A.E.

Thus, the G.S. of reduced homogeneous equation is

$y_x = C_1 2^x + C_2 3^x$

Now particular solution of given non-homogeneous equation is

$$\text{P.S.} = \frac{1}{(E^2 - 5E + 6)}(2)$$

$$= \frac{2}{(E-2)(E-3)}(1^x)$$

$$= \frac{2}{(1-2)(1-3)}$$

$$= 1$$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 2^x + C_2 3^x + 1$

By using initial conditions $y_0 = 1$ and $y_1 = -1$, we get

$C_1 2^0 + C_2 3^0 + 1 = 1$ i.e. $C_1 + C_2 = 0$

& $C_1 2^1 + C_2 3^1 + 1 = -1$ i.e. $2C_1 + 3C_2 = -2$

Solving we get $C_1 = 2$ and $C_2 = -2$

Hence the solution is $y_x = 2^{x+1} - 2.3^x + 1$

==

Ex. Solve the following non-homogeneous linear difference equations:

i) $y_{x+2} - 4y_x = 9x^2$ b) $\Delta y_x + \Delta^2 y_x = \sin x$

Solution: i) Let $y_{x+2} - 4y_x = 9x^2$ i.e. $(E^2 - 4)\, y_x = 9x^2$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - 4 = 0$

i.e. $(m - 2)(m + 2) = 0$

∴ m = 2, -2 are the roots of an A. E.

∴ The G. S. of reduced homogeneous difference equation is

$y_x = C_1 2^x + C_2(-2)^x$

Now particular solution of given non-homogeneous equation is

$$\text{P.S. } \frac{1}{(E^2-4)}(9x^2)$$

$$= \frac{1}{(1+\Delta)^2-4}(9x^2)$$

$$= \frac{9}{-3+2\Delta+\Delta^2}(x^2)$$

$$= \frac{-3}{[1-(\frac{2}{3}\Delta+\frac{1}{3}\Delta^2)]}(x^2)$$

`

$$= -3[1+(\frac{2}{3}\Delta + \frac{1}{3}\Delta^2) + (\frac{2}{3}\Delta + \frac{1}{3}\Delta^2)^2 + \ldots](x^2)$$

$$= -3[1+\frac{2}{3}\Delta + \frac{7}{9}\Delta^2+ \frac{4}{9}\Delta^3+ \ldots](x^2)$$

$$= -3[x^2+\frac{2}{3}(2x) + \frac{7}{9}(2) + 0]$$

$$= -3x^2 - 4x - \frac{14}{3}$$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C_1 2^x + C_2(-2)^x - 3x^2 - 4x - \frac{14}{3}$

ii) Let $\Delta y_x + \Delta^2 y_x = \sin x$

i.e. $(\Delta + \Delta^2) y_x = \sin x$

i.e. $(E - 1 + E^2 - 2E + 1) y_x = \sin x$

i.e. $(E^2 - E) y_x = \sin x$

be the given non-homogeneous linear difference equation.

When we take $y_x = m^x$, the A.E. is

$m^2 - m = 0$

i.e. $m(m - 1) = 0$

∴ m = 0, 1 are the roots of an A. E.

∴ The G. S. of reduced homogeneous difference equation is

$y_x = C_1 0^x + C_2(1)^x$

i.e. $y_x = C$, where $C_2 = C$

Now particular solution given non-homogeneous equation is

P.S. = $\frac{1}{(E^2 - E)}(\sin x)$

$= \text{Imaginary part of } \frac{1}{(E^2 - E)}(e^{ix})$

$= \text{Imaginary part of } \frac{1}{(E^2 - E)}(e^{i})^x$

$= \text{Imaginary part of } \frac{e^{ix}}{(e^{2i} - e^{i})}$

$= \text{Imaginary part of } \frac{e^{i(x-1)}}{(e^{i} - 1)}$

$= \text{Imaginary part of } \frac{e^{i(x-1)}}{(e^{i} - 1)} \text{ x } \frac{(e^{-i} - 1)}{(e^{-i} - 1)}$

$= \text{Imaginary part of } \frac{e^{i(x-2)} - e^{i(x-1)}}{(1 - e^{i} - e^{-i} + 1)}$

$= \text{Imaginary part of } \left[\frac{\cos(x-2) + i\sin(x-2) - \cos(x-1) - i\sin(x-1)}{2 - \cos 1 - i\sin 1 - \cos 1 + i\sin 1}\right]$

$= \frac{\sin(x-2) - \sin(x-1)}{2 - 2\cos 1}$

$= \frac{\sin(x-2) - \sin(x-1)}{2(1 - \cos 1)}$

Hence G.S. of given equation is y_x = G.S. + P.S.

i.e. $y_x = C + \frac{\sin(x-2) - \sin(x-1)}{2(1 - \cos 1)}$

==

MULTIPLE CHOICE QUESTIONS [MCQ'S]

1) Shift operator is denoted by E and defined as Ef(x) = ……

A) f(x-h) B) f(x) C) f(x + h) D) None of these

2) If E is a shift operator, then $E^n f(x)$ = ……

A) f(x + nh) B) f(x) C) f(x − nh) D) None of these

3) Forward difference operator is denoted by Δ and defined as Δf(x) = ……

A) f(x+h) – f(x) B) f(x) - f(x + h) C) f(x + h) + f(x) D) None of these

4) If Δ is a forward difference operator, then $\Delta^2 f(x)$ = ……

A) f(x+2h) – f(x) B) f(x + 2h) - 2f(x + h) + f(x)

C) f(x + 2h) + 2f(x + h) + f(x) D) None of these

5) If Δ is a forward difference operator, then $\Delta^3 f(x)$ = ……

A) f(x + 3h) + 3f(x + 2h) + 3f(x + h) + f(x) B) f(x + 2h) - 2f(x + h) + f(x)

C) f(x + 3h) − 3f(x + 2h) + 3f(x + h) – f(x) D) None of these

6) Relation between forward difference operator Δ and shift operator E is ……

A) Δ = E - 1 B) Δ = E + 1 C) Δ = 1 − E D) None of these

7) A relation of the form F [x, y, $\frac{\Delta y}{\Delta x}, \frac{\Delta^2 y}{\Delta x^2}$, ……, $\frac{\Delta^n y}{\Delta x^n}$] = 0 is called a ……

A) differential equation B) difference equation

C) linear equation D) None of these

8) For y = f(x) the relation of the form φ[x, f(x), f(x+h), f(x+2h), ….. f(x+nh)] is called a …….

A) differential equation B) linear equation

C) difference equation D) None of these

9) If E and Δ are shift and forward difference operators respectively and h is interval difference, then $\frac{\Delta^n y}{\Delta x^n}$ = ……

A) $\frac{E^n f(x)}{h^n}$ B) $\frac{(E-1)^n f(x)}{h^n}$ C) $\frac{(E+1)^n f(x)}{h^n}$ D) None of these

10) The difference between the largest and smallest arguments for the function involved divided by h is called …… of a difference equation.

A) order B) solution C) root D) None of these

11) The order of the difference equation $f(x+2h) - 5f(x+h) + 6f(x)$ = 0 is …

A) 1 B) 2 C) 3 D) None of these

12) Order of a difference equation $\varphi[x, f(x), f(x+h), f(x+2h), \ldots f(x+nh)] = 0$ is ……

A) 1 B) n-1 C) n D) None of these

13) The order of the difference equation $y_{x+2} - 7y_x = 5$ is ……

A) 1 B) 2 C) 3 D) None of these

14) The order of the difference equation $y_{x+4} - 5y_{x+1} + 6y_x = 0$. is ……

A) 4 B) 5 C) 6 D) None of these

15) The order of the difference equation $\Delta^2 y_x + 3\Delta y_x = x$ is ……

A) 1 B) 2 C) 3 D) None of these

16) The order of the difference equation $\Delta^3 y_x + 2\Delta y_x + y_x = x + 3$ is ……

A) 1 B) 2 C) 3 D) None of these

17) The order of the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ is ……

A) 1 B) 2 C) 3 D) None of these

18) The difference equation $f(x+2h) - 5f(x+h) + 6f(x) = 0$ is written in subscript form as ……

A) $y_{x+2} - 5y_{x-1} + 6y_x = 0$ B) $y_{x+2} - 5y_{x+1} + 6y_{x-1} = 0$

C) $y_{x+2} - 5y_{x+1} + 6y_x = 0$ D) None of these

19) The difference equation $\Delta y_k - 2y_k = 3$ is written in subscript form as ……

A) $y_{k+1} - 2y_k = 3$ B) $y_{k+1} - 3y_k = 3$ C) $y_{k+1} - y_k = 3$ D) $y_{k+1} + y_k = 3$

20) The difference equation $\Delta^3 y_k - \Delta^2 y_k + \Delta y_k + y_k = 0$ is written in subscript form as …

A) $y_{k+3} - 4y_{k+2} + 6y_{k+1} = 0$ B) $y_{k+3} - 2y_{k+2} + 2y_{k+1} = 0$

C) $y_{k+3} - y_{k+2} + y_{k+1} + y_k = 0$ D) None of these

21) $y_x = \frac{x(x-1)}{2}$ is the solution of the difference equation ……

A) $y_{x+1} + 2y_x = 0$ B) $y_{x+1} + y_x = 0$

C) $y_{x+1} - y_x = x$ D) None of these

22) $y_x = 1 - \frac{2}{x}$ is the solution of the difference equation ……

A) $(x+1)y_{x+1} + xy_x = 2x - 3$ B) $(x+1)y_{x+1} + xy_x = 2x$

C) $(x+1)y_{x+1} + xy_x = 0$ D) None of these

23) $y_x = C_1 + C_2 2^x - x$ is the solution of the difference equation ……

A) $y_{x+2} - 3y_{x+1} + 2y_x = 0$ B) $y_{x+2} - 3y_{x+1} + 2y_x = 1$

C) $y_{x+2} - 3y_{x+1} + 2y_x = x$ D) None of these

24) An equation of the form $a_0(k)E^n y_k + a_1(k)E^{n-1}y_k + a_2(k)E^{n-2}y_k + \ldots\ldots + a_n(k)y_k = R(k)$, $a_0(k) \neq 0$ and $a_i(k)$ (i = 0, 1, 2, ……) are constants, is called a ……with constant coefficients.

A) linear differential equation B) linear difference equation

C) non-linear difference equation D) None of these

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25) $(E^3 - 6E^2 + 12)\, y_k = 0$ is a ……with constant coefficients.

A) linear difference equation B) linear differential equation

C) non-linear difference equation D) None of these

26) $(kE^2 - kE + 4)\, y_k = 4k + 1$ is a ……with a variable coefficient.

A) linear difference equation B) linear differential equation

C) non-linear difference equation D) None of these

27) $y_k^2 + y_k y_{k+1} = 10k$ is a ……

A) linear difference equation B) linear differential equation

C) non-linear difference equation D) None of these

28) If the solution of difference equation contains n arbitrary constants, then order of difference equation is ……

A) n - 1 B) n C) n + 1 D) n + 2

29) The order of the difference equation formed from the solution $y_n = A3^n + B5^n$ is ……

A) 2 B) 1 C) 0 D) 3

30) The order of the difference equation formed from the solution $y_n = ax^2 + bx - 3$ is …

A) 1 B) 2 C) 3 D) 4

31) If $a_2 \neq 0$, then $y_{x+2} + a_1 y_{x+1} + a_2 y_n = 0$ is called …… difference equation

A) homogenous B) non- homogenous

C) linear D) None of these

32) If $a_n \neq 0$, then $y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \ldots\ldots + a_{n-1}y_{x+1} + a_n y_x = 0$ is called n^{th} order …… difference equation

A) homogenous B) non- homogenous

C) linear D) None of these

33) If m_1 and m_2 are distinct real roots of an auxiliary equation of the difference equation, then the solution y_x = ……

A) $c_1 m_1^x + c_2 m_2^x$ B) $c_1 x^{m_1 x} + c_2 x^{m_2 x}$

C) $(m_1 + m_2)x$ D) None of these

34) If an auxiliary equation $m^2 + a_1 m + a_2 = 0$ of given second order homogenous difference equation has equal real roots m_1 and m_2, then the solution is y_x = ……

A) $c_1 m_1^x + c_2 m_2^x$ B) $c_1 x^{m_1 x} + c_2 x^{m_2 x}$

C) $(c_1 + c_2 x)\, m_1^x$ D) None of these

35) If $m = \alpha \pm i\beta$ are the complex roots of an auxiliary equation $m^2 + a_1 m + a_2 = 0$ with $\rho = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\frac{\beta}{\alpha})$ and c_1, c_2 are constants, of given second order homogenous difference equation, then the solution is y_x = …

A) $\rho^x (c_1 \cos x\theta + c_2 \sec x\theta)$ B) $\rho^x (c_1 \cos x\theta + c_2 \sin x\theta)$

C) $\rho^x (c_1 \operatorname{cosec} x\theta + c_2 \sin x\theta)$ D) None of these

36) If $m_1, m_2, \ldots, m_n$ are distinct real roots of an auxiliary equation of given n^{th} order homogenous difference equation, then the solution is

A) $c_1 m_1^x + c_2 m_2^x + \ldots\ldots + c_k m_k^x$ B) $c_1 x^{m_1 x} + c_2 x^{m_2 x} + \ldots + c_k x^{m_k x}$

C) $(c_1 + c_2 x + \ldots + c_k x^{k-1})\, m_1^x$ D) None of these

37) If $m_1, m_2, \ldots, m_k$ are equal real roots of an auxiliary equation of given n^{th} order homogenous difference equation repeated k times, then the solution is

A) $c_1 m_1^x + c_2 m_2^x + \ldots\ldots + c_k m_k^x$ B) $c_1 x^{m_1 x} + c_2 x^{m_2 x} + \ldots + c_k x^{m_k x}$

C) $(c_1 + c_2 x + \ldots + c_k x^{k-1})\, m_1^x$ D) None of these

38) The solution of the difference equation $y_{x+2} - 7y_{x+1} + 12y_x = 0$ is y_x = ……

A) $c_1 m_1^3 + c_2 m_2^4$ B) $c_1 3^x + c_2 4^x$

C) $(3^x + 4^x)x$ D) None of these

39) The solution of the difference equation $2y_{x+2} - 5y_{x+1} + 2y_x = 0$ is y_x = ……

A) $c_1 2^{-x} + c_2 2^x$ B) $c_1 2^x + c_2 5^x$

C) $c_1 2^{-x} + c_2 5^x$ D) None of these

40) The solution of the difference equation $9y_{x+2} - 6y_{x+1} + y_x = 0$ is y_x = ……

A) $c_1 3^{-x} + c_2 4^x$ B) $c_1 3^x + c_2 4^{-x}$

C) $(c_1 + c_2x)3^{-x}$ D) None of these

41) The solution of the difference equation $y_{k+2} - 6y_{k+1} + 8y_k = 0$ is $y_k = \ldots\ldots$

A) $c_1 3^{-k} + c_2 4^k$ B) $c_1 2^k + c_2 4^k$

C) $(c_1 + c_2k)3^{-k}$ D) None of these

42) The solution of the difference equation $16y_{k+2} - 8y_{k+1} + y_k = 0$ is $y_k = \ldots\ldots$

A) $c_1 3^{-k} + c_2 4^k$ B) $c_1 2^k + c_2 4^k$

C) $(c_1 + c_2k)4^{-k}$ D) None of these

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."