Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: S.Y.B.SC SEM.-IV

SUBJECT: MTH-401: COMPLEX VARIABLES

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MTH -401: COMPLEX VARIABLES

Unit-1: Complex numbers **Marks-15**

1.1 Complex numbers, modulus and amplitude, polar form

1.2 Triangle inequality and Argand's diagram

1.3 DeMoivre's theorem for rational indices and applications

1.4 nth roots of a complex number

1.5 Elementary functions: Trigonometric functions, Hyperbolic functions of a complex variables (definitions only).

Unit-2: Functions of complex variables **Marks-15**

2.1 Limits, Continuity and Derivative.

2.2 Analytic functions, A Necessary and sufficient conditions for analytic functions.

2.3 Cauchy Riemann equations.

2.4 Laplace equations and Harmonic functions

2.5 Construction of analytic functions

Unit-3: Complex integrations **Marks-15**

3.1 Line integral and theorems on it.

3.2 Statement and verification of Cauchy-Gaursat's Theorem.

3.3 Cauchy's integral formulae for $f(a)$,$f'(a)$ and $f^n(a)$

3.4 Taylor's and Laurent's series.

Unit-4: Calculus of Residues **Marks-15**

4.1 Zeros and poles of a function.

4.2 Residue of a function

4.3 Cauchy's residue theorem

4.4 Evaluation of integrals by using Cauchy's residue theorem

4.5 Contour integrations of the type $\int_0^{2\pi} f(cos\theta, sin\theta)\, d\theta$ and $\int_{-\infty}^{\infty} f(x)\, dx$

Recommended book:

1. Complex Variables and Applications; J. W. Brownand, R. V. Churchill. 7th Edition. (McGraw-Hill) (Capter1, chapter 2, chapter 3, chapter 4, chapter 6)

Reference Books:

1. Theory of Functions of Complex Variables: Shanti Narayan, S. Chand and Company New Delhi.
2. Complex variables: Schaum's Outline Series.

Learning Outcomes:

a) The course is aimed to introduce the theory for functions of complex variables

b) Students will understand the concept of analytic function

c) Students will understand the Cauchy Riemann Equations

d) Students will understand harmonic functions

e) Students will understand complex integrations

f) Students will understand calculus of residues.

g) Students will acquire the skill of contour integrations.

UNIT-1: COMPLEX NUMBERS

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Complex number: A number $z = x + iy$, $x, y \in R$ is called a complex number.

Where $i = \sqrt{-1}$.

Equality of Complex numbers: Two complex numbers $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$ are equal i.e $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Operations with Complex numbers: If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$ be any two complex numbers then I) Addition : $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \in C$

II) Subtractions : $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \in C$

III) Multiplication : $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \in C$

IV) Division : $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2{}^2 + y_2{}^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2{}^2 + y_2{}^2}\right) \in C$

Remark: If z_1, z_2 & $z_3 \in C$ then

I) Addition and Multiplication in C are commutative

i.e. $z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$

II) Addition and Multiplication in C are associative in C

i.e. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1z_2)z_3 = z_1(z_2z_3)$

III) 0 is the identity element in C w.r.t. addition and 1 is the identity element in C w.r.t. multiplication

i.e. $z + 0 = 0 + z = z$ and $z.1 = 1.z = z \ \forall z \in C$.

IV) For any $z = x + iy \in C$, $-z = -x - iy$ is additive inverse of z in C and if $z \neq 0$ then $z^{-1} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$ is multiplicative inverse of z in C

V) Multiplication is distributive over addition

i.e. $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

VI) Cancellation law: If $z_1 \neq 0$, then $z_1.z_2 = z_1z_3 \Rightarrow z_2 = z_3$

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Ex. Prove that for any complex number z, $I(iz) = R(z)$ & $R(iz) = -I(z)$

Proof: Let $z = x + iy$, then $iz = ix + i^2y = -y + ix$

$\therefore I(iz) = x = R(z)$ & $R(iz) = -y = -I(z)$ Hence proved.

Ex. Show that the complex number $1 = 1 + i0$ is the only multiplicative identity in C.

Proof: Suppose $u + iv$ another multiplicative identity in C.

$\therefore (u+iv)1 = 1$ …….(1) $\because 1 \in C$.

Also 1 is multiplicative identity in C.

$\therefore (u+iv)1 = u+iv$ …….(2) $\because u + iv \in C$.

By equation (1) and (2)

$u+iv = 1 = 1 + i0$

Hence $1 = 1 + i0$ is the only multiplicative identity in C is proved.

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Ex. Show that $i^m = 1$, $m \in \mathbb{Z}$ is multiple of four and $i^m = -1$, if $m \in \mathbb{Z}$ is an even integer, but not multiple of four.

Proof: Case (i) Suppose $m \in \mathbb{Z}$ is multiple of four.

i.e. $m = 4k$, $k \in \mathbb{Z}$

$\therefore i^m = i^{4k} = (i^4)^k = 1^k = 1 \quad \because i^4 = 1$

Case (ii) Suppose $m \in \mathbb{Z}$ is an even integer, but not multiple of four.

i.e. $m = 2k$, $k \in \mathbb{Z}$ is odd.

$\therefore i^m = i^{2k} = (i^2)^k = (-1)^k = -1 \qquad \because$ k is odd.

Hence proved.

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Ex. Express in the form x + iy where x, y ∈ ℝ. a) $(2+i)^4$ b) $(1+2i)(3+4i)$ c) $\frac{1}{3+2i}$ d) $\frac{3+4i}{1+2i}$

Solution: a) Let $z = (2+i)^4$

$= [(2+i)^2]^2$

$= [4+4i-1]^2$

$= (3+4i)^2$

$= 9+24i-16$

$= -7+24i$

b) Let $z = (1+2i)(3+4i)$

$= 3+4i+6i-8$

$= -5+10i$

c) Let $z = \frac{1}{3+2i} \times \frac{3-2i}{3-2i}$

$= \frac{3-2i}{9+4}$

$= \frac{3}{13} - \frac{2}{13}i$

b) Let $z = \frac{3+4i}{1+2i}$

$= \frac{3+4i}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{3-6i+4i+8}{1+4}$

$= \frac{11}{5} - \frac{2}{5}i$

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Ex. Show that $\frac{5}{(1-i)(2-i)(3-i)} = \frac{1}{2}i$

Proof: Consider

$$\text{LHS} = \frac{5}{(1-i)(2-i)(3-i)}$$

$$= \frac{5}{(1-i)(6-2i-3i-1)}$$

$= \frac{5}{(1-i)(5-5i)}$

$= \frac{1}{(1-i)(1-i)}$

$= \frac{1}{(1-2i-1)}$

$= \frac{1}{-2i} \times \frac{i}{i}$

$= \frac{1}{2} i$

= RHS.

Hence proved.

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Ex. Show that $(3+i)(3-i)(\frac{1}{5} + \frac{i}{10}) = 2 + i$

Proof: Consider

LHS $= (3+i)(3-i)(\frac{1}{5} + \frac{i}{10})$

$= (9+1)(\frac{1}{5} + \frac{i}{10})$

$= 10(\frac{1}{5} + \frac{i}{10})$

$= 2 + i$

= RHS.

Hence proved.

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Conjugate of Complex number: A number $\bar{z} = x - iy$, $x, y \in \mathbb{R}$ is called a conjugate of complex number $z = x + iy$.

Remark: 1) If $z = x + iy$ and $\bar{z} = x - iy$ then $\bar{\bar{z}} = x + iy = z$ i.e. z and $\bar{z}$ are complex conjugate of each other.

2) $z + \bar{z} = x + iy + x - iy = 2x = 2R(z) = 2R(\bar{z})$ i.e. $R(z) = R(\bar{z}) = x = \frac{z+\bar{z}}{2}$

3) $z - \bar{z} = x + iy - x + iy = 2iy = 2i\text{Im}(z)$ i.e. $\text{Im}(z) = y = \frac{z-\bar{z}}{2i}$

4) $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

5) If $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

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Proposition: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers then

i) $\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$, ii) $\overline{z_1-z_2} = \bar{z_1} - \bar{z_2}$, iii) $\overline{z_1z_2} = \bar{z_1}\bar{z_2}$, ii) $\overline{(\frac{z_1}{z_2})} = \frac{\bar{z_1}}{\bar{z_2}}$

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers then

$\bar{z_1} = x_1 - iy_1$ and $\bar{z_2} = x_2 - iy_2$

i) We have $z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$

$\therefore \overline{z_1+z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z_1} + \bar{z_2}$

ii)We have $z_1 - z_2 = x_1+iy_1-x_2-iy_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\therefore\ \overline{z_1 - z_2} = (x_1 - x_2) - i(y_1 - y_2) = (x_1 - iy_1) - (x_2 - iy_2) = \bar{z}_1 - \bar{z}_2$

iii) We have $z_1z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2) + i(x_1y_2 + x_2y_1)$

$\therefore\ \overline{z_1z_2} = (x_1x_2-y_1y_2) - i(x_1y_2 + x_2y_1) = (x_1 - iy_1)(x_2 - iy_2) = \bar{z}_1\bar{z}_2$

iv) We have $\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$

$\therefore\ \overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) - i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right) = \frac{(x_1 - iy_1)(x_2 + iy_2)}{(x_2 - iy_2)(x_2 + iy_2)} = \frac{x_1 - iy_1}{x_2 - iy_2} = \frac{\bar{z}_1}{\bar{z}_2}$

Hence proved.

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Ex. Find Re(z), Im(z) and complex conjugate of z, where z is

a) $\frac{1}{2+3i}$ b) $\frac{3}{i} + \frac{7}{2}$ c) $i^{15} + i^{19}$ d) $\left(\frac{2+i}{3-2i}\right)^2$

Solution: a) Let $z = \frac{1}{2+3i} = \frac{1}{2+3i} \times \frac{2-3i}{2-3i}$

$= \frac{2-3i}{4+9}$

$= \frac{2}{13} - \frac{3}{13}i$

$\therefore$ Re(z) $= \frac{2}{13}$, Im(z) $= -\frac{3}{13}$ and $\bar{z} = \frac{2}{13} + \frac{3}{13}i$

b) Let $z = \frac{3}{i} + \frac{7}{2}$

$= \frac{7}{2} + \frac{3}{i} \times \frac{i}{i}$

$= \frac{7}{2} - 3i$

$\therefore$ Re(z) $= \frac{7}{2}$, Im(z) $= -3$ and $\bar{z} = \frac{7}{2} + 3i$

c) Let $z = i^{15} + i^{19} = (i^2)^7 i + (i^2)^9 i$

$= (-1)^7 i + (-1)^9 i$

$= -i - i = -2i$

$\therefore$ Re(z) $= 0$, Im(z) $= -2$ and $\bar{z} = 2i$

d) Let $z = \left(\frac{2+i}{3-2i}\right)^2 = \frac{(2+i)^2}{(3-2i)^2}$

$= \frac{4+4i-1}{9-12i-4}$

$= \frac{3+4i}{5-12i}$

$= \frac{3+4i}{5-12i} \times \frac{5+12i}{5+12i}$

$= \frac{15+36i+20i-48}{25+144}$

$= -\frac{33}{169} + \frac{56}{169}i$

$\therefore$ Re(z) $= -\frac{33}{169}$, Im(z) $= \frac{56}{169}$ and $\bar{z} = -\frac{33}{169} - \frac{56}{169}i$

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Ex. Show that z is real if and only if $\bar{z} = z$

Proof: Suppose that z is a real. i.e. z = x

$\therefore$ Im(z) = 0 i.e. y = 0

Now $\bar{z} = x - iy = x = z$

Conversely, Suppose $\bar{z} = z$

$\therefore x - iy = x + iy$

$\therefore -2iy = 0$

$\therefore y = 0$

$\therefore z = x + i0 = x$

$\therefore$ z is real.

Hence proved.

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Modulus or Absolute Value a complex number: The positive number $r = |z| = \sqrt{x^2 + y^2}$ is called modulus or absolute value of a complex number z = x + iy.

Argument or Amplitude of a complex number: The angle $\theta = \arg z = \tan^{-1}\frac{y}{x}$ is called an argument or amplitude of a complex number z = x + iy.

Polar form of a complex number: If r is modulus and angle θ an argument of a complex number z = x + iy, then $z = r(\cos\theta + i\sin\theta)$ is called polar form or modulus-amplitude form of a complex number.

Remark: i) $|z| = \sqrt{x^2 + y^2} \geq 0$ and $|z| = 0$ iff $z = 0$.

ii) $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

iii) $\theta = \arg z = \tan^{-1}\frac{y}{x} \in (-\pi, \pi)$ is called principal argument of a complex number z = x + iy.

iv) If $\theta = \arg z$ then $\theta + 2n\pi$, $n \in N$ is called general argument of a complex number z = x + iy.

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Ex. Compute the modulus and principal argument of each of the following complex numbers. a) $i^7 + i^{10}$ b) $\frac{1}{1+i}$ c) $(-1+i)^3$ d) $\frac{(1+i)^3}{(1-i)^2}$

Solution: a) Let $z = i^7 + i^{10}$

$= (i^2)^3 i + (i^2)^5$

$= (-1)^3 i + (-1)^5$

$= -i - 1$

$= -1 - i$

$\therefore x = -1$ and $y = -1$

$\therefore |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(-1)}{(-1)} = \tan^{-1}1 = -\frac{3\pi}{4} \in (-\pi, \pi)$

b) Let $z = \frac{1}{1+i}$

$= \frac{1}{1+i} \times \frac{1-i}{1-i}$

$= \frac{1-i}{1+1}$

$= \frac{1}{2} - \frac{1}{2}i$

$\therefore x = \frac{1}{2}$ and $y = -\frac{1}{2}$

$\therefore |z| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(-\frac{1}{2})}{(\frac{1}{2})} = \tan^{-1}(-1) = -\frac{\pi}{4} \in (-\pi, \pi)$

c) Let $z = (-1+i)^3$

$= -1+3i+3-i$

$= 2 + 2i$

$\therefore x = 2$ and $y = 2$

$\therefore |z| = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(2)}{(2)} = \tan^{-1}1 = \frac{\pi}{4} \in (-\pi, \pi)$

d) Let $z = \frac{(1+i)^3}{(1-i)^2}$

$= \frac{1+3i-3-i}{1-2i-1}$

$= \frac{2i-2}{-2i}$

$= -1 + \frac{1}{i}$

$= -1-i$

$\therefore x = -1$ and $y = -1$

$\therefore |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(-1)}{(-1)} = \tan^{-1}1 = -\frac{3\pi}{4} \in (-\pi, \pi)$

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Ex. Find the modulus argument form (polar form) of

a) $1+ i$ b) $-i$ c) $3+4i$ d) $\sqrt{3} - i$ e) $1+i\sqrt{3}$

Solution: a) Let $z = 1+i$ $\therefore x = 1$ and $y = 1$

$\therefore r = |z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(1)}{(1)} = \tan^{-1}1 = \frac{\pi}{4} \in (-\pi, \pi)$

General value of $\theta = 2n\pi + \frac{\pi}{4}$

$\therefore$ polar form is $z = \sqrt{2}\,[\cos(2n\pi + \frac{\pi}{4}) + i\sin(2n\pi + \frac{\pi}{4})]$, where $n \in \mathbb{Z}$

b) Let $z = -i$ $\quad\therefore x = 0$ and $y = -1$

$\therefore r = |z| = \sqrt{(0)^2 + (-1)^2} = 1$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(-1)}{(0)} = \tan^{-1}(-\infty) = -\frac{\pi}{2} \in (-\pi, \pi)$

General value of $\theta = 2n\pi - \frac{\pi}{2}$

$\therefore$ polar form is $z = \cos(2n\pi - \frac{\pi}{2}) + i\sin(2n\pi - \frac{\pi}{2})$, where $n \in N$

c) Let $z = 3+4i$ $\quad\therefore x = 3$ and $y = 4$

$\therefore r = |z| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{3} \in (-\pi, \pi)$

General value of $\theta = 2n\pi + \tan^{-1}\frac{4}{3}$

$\therefore$ polar form is $z = 5\,[\cos(2n\pi + \tan^{-1}\frac{4}{3}) + i\sin(2n\pi + \tan^{-1}\frac{4}{3})]$, where $n \in N$

d) Let $z = \sqrt{3} - i$ $\quad\therefore x = \sqrt{3}$ and $y = -1$

$\therefore r = |z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}(\frac{-1}{(\sqrt{3})}) = -\frac{\pi}{6} \in (-\pi, \pi)$

General value of $\theta = 2n\pi - \frac{\pi}{6}$

$\therefore$ polar form is $z = 2\,[\cos(2n\pi - \frac{\pi}{6}) + i\sin(2n\pi - \frac{\pi}{6})]$, where $n \in N$

e) Let $z = 1 + i\sqrt{3}$ $\therefore x = 1$ and $y = \sqrt{3}$

$\therefore r = |z| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}(\frac{\sqrt{3}}{(1)}) = \frac{\pi}{3} \in (-\pi, \pi)$

General value of $\theta = 2n\pi + \frac{\pi}{3}$

$\therefore$ polar form is $z = 2\,[\cos(2n\pi + \frac{\pi}{3}) + i\sin(2n\pi + \frac{\pi}{3})]$, where $n \in N$

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Ex. Prove that $\arg(\frac{2+i}{2-i}) = \tan^{-1}(\frac{4}{3})$

Proof: Let $z = \frac{2+i}{2-i}$

$= \frac{2+i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{4+4i-1}{4+1}$

$= \frac{3+4i}{5}$

$= \frac{3}{5} + i\frac{4}{5}$

$\therefore$ x = $\frac{3}{5}$ and y = $\frac{4}{5}$

$\therefore \theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{(\frac{4}{5})}{(\frac{3}{5})}$

$\therefore \arg(\frac{2+i}{2-i}) = \tan^{-1}(\frac{4}{3})$

Hence proved.

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Ex. Find the modulus and principle value of the argument of $\frac{(1+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}}$

Solution: Let $z = \frac{(1+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}}$

$= \frac{(-i^2+i\sqrt{3})^{13}}{(\sqrt{3}-i)^{11}}$

$= \frac{i^{13}(\sqrt{3}-i)^{13}}{(\sqrt{3}-i)^{11}}$

$= (i^2)^6 i(\sqrt{3}-i)^2$

$= i(3\text{-}2\sqrt{3}i - 1)$

$= i(-2\sqrt{3}i + 2)$

$= 2\sqrt{3} + 2i$

$\therefore x = 2\sqrt{3}$ and $y = 2$

$\therefore r = |z| = \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{12+4} = 4$

$\therefore \theta = \arg z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{2\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \in (-\pi, \pi)$ is the principal argument.

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Argand's diagram: The representation of complex numbers by points in a plane is called an Argand's diagram.

Remark: The complex numbers z = x + iy is represented by point (x, y) in an Argand's diagram.

Triangle inequality:

Theorem: For any two complex numbers z_1 and z_2:

i) $|z_1 + z_2| \leq |z_1| + |z_2|$, ii) $|z_1 - z_2| \leq |z_1| + |z_2|$

Proof: Let $|z_1| = r_1$, $\arg z_1 = \theta_1$ and $|z_2| = r_2$, $\arg z_2 = \theta_2$

$\therefore z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore |z_1 + z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$

$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$

$+2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

$\leq r_1^2 + r_2^2 + 2r_1r_2$ $\quad\because r_1, r_2 \geq 0$ and $\cos(\theta_1 - \theta_2) \leq 1$

$\therefore |z_1 + z_2|^2 \leq (r_1 + r_2)^2$

i) Taking positive square root, we get,

$|z_1 + z_2| \leq r_1 + r_2$

i.e. $|z_1 + z_2| \leq |z_1| + |z_2|$ ……(1)

ii)) Replacing $-z_2$ for z_2 in (1), we get,

$|z_1 + (-z_2)| \leq |z_1| + |-z_2|$

i.e. $|z_1 - z_2| \leq |z_1| + |z_2|$ $\because |-z_2| = |z_2|$

Hence proved.

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Theorem: For any $z_1, z_2 \in C$: $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

Proof: Let $|z_1| = r_1$, $\arg z_1 = \theta_1$ and $|z_2| = r_2$, $\arg z_2 = \theta_2$

$\therefore z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_1 - z_2 = (r_1\cos\theta_1 - r_2\cos\theta_2) + i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$\therefore |z_1 - z_2|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$

$-2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$\geq r_1^2 + r_2^2 - 2r_1r_2$ $\because r_1, r_2 \geq 0$ and $\cos(\theta_1 - \theta_2) \leq 1$

$\therefore |z_1 - z_2|^2 \geq (r_1 - r_2)^2$

Taking positive square root, we get,

$|z_1 - z_2| \geq |r_1 - r_2|$

i.e. $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ Hence proved.

==

Theorem: If $z_1, z_2 \in C$, then

$|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$

Proof: Let $|z_1| = r_1$, $\arg z_1 = \theta_1$ and $|z_2| = r_2$, $\arg z_2 = \theta_2$

$\therefore z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_1 z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$\therefore z_1 z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore |z_1z_2| = r_1r_2 = |z_1||z_2|$ and

$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

Hence proved.

==

Theorem: If $z_1, z_2 \in C$ and $z_2 \neq 0$, then

$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Proof: Let $|z_1| = r_1$, $\arg z_1 = \theta_1$ and $|z_2| = r_2$, $\arg z_2 = \theta_2$

$\therefore z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$

$= \frac{r_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{r_2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$

$= \frac{r_1(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2+\sin^2\theta_2)}$

$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$

$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ and

$\arg(\frac{z_1}{z_2}) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

Hence proved.

==

Ex. If $|z_1|=|z_2|=|z_3|=5$ and $z_1+z_2+z_3=0$ then prove that $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=0$

Proof: Let $|z_1|=|z_2|=|z_3|=5$ and $z_1+z_2+z_3=0$……(1)

Consider $\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}$

$= \frac{\overline{z_1}}{z_1\overline{z_1}}+\frac{\overline{z_2}}{z_2\overline{z_2}}+\frac{\overline{z_3}}{z_3\overline{z_3}}$

$= \frac{\overline{z_1}}{|z_1|^2}+\frac{\overline{z_2}}{|z_2|^2}+\frac{\overline{z_3}}{|z_3|^2}$

$= \frac{\overline{z_1}}{25}+\frac{\overline{z_2}}{25}+\frac{\overline{z_3}}{25}$ by (1)

$= \frac{1}{25}[\overline{z_1}+\overline{z_2}+\overline{z_3}]$

$= \frac{1}{25}[\overline{z_1+z_2+z_3}]$

$= \frac{1}{25}[\overline{0}]$ by (1)

$= 0$

Hence proved.

==

Ex. Prove that $\left|\frac{z-1}{1-\bar{z}}\right|=1$

Proof: Let $z = x+iy$ then $\bar{z} = x - iy$

Consider

L.H.S. $= \left|\frac{z-1}{1-\bar{z}}\right| = \left|\frac{x+iy-1}{1-x+iy}\right| = \frac{|x-1+iy|}{|1-x+iy|}$

$= \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(1-x)^2+y^2}}$

$= 1 \qquad \because (x-1)^2=(1-x)^2$

= R.H.S.

==

Ex. Prove that for any two complex numbers z_1 and z_2

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

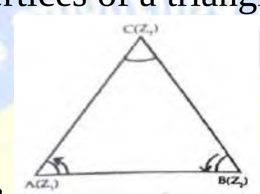
Proof: Let us consider

$$\begin{aligned}
\text{L.H.S.} &= |z_1 + z_2|^2 + |z_1 - z_2|^2 \\
&= (z_1 + z_2)(\overline{z_1 + z_2}) + (z_1 - z_2)(\overline{z_1 - z_2}) \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} \\
&= 2z_1\overline{z_1} + 2z_2\overline{z_2} \\
&= 2|z_1|^2 + 2|z_2|^2 \\
&= \text{R.H.S.}
\end{aligned}$$

Hence proved.

==

Remark: If A, B and C are the vertices of a triangle represented by the complex



numbers z_1, z_2 and z_3 respectively,

then $l(AB) = |z_2 - z_1|$, $l(BC) = |z_3 - z_2|$, $l(AC) = |z_3 - z_1|$

and $m\angle A = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$, $m\angle B = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$, $m\angle C = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$

==

Ex. If z_1, z_2, z_3 represents vertices of an equilateral triangle, prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Proof: Let A, B and C are the vertices of an equilateral triangle represented by the complex numbers z_1, z_2 and z_3 respectively,

$\therefore l(AB) = |z_2 - z_1|$, $l(BC) = |z_3 - z_2|$, $l(AC) = |z_3 - z_1|$ and

$m\angle A = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$, $m\angle B = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$, $m\angle C = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$

As ΔABC is an equilateral triangle

$\therefore l(AB) = l(BC) = l(AC)$ i.e. $|z_2 - z_1| = |z_3 - z_2| = |z_3 - z_1|$

$\therefore \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| = 1 \ldots\ldots(1)$

and $m\angle A = m\angle B = m\angle C = \frac{\pi}{3}$

i.e. $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{3} \ldots\ldots(2)$

By (1) and (2)

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_1 - z_2}{z_3 - z_2}$$

i.e. $z_3^2 - z_3z_2 - z_1z_3 + z_1z_2 = z_1z_2 - z_1^2 - z_2^2 + z_2z_1$

$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

Hence proved.

Ex. If $\frac{z-1}{z+i}$ is purely imaginary, find the locus of z.

Solution: Let z = x + iy

$\therefore \frac{z-1}{z+i} = \frac{x+iy-1}{x+iy+i} = \frac{(x-1)+iy}{x+i(y+1)} \times \frac{x-i(y+1)}{x-i(y+1)}$

$= \frac{x(x-1)+y(y+1)+i[yx-(x-1)(y+1)]}{x^2+(y+1)^2}$

is purely imaginary

$\therefore$ The real part of $\frac{z-1}{z+i} = 0$

$\therefore \frac{x(x-1)+y(y+1)}{x^2+(y+1)^2} = 0$

$\therefore x(x-1) + y(y+1) = 0$

$\therefore x^2 - x + y^2 + y = 0$

$\therefore (x-\frac{1}{2})^2 - \frac{1}{4} + (y+\frac{1}{2})^2 - \frac{1}{4} = 0$

$\therefore (x-\frac{1}{2})^2 + (y+\frac{1}{2})^2 = \frac{1}{2}$

$\therefore (x-\frac{1}{2})^2 + (y+\frac{1}{2})^2 = (\frac{1}{\sqrt{2}})^2$

i.e. locus of point z is a circle with Centre $(\frac{1}{2}, -\frac{1}{2})$ and radius $\frac{1}{\sqrt{2}}$

==

Ex. Show that the locus of the point z, which satisfies $|z-3|+|z+3|=4$ represents an hyperbola.

Proof: Let z = x + iy

$\therefore |z-3|+|z+3|=4$ gives

$|x + iy - 3|+|x + iy + 3|=4$

i.e. $|(x-3) + iy|+|(x+3) + iy|=4$

$\therefore \sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 4$

$\therefore \sqrt{(x+3)^2 + y^2} = 4 - \sqrt{(x-3)^2 + y^2}$

Squaring both sides, we get,

$(x+3)^2 + y^2 = 16 - 8\sqrt{(x-3)^2 + y^2} + (x-3)^2 + y^2$

$\therefore x^2+6x+9+y^2 = 16 - 8\sqrt{(x-3)^2 + y^2} + x^2-6x+9+y^2$

$\therefore 12x-16 = -8\sqrt{(x-3)^2 + y^2}$

$\therefore 3x-4 = -2\sqrt{x^2 - 6x + 9 + y^2}$

Again squaring both sides, we get,

$9x^2-24x+16 = 4(x^2 - 6x + 9 + y^2)$

$\therefore 9x^2-24x+16 = 4x^2 - 24x + 36 + 4y^2$

$\therefore 5x^2-4y^2 = 20$

$\therefore \frac{x^2}{4} - \frac{y^2}{5} = 1$

i.e. locus of point z is a hyperbola is proved.

Ex. Determine the region in the z-plane represented by $|z-3|+|z+3|=10$

Proof: Let z = x + iy

$\therefore |z-3|+|z+3|=10$ gives

$|x+iy-3|+|x+iy+3|=10$

i.e. $|(x-3)+iy|+|(x+3)+iy|=10$

$\therefore \sqrt{(x-3)^2+y^2}+\sqrt{(x+3)^2+y^2}=10$

$\therefore \sqrt{(x+3)^2+y^2}=10-\sqrt{(x-3)^2+y^2}$

Squaring both sides, we get,

$(x+3)^2+y^2=100-20\sqrt{(x-3)^2+y^2}+(x-3)^2+y^2$

$\therefore x^2+6x+9+y^2=100-20\sqrt{(x-3)^2+y^2}+x^2-6x+9+y^2$

$\therefore 12x-100=-20\sqrt{(x-3)^2+y^2}$

$\therefore -5\sqrt{x^2-6x+9+y^2}=3x-25$

Again squaring both sides, we get,

$25(x^2-6x+9+y^2)=9x^2-150x+625$

$\therefore 25x^2-150x+225+25y^2=9x^2-150x+625$

$\therefore 16x^2+25y^2=400$

$\therefore \frac{x^2}{25}+\frac{y^2}{16}=1$ i.e. The region in the z-plane is the ellipse.

DeMoivre's theorem for rational indices: If n is a rational number, then

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

Proof: To prove the theorem we consider four cases.

Case-i) Let n be the positive integer. In this case we prove the result by mathematical induction.

Let P(n) : $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ where $n\in N$.

Step 1: If n = 1, then we have

$(\cos\theta+i\sin\theta)^1=\cos\theta+i\sin\theta=\cos 1\theta+i\sin 1\theta$

i.e. P(1) is true.

Step 2: Suppose P(k) is true

i.e. $(\cos\theta+i\sin\theta)^k=\cos k\theta+i\sin k\theta$ ……(1)

Consider

$$\begin{aligned}(\cos\theta+i\sin\theta)^{k+1} &= (\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)\\ &= (\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta) \qquad \text{by (1)}\\ &= (\cos k\theta\cos\theta-\sin k\theta\sin\theta)+i(\sin k\theta\cos\theta+\cos k\theta\sin\theta)\\ &= \cos(k\theta+\theta)+i\sin(k\theta+\theta)\\ &= \cos(k+1)\theta+i\sin(k+1)\theta\end{aligned}$$

i.e. P(k) is true $\Rightarrow$ P(k+1) is true

∴ By principle of mathematical induction P(n) is true ∀ n ∈ N.

i.e.$(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$ ∀ n ∈ N.

Case-ii) If n = 0, then

$(\cos\theta+i\sin\theta)^0 = 1= \cos 0+i\sin 0 = \cos 0\theta+i\sin 0\theta$

i.e. Result is true for n = 0.

Case-iii) Let n be the negative integer. Suppose n = -m, where m is positive integer.

We have $(\cos\theta+i\sin\theta)^n = (\cos\theta+i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta+i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta+i\sin m\theta} \qquad \because \text{m is positive integer.}$$

$$= \frac{1}{\cos m\theta+i\sin m\theta} \times \frac{\cos m\theta-i\sin m\theta}{\cos m\theta-i\sin m\theta}$$

$$= \frac{\cos m\theta-i\sin m\theta}{\cos^2 m\theta+\sin^2 m\theta}$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta$$

i.e. Result is true for negative integer n.

Case-iv) Let n is rational number.

Suppose $n = \frac{p}{q}$, where q is positive integer and p is positive or negative integer.

Consider $[\cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})]^q = \cos(q.\frac{\theta}{q}) + i\sin(q.\frac{\theta}{q})$

$$= \cos\theta + i\sin\theta$$

i.e. $\cos\theta + i\sin\theta = [\cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})]^q$

By taking q^{th} root, we get,

$(\cos\theta + i\sin\theta)^{1/q} = \cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})$

Raising both sides to the power p, we get,

$(\cos\theta + i\sin\theta)^{p/q} = [\cos(\frac{\theta}{q}) + i\sin(\frac{\theta}{q})]^p$

$$= [\cos(p.\frac{\theta}{q}) + i\sin(p.\frac{\theta}{q})]$$

$$= \cos(\frac{p}{q})\theta + i\sin(\frac{p}{q})\theta$$

i.e. $(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$

i.e. Result is true for rational number n.

Hence proved.

==

Remark: If n > 0 then i)$(\cos\theta-i\sin\theta)^n = \cos n\theta-i\sin n\theta$ ii) $(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta$ and iii) $(\cos\theta-i\sin\theta)^{-n}=\cos n\theta+i\sin n\theta$

==

Ex. Simplify using DeMoivre's Theorem,

i) $(\cos3\theta+i\sin3\theta)^8(\cos4\theta-i\sin4\theta)^{-2}$ ii) $\frac{(\cos2\theta-i\sin2\theta)^7(\cos3\theta+i\sin3\theta)^{-5}}{(\cos4\theta+i\sin4\theta)^{12}(\cos5\theta-i\sin5\theta)^{-6}}$

Solution: i) Consider

$$(\cos3\theta+i\sin3\theta)^8(\cos4\theta-i\sin4\theta)^{-2}$$
$$= [(\cos\theta+i\sin\theta)^3]^8[(\cos\theta+i\sin\theta)^{-4}]^{-2}$$
$$= (\cos\theta+i\sin\theta)^{24}(\cos\theta+i\sin\theta)^8$$
$$= (\cos\theta+i\sin\theta)^{32}$$
$$= \cos32\theta+i\sin32\theta$$

ii) Consider

$$\frac{(\cos2\theta-i\sin2\theta)^7(\cos3\theta+i\sin3\theta)^{-5}}{(\cos4\theta+i\sin4\theta)^{12}(\cos5\theta-i\sin5\theta)^{-6}}$$
$$= \frac{[(\cos\theta+i\sin\theta)^{-2}]^7[(\cos\theta+i\sin\theta)^3]^{-5}}{[(\cos\theta+i\sin\theta)^4]^{12}[(\cos\theta+i\sin\theta)^{-5}]^{-6}}$$
$$= \frac{(\cos\theta+i\sin\theta)^{-14}(\cos\theta+i\sin\theta)^{-15}}{(\cos\theta+i\sin\theta)^{48}(\cos\theta+i\sin\theta)^{30}}$$
$$= (\cos\theta+i\sin\theta)^{-14-15-48-30}$$
$$= (\cos\theta+i\sin\theta)^{-107}$$
$$= \cos107\theta-i\sin107\theta$$

==

Ex. Prove that $(1+i\sqrt{3})^8+(1-i\sqrt{3})^8 = -256$

Proof: i) Consider

$$(1+i\sqrt{3})^8+(1-i\sqrt{3})^8$$
$$= [2(\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2})]^8+[2(\tfrac{1}{2}-i\tfrac{\sqrt{3}}{2})]^8 \qquad \because |1+i\sqrt{3}|=\sqrt{(1)^2+(\sqrt{3})^2}=2$$
$$= [2(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3})]^8+[2(\cos\tfrac{\pi}{3}-i\sin\tfrac{\pi}{3})]^8$$
$$= 2^8(\cos\tfrac{8\pi}{3}+i\sin\tfrac{8\pi}{3})+2^8(\cos\tfrac{8\pi}{3}-i\sin\tfrac{8\pi}{3})$$
$$= 2^8(2\cos\tfrac{8\pi}{3})$$
$$= 512\cos(3\pi-\tfrac{\pi}{3})$$
$$= 512[-\cos(\tfrac{\pi}{3})]$$
$$= 512(-\tfrac{1}{2})$$
$$= -256$$

==

Ex. Simplify $\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^{10}+\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^{10}$

Solution: Consider

$(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})^{10}+(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}})^{10}$

$= (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^{10} + (\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})^{10}$

$= \cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4} + \cos\frac{10\pi}{4} - i\sin\frac{10\pi}{4}$

$= 2\cos\frac{5\pi}{2}$

$= 2\,(0)$

$= 0$

===

n^{th} root of complex number: A complex number ω is said be n^{th} root of complex number $z = x + iy$ if $\omega^n = z$.

===

Remark: To find n^{th} root of complex number $z = x + iy$ first express it into polar form

$z = r(\cos\theta + i\sin\theta) = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$, then

$\omega = z^{1/n} = r^{1/n}[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{1/n}$

i.e. $\omega = r^{1/n}[\cos(\frac{\theta+2k\pi}{n}) + i\sin(\frac{\theta+2k\pi}{n})]$

By putting $k = 0, 1, 2, \ldots\ldots(n-1)$ we get n-n^{th} roots of complex number $z = x + iy$.

===

Ex. Show that i) the n-n^{th} roots of unity form geometrical progression,

ii) the sum of n-n^{th} roots of unity is zero.

Proof: Let ω be the n^{th} root of unity.

$\therefore \omega^n = 1 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = \cos(2k\pi) + i\sin(2k\pi)$

$\therefore \omega_k = [\cos(2k\pi) + i\sin(2k\pi)]^{1/n}$

$= \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$, where $k = 0, 1, 2, 3, \ldots\ldots(n-1)$.

Putting $k = 0, 1, 2, 3, \ldots\ldots(n-1)$ we get n-n^{th} roots of unity as

$\omega_0 = \cos 0 + i\sin 0 = 1,$

$\omega_1 = \cos(\frac{2\pi}{n}) + i\sin\left(\frac{2\pi}{n}\right) = \omega$

$\omega_2 = \cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right) = \omega^2$

$\omega_3 = \cos\left(\frac{6\pi}{n}\right) + i\sin\left(\frac{6\pi}{n}\right) = \omega^3$

…

…

$\omega_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = \omega^{n-1}$

i) $1, \omega, \omega^2, \omega^3, \ldots\ldots, \omega^{n-1}$ are the n-n^{th} roots of unity,

where $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ and $\omega^n = 1$

This shows that the n-n^{th} roots of unity form geometrical progression.

ii) Let 1, $\omega, \omega^2, \omega^3, \ldots\ldots, \omega^{n-1}$ are the n-n^{th} roots of unity,

where $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ and $\omega^n = 1$

$\therefore 1+\omega+\omega^2+\omega^3+\ldots\ldots+\omega^{n-1} = \frac{1-\omega^n}{1-\omega}$ $\quad\because \omega \neq 1$

$= \frac{1-1}{1-\omega}$

$= 0$

Thus the sum of n-n^{th} roots of unity is zero is proved.

==

Ex. Find the cube roots of unity.

Proof: Let ω be the cube root of unity.

$\therefore \omega^3 = 1 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = \cos(2k\pi) + i\sin(2k\pi)$

$\therefore \omega_k = [\cos(2k\pi) + i\sin(2k\pi)]^{1/3}$

$= \cos(\frac{2k\pi}{3}) + i\sin(\frac{2k\pi}{3})$, where k = 0, 1, 2.

Putting k = 0, 1, 2. we get cube roots of unity as

$\omega_0 = \cos 0 + i\sin 0 = 1$,

$\omega_1 = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$

$\omega_2 = \cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}) = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}$

1, $\frac{-1\pm i\sqrt{3}}{2}$ are the cube roots of unity.

==

Ex. Find the five-fifth roots of unity.

Proof: Let ω be the fifth root of unity.

$\therefore \omega^5 = 1 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = \cos(2k\pi) + i\sin(2k\pi)$

$\therefore \omega_k = [\cos(2k\pi) + i\sin(2k\pi)]^{1/5}$

$= \cos(\frac{2k\pi}{5}) + i\sin(\frac{2k\pi}{5})$, where k = 0, 1, 2, 3, 4.

Putting k = 0, 1, 2, 3, 4. we get fifth roots of unity as

$\omega_0 = \cos 0 + i\sin 0 = 1$

$\omega_1 = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$

$\omega_2 = \cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5})$

$\omega_3 = \cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5}) = \cos(\frac{4\pi}{5}) - i\sin(\frac{4\pi}{5})$ $\quad\because \frac{6\pi}{5} = 2\pi - \frac{4\pi}{5}$

$\omega_2 = \cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5}) = \cos(\frac{2\pi}{5}) - i\sin(\frac{2\pi}{5})$ $\quad\because \frac{8\pi}{5} = 2\pi - \frac{2\pi}{5}$

1, $\cos(\frac{2\pi}{5}) \pm i\sin(\frac{2\pi}{5})$ & $\cos(\frac{4\pi}{5}) \pm i\sin(\frac{4\pi}{5})$ are the five-fifth roots of unity.

==

Ex. Find the five-fifth roots of -1.

Proof: Let ω be the fifth root of -1.

$\therefore \omega^5 = -1 = \cos\pi + i\sin\pi = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$

$\therefore \omega_k = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{1/5}$

$= \cos\left(\frac{\pi+2k\pi}{5}\right) + i\sin\left(\frac{\pi+2k\pi}{5}\right)$, where k = 0, 1, 2, 3, 4.

Putting k = 0, 1, 2, 3, 4. we get fifth roots of -1 as

$\omega_0 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$

$\omega_1 = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)$

$\omega_2 = \cos(\pi) + i\sin(\pi) = -1$

$\omega_3 = \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right) = \cos\left(\frac{3\pi}{5}\right) - i\sin\left(\frac{3\pi}{5}\right) \qquad \because \frac{7\pi}{5} = 2\pi - \frac{3\pi}{5}$

$\omega_2 = \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right) \quad \because \frac{9\pi}{5} = 2\pi - \frac{\pi}{5}$

-1, $\cos\left(\frac{\pi}{5}\right) \pm i\sin\left(\frac{\pi}{5}\right)$ & $\cos\left(\frac{3\pi}{5}\right) \pm i\sin\left(\frac{3\pi}{5}\right)$ are the five-fifth roots of -1.

==

Ex. Find all the values of $(1 - i\sqrt{3})^{1/4}$.

Solution: Let $z = 1 - i\sqrt{3}$

$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= 2\left[\cos\left(\frac{\pi}{3} + 2k\pi\right) - i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right]$

$= 2\left[\cos\left(\frac{\pi+6k\pi}{3}\right) - i\sin\left(\frac{\pi+6k\pi}{3}\right)\right]$

$\therefore \omega_k = z^{1/4} = (1 - i\sqrt{3})^{1/4} = 2^{1/4}\left[\cos\left(\frac{\pi+6k\pi}{3}\right) - i\sin\left(\frac{\pi+6k\pi}{3}\right)\right]^{1/4}$

$= 2^{1/4}\left[\cos\left(\frac{\pi+6k\pi}{12}\right) - i\sin\left(\frac{\pi+6k\pi}{12}\right)\right]$, where k = 0, 1, 2, 3.

Putting k = 0, 1, 2, 3. we get all the values of $(1 - i\sqrt{3})^{1/4}$ as

$\omega_0 = 2^{1/4}\left[\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right]$,

$\omega_1 = 2^{1/4}\left[\cos\left(\frac{7\pi}{12}\right) - i\sin\left(\frac{7\pi}{12}\right)\right]$,

$\omega_2 = 2^{1/4}\left[\cos\left(\frac{13\pi}{12}\right) - i\sin\left(\frac{13\pi}{12}\right)\right] = 2^{1/4}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right] \qquad \because \frac{13\pi}{12} = 2\pi - \frac{11\pi}{12}$

& $\omega_3 = 2^{1/4}\left[\cos\left(\frac{19\pi}{12}\right) - i\sin\left(\frac{19\pi}{12}\right)\right] = 2^{1/4}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right] \qquad \because \frac{19\pi}{12} = 2\pi - \frac{5\pi}{12}$

==

Ex. Find all the values of $(1 + i)^{1/5}$. Show that their continued product is 1 + i.

Proof: Let $z = 1 + i$

$= \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$

$= 2^{1/2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$= 2^{1/2}[\cos(\frac{\pi}{4} + 2k\pi) + i\sin(\frac{\pi}{4} + 2k\pi)]$

$= 2^{1/2}[\cos(\frac{\pi+8k\pi}{4}) + i\sin(\frac{\pi+8k\pi}{4})]$

$\therefore \omega_k = z^{1/5} = (1+i)^{1/5} = 2^{1/10}[\cos(\frac{\pi+8k\pi}{4}) + i\sin(\frac{\pi+8k\pi}{4})]^{1/5}$

$= 2^{1/10}[\cos(\frac{\pi+8k\pi}{20}) + i\sin(\frac{\pi+8k\pi}{20})]$, where k = 0, 1, 2, 3, 4.

Putting k = 0, 1, 2, 3, 4. we get all the values of $(1 + i)^{1/5}$ as

$\omega_0 = 2^{1/10}[\cos(\frac{\pi}{20}) + i\sin(\frac{\pi}{20})]$,

$\omega_1 = 2^{1/10}[\cos(\frac{9\pi}{20}) + i\sin(\frac{9\pi}{20})]$,

$\omega_2 = 2^{1/10}[\cos(\frac{17\pi}{20}) + i\sin(\frac{17\pi}{20})]$,

$\omega_3 = 2^{1/10}[\cos(\frac{25\pi}{20}) + i\sin(\frac{25\pi}{20})]$,

& $\omega_4 = 2^{1/10}[\cos(\frac{33\pi}{20}) + i\sin(\frac{33\pi}{20})]$.

The continued product of these values is

$\omega_0.\omega_1.\omega_2.\omega_3.\omega_4 = 2^{5/10}[\cos(\frac{\pi}{20}+\frac{9\pi}{20}+\frac{17\pi}{20}+\frac{25\pi}{20}+\frac{33\pi}{20}) + i\sin(\frac{\pi}{20}+\frac{9\pi}{20}+\frac{17\pi}{20}+\frac{25\pi}{20}+\frac{33\pi}{20})]$

$= 2^{1/2}[\cos(\frac{85\pi}{20}) + i\sin(\frac{85\pi}{20})]$

$= \sqrt{2}[\cos(\frac{17\pi}{4}) + i\sin(\frac{17\pi}{4})]$

$= \sqrt{2}[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})] \quad \because \frac{17\pi}{4} = 4\pi + \frac{\pi}{4}$

$= \sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$

$= 1 + i$

Hence proved

==

Ex. Solve the equation $x^2 - i = 0$.

Solution: Let $x^2 - i = 0$

i.e. $x^2 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \cos(\frac{\pi}{2} + 2k\pi) + i\sin(\frac{\pi}{2} + 2k\pi)$

$\therefore x = [\cos(\frac{\pi+4k\pi}{2}) + i\sin(\frac{\pi+4k\pi}{2})]^{1/2}$

$= \cos(\frac{\pi+4k\pi}{4}) + i\sin(\frac{\pi+4k\pi}{4})$ Where k = 0, 1.

Putting k = 0, 1. we get roots of $x^2 - i = 0$ as

$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and , $\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}) = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Ex. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$.

Solution: Let $x^9 - x^5 + x^4 - 1 = 0$

i.e. $x^5 (x^4 - 1) + (x^4 - 1) = 0$

i.e. $(x^4 - 1)(x^5 + 1) = 0$ be the given equation with

$x^4 - 1 = 0$ or $x^5 + 1 = 0$

Now $x^4 - 1 = 0$ gives

$x^4 = 1 = \cos 0 + i\sin 0 = \cos(2k\pi) + i\sin(2k\pi)$

$\therefore x = [\cos(2k\pi) + i\sin(2k\pi)]^{1/4}$

$= \cos(\frac{2k\pi}{4}) + i\sin(\frac{2k\pi}{4})$

$= \cos(\frac{k\pi}{2}) + i\sin(\frac{k\pi}{2})$ where k = 0, 1, 2, 3.

Putting k = 0, 1, 2, 3. we get solution of $x^4 - 1 = 0$ as

$\cos 0 + i\sin 0 = 1$, $\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$, $\cos\pi + i\sin\pi = -1$ and

$\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = -i$

And $x^5 + 1 = 0$ gives

$\therefore x^5 = -1 = \cos\pi + i\sin\pi = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$

$\therefore x = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{1/5}$

$= \cos(\frac{\pi+2k\pi}{5}) + i\sin(\frac{\pi+2k\pi}{5})$, where k = 0, 1, 2, 3, 4.

Putting k = 0, 1, 2, 3, 4. we get fifth roots of -1 as

$\cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5})$, $\cos(\frac{3\pi}{5}) + i\sin(\frac{3\pi}{5})$, $\cos(\pi) + i\sin(\pi) = -1$

$\cos(\frac{7\pi}{5}) + i\sin(\frac{7\pi}{5}) = \cos(\frac{3\pi}{5}) - i\sin(\frac{3\pi}{5})$ $\because \frac{7\pi}{5} = 2\pi - \frac{3\pi}{5}$

and $\cos(\frac{9\pi}{5}) + i\sin(\frac{9\pi}{5}) = \cos(\frac{\pi}{5}) - i\sin(\frac{\pi}{5}) \because \frac{9\pi}{5} = 2\pi - \frac{\pi}{5}$

$\pm 1, \pm i, \cos(\frac{\pi}{5}) \pm i\sin(\frac{\pi}{5})$ & $\cos(\frac{3\pi}{5}) \pm i\sin(\frac{3\pi}{5})$ are the roots of given equation.

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Ex. Solve the equation $x^4 + x^3 + x^2 + x + 1 = 0$.

Solution: We have $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$ …… (1)

Consider the equation $x^5 - 1 = 0$

$\therefore x^5 = 1 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = \cos(2k\pi) + i\sin(2k\pi)$

$\therefore x_k = [\cos(2k\pi) + i\sin(2k\pi)]^{1/5}$

$= \cos(\frac{2k\pi}{5}) + i\sin(\frac{2k\pi}{5})$, where k = 0, 1, 2, 3, 4.

Putting k = 0, 1, 2, 3, 4. we get fifth roots of unity as

$x_0 = \cos 0 + i\sin 0 = 1$,

$x_1 = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$,

$x_2 = \cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5})$,

$x_3 = \cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5}) = \cos(\frac{4\pi}{5}) - i\sin(\frac{4\pi}{5})$ $\because \frac{6\pi}{5} = 2\pi - \frac{4\pi}{5}$

and $x_4 = \cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5}) = \cos(\frac{2\pi}{5}) - i\sin(\frac{2\pi}{5})$ $\because \frac{8\pi}{5} = 2\pi - \frac{2\pi}{5}$

1, $\cos(\frac{2\pi}{5}) \pm i\sin(\frac{2\pi}{5})$ & $\cos(\frac{4\pi}{5}) \pm i\sin(\frac{4\pi}{5})$ are the five-fifth roots of $z^5 = 1$.

Out of these $x_0 = 1$ corresponds to the factor $x - 1 = 0$ in equation (1).

Hence $\cos(\frac{2\pi}{5}) \pm i\sin(\frac{2\pi}{5})$ & $\cos(\frac{4\pi}{5}) \pm i\sin(\frac{4\pi}{5})$ are the roots of given equation.

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Ex. Solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$.

Solution: We have $x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1)$ …… (1)

Consider the equation $x^5 + 1 = 0$

$\therefore x^5 = -1 = = \cos\pi + i\sin\pi = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$

$\therefore x_k = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{1/5}$

$= \cos(\frac{\pi+2k\pi}{5}) + i\sin(\frac{\pi+2k\pi}{5})$, where $k = 0, 1, 2, 3, 4$.

Putting $k = 0, 1, 2, 3, 4$. we get roots of equation $x^5 + 1 = 0$ as

$x_0 = \cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5})$,

$x_1 = \cos(\frac{3\pi}{5}) + i\sin(\frac{3\pi}{5})$,

$x_2 = \cos(\pi) + i\sin(\pi) = -1$,

$x_3 = \cos(\frac{7\pi}{5}) + i\sin(\frac{7\pi}{5}) = \cos(\frac{3\pi}{5}) - i\sin(\frac{3\pi}{5})$ $\because \frac{7\pi}{5} = 2\pi - \frac{3\pi}{5}$

and $x_4 = \cos(\frac{9\pi}{5}) + i\sin(\frac{9\pi}{5}) = \cos(\frac{\pi}{5}) - i\sin(\frac{\pi}{5})$ $\because \frac{9\pi}{5} = 2\pi - \frac{3\pi}{5}$

Out of these $x_2 = -1$ corresponds to the factor $x + 1 = 0$ in equation (1).

Hence $\cos(\frac{\pi}{5}) \pm i\sin(\frac{\pi}{5})$ & $\cos(\frac{3\pi}{5}) \pm i\sin(\frac{3\pi}{5})$ are the roots of given equation.

==

Ex. Solve the equation $x^8 - x^4 + 1 = 0$.

Solution: Let $x^8 - x^4 + 1 = 0$ …… (1) be the given equation.

Put $x^4 = z$, we get,

$z^2 - z + 1 = 0$ having roots $z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore x^4 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = \cos(\frac{\pi}{3} + 2k\pi) \pm i\sin(\frac{\pi}{3} + 2k\pi)$

$\therefore x_k = [\cos(\frac{\pi+6k\pi}{3}) \pm i\sin(\frac{\pi+6k\pi}{3})]^{1/4}$

$= \cos(\frac{\pi+6k\pi}{12}) \pm i\sin(\frac{\pi+6k\pi}{12})$, where $k = 0, 1, 2, 3$.

Putting $k = 0, 1, 2, 3$. we get,

$x_0 = \cos(\frac{\pi}{12}) \pm i\sin(\frac{\pi}{12})$,

$x_1 = \cos(\frac{7\pi}{12}) \pm i\sin(\frac{7\pi}{12})$,

$x_2 = \cos(\frac{13\pi}{12}) \pm i\sin(\frac{13\pi}{12})$,

and $x_3 = \cos(\frac{19\pi}{12}) \pm i\sin(\frac{19\pi}{12})$ are the roots of given equation.

Application of DeMoivre's Theorem to Prove Trignometric Identities:

1) To express $\sin n\theta$ and $\cos n\theta$ in powers of $\sin\theta$ & $\cos\theta$, we use DeMoivre's Theorem as $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$$= \cos^n\theta + {}^nc_1\cos^{n-1}\theta(i\sin\theta) + {}^nc_2\cos^{n-2}\theta(i\sin\theta)^2 + \ldots$$
$$+ {}^nc_{n-1}\cos\theta(i\sin\theta)^{n-1} + (i\sin\theta)^n \quad \text{by using binomial theorem.}$$

By simplifying and equating real and imaginary parts we get required expansions.

2) Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$ and

$x^m = \cos m\theta + i\sin m\theta$, $\frac{1}{x^m} = \cos m\theta - i\sin m\theta$ by DeMoivre's Theorem.

Now $x + \frac{1}{x} = 2\cos\theta$, $x - \frac{1}{x} = 2i\sin\theta$,

$$x^m + \frac{1}{x^m} = 2\cos m\theta, \; x^m - \frac{1}{x^m} = 2i\sin m\theta \quad \ldots\ldots (1)$$

i) To express $\sin^n\theta$ in terms of multiple angle of sine, consider

$$(2i\sin\theta)^n = (x - \tfrac{1}{x})^n = x^n + {}^nc_1x^{n-1}(-\tfrac{1}{x}) + {}^nc_2x^{n-2}(-\tfrac{1}{x})^2 + \ldots + {}^nc_{n-1}x(-\tfrac{1}{x})^{n-1} + (-\tfrac{1}{x})^n$$

by using binomial theorem.

ii) To express $\cos^n\theta$ in terms of multiple angle of cosine, consider

$$(2\cos\theta)^n = (x + \tfrac{1}{x})^n = x^n + {}^nc_1x^{n-1}(\tfrac{1}{x}) + {}^nc_2x^{n-2}(\tfrac{1}{x})^2 + \ldots + {}^nc_{n-1}x(\tfrac{1}{x})^{n-1} + (\tfrac{1}{x})^n$$

by using binomial theorem.

By simplifying and using equation (1), we get required expansions.

===

Ex. Use DeMoivre's Theorem to prove the following

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Proof: Consider

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + {}^5c_1\cos^4\theta(i\sin\theta) + {}^5c_2\cos^3\theta(i\sin\theta)^2$

$+ {}^5c_3\cos^2\theta(i\sin\theta)^3 + {}^5c_4\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$ by binomial theorem

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

By equating real and imaginary parts, we get,

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Hence proved.

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Ex. Use DeMoivre's Theorem to prove the following

$\cos6\theta = \cos^6\theta - 15\cos^4\theta \sin^2\theta+15\cos^2\theta \sin^4\theta- \sin^6\theta$

$\sin6\theta = 6\cos^5\theta \sin\theta -20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$

Proof: Consider

$$\begin{aligned}
&\text{os}6\theta + i\sin6\theta\\
&= (\cos\theta + i\sin\theta)^6\\
&= \cos^6\theta +{}^6c_1\cos^5\theta(i\sin\theta)+{}^6c_2\cos^4\theta(i\sin\theta)^2+{}^6c_3\cos^3\theta(i\sin\theta)^3\\
&\quad +{}^6c_4\cos^2\theta(i\sin\theta)^4+{}^6c_5\cos\theta(i\sin\theta)^5+(i\sin\theta)^6 \qquad \text{by binomial theorem}\\
&= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta\\
&\quad + 15\cos^2\theta\sin^4\theta+6i\cos\theta\sin^5\theta - \sin^6\theta\\
&= (\cos^6\theta - 15\cos^4\theta \sin^2\theta+15\cos^2\theta \sin^4\theta- \sin^6\theta)\\
&\quad + i(6\cos^5\theta \sin\theta -20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta)
\end{aligned}$$

By equating real and imaginary parts, we get,

$\cos6\theta = \cos^6\theta - 15\cos^4\theta \sin^2\theta+15\cos^2\theta \sin^4\theta- \sin^6\theta$

$\sin6\theta = 6\cos^5\theta \sin\theta -20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$

Hence proved.

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Ex. Express $\frac{\sin7\theta}{\sin\theta}$ in powers of $\sin\theta$ only.

Solution: Consider

$$\begin{aligned}
&\cos7\theta + i\sin7\theta\\
&= (\cos\theta + i\sin\theta)^7\\
&= \cos^7\theta +{}^7c_1\cos^6\theta(i\sin\theta)+{}^7c_2\cos^5\theta(i\sin\theta)^2+{}^7c_3\cos^4\theta(i\sin\theta)^3+{}^7c_4\cos^3\theta(i\sin\theta)^4\\
&\quad +{}^7c_5\cos^2\theta(i\sin\theta)^5+{}^7c_6\cos\theta(i\sin\theta)^6+(i\sin\theta)^7 \qquad \text{by binomial theorem}\\
&= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta\\
&\quad + 35\cos^3\theta\sin^4\theta+ 21i\cos^2\theta\sin^5\theta-7\cos\theta\sin^6\theta -i\sin^7\theta\\
&= (\cos^7\theta - 21\cos^5\theta\sin^2\theta+35\cos^3\theta \sin^4\theta - 7\cos\theta\sin^6\theta)\\
&\quad + i(7\cos^6\theta \sin\theta -35\cos^4\theta \sin^3\theta + 21\cos^2\theta\sin^5\theta-\sin^7\theta)
\end{aligned}$$

By equating real imaginary parts, we get,

$\sin7\theta = 7\cos^6\theta \sin\theta -35\cos^4\theta \sin^3\theta + 21\cos^2\theta\sin^5\theta-\sin^7\theta$

$$\begin{aligned}
\therefore \frac{\sin7\theta}{\sin\theta} &= 7\cos^6\theta -35\cos^4\theta \sin^2\theta + 21\cos^2\theta\sin^4\theta-\sin^6\theta\\
&= 7(1- \sin^2\theta)^3-35(1- \sin^2\theta)^2\sin^2\theta + 21(1- \sin^2\theta)\sin^4\theta-\sin^6\theta\\
&= 7(1- 3\sin^2\theta+3\sin^4\theta-\sin^6\theta)-35\sin^2\theta(1- 2\sin^2\theta+\sin^4\theta)\\
&\quad + 21\sin^4\theta-21\sin^6\theta-\sin^6\theta\\
&= 7 - 21\sin^2\theta+21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta+70\sin^4\theta -35\sin^6\theta
\end{aligned}$$

$+ 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$

$\therefore \frac{\sin7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$

Ex. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then show that

i) $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$ and

$\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

ii) $\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and

$\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

Proof: Given $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$ … … (1)

Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$

$\therefore\ a + b + c = \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0$ by (1)

$\therefore\ a + b + c = 0$ … … (2)

and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$\therefore \frac{bc + ac + ab}{abc} = 0 - i0$ by (1)

$\therefore ab + bc + ac = 0$ … … (3)

i) As $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$\therefore a^3 + b^3 + c^3 - 3abc = 0$ by (2)

$\therefore a^3 + b^3 + c^3 = 3abc$

$\therefore \cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma$

$= 3[\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)]$

$\therefore (\cos3\alpha + \cos3\beta + \cos3\gamma) + i(\sin3\alpha + \sin3\beta + \sin3\gamma$

$= 3\cos(\alpha + \beta + \gamma) + i3\sin(\alpha + \beta + \gamma)$

Equating real and imaginary parts, we get,

$\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha + \beta + \gamma)$ and

$\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

ii) As $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$= 0$ by (2) and (3)

$\therefore \cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0$

$\therefore (\cos2\alpha + \cos2\beta + \cos2\gamma) + i(\sin2\alpha + \sin2\beta + \sin2\gamma) = 0$

Equating real and imaginary parts, we get,

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ and

$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

===

Ex. Prove that $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$\therefore x - \frac{1}{x} = 2i\sin\theta$ and $x^m - \frac{1}{x^m} = 2i\sin m\theta$

$\therefore (2i\sin\theta)^5 = (x - \frac{1}{x})^5$

$\therefore 32i\sin^5\theta = x^5 - 5x^4(\frac{1}{x}) + 10x^3(\frac{1}{x})^2 - 10x^2(\frac{1}{x})^3 + 5x(\frac{1}{x})^4 - (\frac{1}{x})^5$

$= x^5 - 5x^3 + 10x - 10\frac{1}{x} + 5\frac{1}{x^3} - \frac{1}{x^5}$

$= (x^5 - \frac{1}{x^5}) - 5(x^3 - \frac{1}{x^3}) + 10(x - \frac{1}{x})$

$\therefore 32i\sin^5\theta = (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$\therefore \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Hence proved.

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Ex. Express $\cos^6\theta$ in terms of cosines of multiples of θ.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$\therefore x + \frac{1}{x} = 2\cos\theta$ and $x^m + \frac{1}{x^m} = 2\cos m\theta$

$\therefore (2\cos\theta)^6 = (x + \frac{1}{x})^6$

$\therefore 64\cos^6\theta = x^6 + 6x^5(\frac{1}{x}) + 15x^4(\frac{1}{x})^2 + 20x^3(\frac{1}{x})^3 + 15x^2(\frac{1}{x})^4 + 6x(\frac{1}{x})^5 + (\frac{1}{x})^6$

$= x^6 + 6x^4 + 15x^2 + 20 + 15\frac{1}{x^2} + 6\frac{1}{x^4} + \frac{1}{x^6}$

$= (x^6 + \frac{1}{x^6}) + 6(x^4 + \frac{1}{x^4}) + 15(x^2 + \frac{1}{x^2}) + 20$

$\therefore 64\cos^6\theta = (2\cos 6\theta) + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$\therefore \cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

===

Ex. Express $\sin^7\theta$ in terms of sines of multiples of θ.

Solution: Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$\therefore x - \frac{1}{x} = 2i\sin\theta$ and $x^m - \frac{1}{x^m} = 2i\sin m\theta$

$\therefore (2i\sin\theta)^7 = (x - \frac{1}{x})^7$

$\therefore -128i\sin^7\theta = x^7 - 7x^6(\frac{1}{x}) + 21x^5(\frac{1}{x})^2 - 35x^4(\frac{1}{x})^3 + 35x^3(\frac{1}{x})^4 - 21x^2(\frac{1}{x})^5 + 7x(\frac{1}{x})^6 - (\frac{1}{x})^7$

$= x^7 - 7x^5 + 21x^3 - 35x + 35\frac{1}{x} - 21\frac{1}{x^3} + 7\frac{1}{x^5} - \frac{1}{x^7}$

$= (x^7 - \frac{1}{x^7}) - 7(x^5 - \frac{1}{x^5}) + 21(x^3 - \frac{1}{x^3}) - 35(x - \frac{1}{x})$

$\therefore$ -128isin$^7\theta$= (2isin7θ) - 7(2isin5θ) + 21(2isin3θ) - 35(2isinθ)

$\therefore$ sin$^7\theta$ = $-\frac{1}{64}$(sin7θ - 7sin5θ + 21sin3θ - 35sinθ)

Elementary Functions:

I) Trignometric Functions of a Complex Variables:

i) Let z ∈ C, the sine function of a complex variable z is denoted by sinz and defined as sinz = $\frac{e^{iz}-e^{-iz}}{2i}$

ii) Let z ∈ C, the cosine function of a complex variable z is denoted by cosz and defined as cosz = $\frac{e^{iz}+e^{-iz}}{2}$

Remark : Using above definitaions, we get i) tanz = $\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$, ii) cotz = $\frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$,

iii) cosecz = $\frac{2i}{e^{iz}-e^{-iz}}$ and iv) secz = $\frac{2}{e^{iz}+e^{-iz}}$

Periodic Function: A function f(z) is said to be periodic function of period T if f(z+T) = f(z)

Remark : i) sinz and cosz are periodic functions of period 2π,

ii) cosecz and secz are periodic functions of period 2π,

iii) tanz and cotz are periodic functions of period π

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Ex. Using the definition of cosz and sinz, prove that cos^{2}z + sin^{2}z = 1

Proof: We have cosz = $\frac{e^{iz}+e^{-iz}}{2}$ and sinz = $\frac{e^{iz}-e^{-iz}}{2i}$

$$\therefore \cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2+\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$=\left(\frac{e^{2iz}+2+e^{-2iz}}{4}\right)+\left(\frac{e^{2iz}-2+e^{-2iz}}{-4}\right)$$

$$=\left(\frac{e^{2iz}+2+e^{-2iz}-e^{2iz}+2-e^{-2iz}}{4}\right)$$

$$=\frac{4}{4}$$

$\therefore$ cos^{2}z + sin^{2}z = 1   Hence proved.

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Hyperbolic Functions of a Complex Variables:

i) Let z ∈ C, the hyperbolic sine function of a complex variable z is denoted by sinhz and defined as sinhz = $\frac{e^{z}-e^{-z}}{2}$

ii) Let z ∈ C, the hyperbolic cosine function of a complex variable z is denoted by coshz and defined as coshz = $\frac{e^{z}+e^{-z}}{2}$

Remark : Using above definitaions, we get i) tanhz = $\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$, ii) cothz = $\frac{e^{z}+e^{-z}}{e^{z}-e^{-z}}$,

iv) cosechz = $\frac{2}{e^{z}-e^{-z}}$ and iv) sechz = $\frac{2}{e^{z}+e^{-z}}$

Remark: i) sinhz and coshz are periodic functions of period 2πi,

ii) cosechz and sechz are periodic functions of period 2πi,

iii) tanhz and cothz are periodic functions of period πi.

Ex. Show that i) siniz = isinhz ii) cosiz = coshz iii) taniz = itanhz

iv) sinhiz = isinz v) coshiz = cosz vi) tanhiz = itanz

Proof: i) $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

$\therefore \sin iz = \left(\frac{e^{i(iz)}-e^{-i(iz)}}{2i}\right) = i\left(\frac{e^{-z}-e^{z}}{-2}\right) = i\left(\frac{e^{z}-e^{-z}}{2}\right) = i\sinh z$

ii) $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

$\therefore \cos iz = \left(\frac{e^{i(iz)}+e^{-i(iz)}}{2}\right) = \left(\frac{e^{-z}+e^{z}}{2}\right) = \left(\frac{e^{z}+e^{-z}}{2}\right) = \cosh z$

iii) $\tan z = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$

$\therefore \tan iz = \frac{e^{i(iz)}-e^{-i(iz)}}{i(e^{i(iz)}+e^{-i(iz)})} = i\frac{e^{-z}-e^{z}}{-(e^{-z}+e^{z})} = i\left(\frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}\right) = i\tanh z$

iv) $\sinh z = \frac{e^{z}-e^{-z}}{2}$

$\therefore \sinh iz = \left(\frac{e^{(iz)}-e^{-(iz)}}{2}\right) = i\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = i\sin z$

v) $\cosh z = \frac{e^{z}+e^{-z}}{2}$

$\therefore \cosh iz = \left(\frac{e^{iz}+e^{-iz}}{2}\right) = \cos z$

vi) $\tanh z = \frac{e^{z}-e^{-z}}{(e^{z}+e^{-z})}$

$\therefore \tanh iz = \frac{e^{iz}-e^{-iz}}{(e^{iz}+e^{-iz})} = i\left[\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\right] = i\tan z$

Hence proved.

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Remark: i) siniz = isinhz ∴ sinhz = -isiniz

ii) cosiz = coshz ∴ coshz = cosiz

iii) taniz = itanhz ∴ tanhz = -itaniz

iv) sinhiz = isinz ∴ sinz = -isinhiz

v) coshiz = cosz ∴ cosz = coshiz

vi) tanhiz = itanz ∴ tanz = -itanhiz

==

Ex. Using the definition of coshz and sinhz, prove that $\cosh^2 z - \sinh^2 z = 1$

Proof: We have $\cosh z = \frac{e^{z}+e^{-z}}{2}$ and $\sinh z = \frac{e^{z}-e^{-z}}{2}$

$\therefore \cosh^2 z - \sinh^2 z = \left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right)^2$

$= \left(\frac{e^{2z}+2+e^{-2z}}{4}\right) - \left(\frac{e^{2z}-2+e^{-2z}}{4}\right)$

$= \left(\frac{e^{2z}+2+e^{-2z}-e^{2z}+2-e^{-2z}}{4}\right)$

$= \frac{4}{4}$

$\therefore \cosh^2 z - \sinh^2 z = 1$ Hence proved.

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MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) $i^m = 1$, if $m \in \mathbb{Z}$ is multiple of …….

a) 2 b) 3 c) 4 d) 5

2) $i^m = -1$, if $m \in \mathbb{Z}$ is an even integer, but not multiple of …….

a) 2 b) 3 c) 4 d) 5

3) If z is real then …….

a) $\bar{z} = z$ b) $\bar{z} = -z$ c) $z = -z$ d) $z = 0$

4) If z is purely imaginary then …….

a) $\bar{z} = z$ b) $\bar{z} = -z$ c) $z = -z$ d) $z = 0$

5) Re(z) = ……. & Im(z) = ……

a) $\frac{z+\bar{z}}{2i}$ & $\frac{z-\bar{z}}{2}$ b) $\frac{z+\bar{z}}{2i}$ & $\frac{z-\bar{z}}{2i}$ c) $\frac{z+\bar{z}}{2}$ & $\frac{z-\bar{z}}{2i}$ d) None of these.

6) R(iz)= ……. & I(iz) = ……

a) I(z) & -R(z) b) -I(z) & R(z) c) -I(z) & -R(z) d) None of these.

7) Real and imaginary parts of $(1+i)^4$ are…and…. respectively.

a) 1 and 1 b) 1 and 0 c) -4 and 0 d) None of these.

8) Real and imaginary parts of $\frac{1}{2+3i}$ are…an…. respectively.

a) $\frac{2}{13}$ and $-\frac{3}{13}$ b) $-\frac{2}{13}$ and $\frac{3}{13}$ c) $-\frac{2}{13}$ and $-\frac{3}{13}$ d) None of these.

9) Real and imaginary parts of $i + i^2 + i^3 + i^4$ are…and... respectively.

a) 1 and 1 b) 1 and 0 c) 0 and 0 d) None of these.

10) Modulus and argument of complex number z = x+iy are ……. & ……. respectively.

a) x & $\tan^{-1}\frac{x}{y}$ b) y & $\sin^{-1}\frac{y}{x}$ c) $\sqrt{x^2+y^2}$ & $\tan^{-1}\frac{y}{x}$ d) None of these.

11) Modulus and argument of 1+ i are ……. & ……. respectively.

a) $\sqrt{2}$ & $-\frac{3\pi}{4}$ b) $\sqrt{2}$ & $\frac{\pi}{4}$ c) $\sqrt{2}$ & $-\frac{\pi}{4}$ d) None of these

12) Modulus and argument of $i^7 + i^{10}$ are ……. & ……. respectively.

a) $\sqrt{2}$ & $-\frac{3\pi}{4}$ b) $\sqrt{2}$ & $\frac{\pi}{4}$ c) $\sqrt{2}$ & $-\frac{\pi}{4}$ d) None of these

13) Modulus $z = (-1, \sqrt{3})$ is …….

a) -1 b) 3 c) 1 d) 2

14) Modulus $z = 1+i\sqrt{3}$ is …….

a) -1 b) 2 c) 1 d) 0

15) $\frac{5}{(1-i)(2-i)(3-i)}$ = …….

a) $-\frac{1}{2}i$ b) $\frac{1}{2}i$ c) i d) None of these

16) $(3+i)(3-i)(\frac{1}{5}+\frac{i}{10}) = \ldots\ldots.$

a) $2 + i$ b) $2 - i$ c) i d) None of these

17) Complex conjugate of $\frac{1}{2+3i}$ is…….

a) 2-3i b) $\frac{2}{13}+\frac{3}{13}i$ c) $\frac{2}{13}+\frac{3}{13}i$ d) 2+3i

18) Complex conjugate of $\frac{3}{i}+\frac{7}{2}$ is…….

a) $\frac{7}{2}+3i$ b) $\frac{7}{2}-3i$ c) $-\frac{3}{i}-\frac{7}{2}$ d) 3+7i

19) Complex conjugate of $i^{15}+i^{19}$ is…….

a) -2i b) 2i c) 2 d) -2

20) For any two complex numbers z_1 and z_2, $|z_1+z_2|^2+|z_1-z_2|^2 = \ldots\ldots$

a) $2|z_1|^2+2|z_2|^2$ b) $|z_1|^2+|z_2|^2$ c) $2|z_1|^2-2|z_2|^2$ d) None of these

21) Multiplicative identity in a set of complex number is …….

a) 1 b) 0 c) -i d) i

22) For any two complex numbers z_1 and z_2: $|z_1+z_2| \ldots. |z_1|+|z_2|$

a) < b) ≤ c) > d) =

23) For any two complex numbers z_1 and z_2: $|z_1-z_2| \ldots. \big||z_1|-|z_2|\big|$

a) < b) ≤ c) > d) ≥

24) For $z_1, z_2 \in C$, $|z_1 z_2| = \ldots\ldots$

a) $|z_1|\,|z_2|$ b) $|z_1|+|z_2|$ c) $|z_1|-|z_2|$ d) None of these

25) For $z_1, z_2 \in C$, $\arg(z_1 z_2) = \ldots\ldots$

a) $\arg z_1+\arg z_2$ b) $\arg z_1-\arg z_2$ c) $\arg z_1 . \arg z_2$ d) None of these

26) For $z_1, z_2 \in C$, $\left|\frac{z_1}{z_2}\right| = \ldots\ldots$

a) $|z_1|\,|z_2|$ b) $\frac{|z_2|}{|z_1|}$ c) $\frac{|z_1|}{|z_2|}$ d) None of these

27) For $z_1, z_2 \in C$, $\arg(\frac{z_1}{z_2}) = \ldots\ldots$

a) $\arg z_1+\arg z_2$ b) $\arg z_1-\arg z_2$ c) $\arg z_1 . \arg z_2$ d) None of these

28) ω is said to be n^{th} root of complex number z if …...

a) $\omega^n = z$ b) $\omega = z^n$ c) $\omega = z$ d) None of these

29) ω is said to be n^{th} root of unity if …...

a) $\omega^n = 1$ b) $\omega = 0$ c) $\omega = -1$ d) None of these

30) n - n^{th} roots of unity are in ……. progression.

a) arithmetic b) geometric c) harmonic d) None of these

31) Sum of all n - n^{th} roots of unity is …….

a) 1 b) 0 c) -1 d) None of these

32) One of n^{th} root of unity is …….

a) -1 b) 0 c) 1 d) None of these

33) $i + i^2 + i^3 + i^4 = \ldots\ldots.$

a) 1 b) 0 c) -1 d) i

34) cosz =……. & sinz =…….

a) $\frac{e^{iz}+e^{-iz}}{2}$ & $\frac{e^{iz}-e^{-iz}}{2i}$ b) $\frac{e^{iz}+e^{-iz}}{2i}$ & $\frac{e^{iz}-e^{-iz}}{2}$ c) $\frac{e^{iz}-e^{-iz}}{2i}$ & $\frac{e^{iz}+e^{-iz}}{2i}$ d) $\frac{e^{iz}+e^{-iz}}{2i}$ & $\frac{e^{iz}+e^{-iz}}{2}$

35) If $z = e^{i\theta}$, then cosθ =……. & sinθ =…….

a) $\frac{1}{2}(z+\frac{1}{z})$ & $\frac{1}{2i}(z+\frac{1}{z})$ b) $\frac{1}{2}(z+\frac{1}{z})$ & $\frac{1}{2i}(z-\frac{1}{z})$

c) $\frac{1}{2}(z-\frac{1}{z})$ & $\frac{1}{2}(z-\frac{1}{z})$ d) $\frac{1}{2}(z+\frac{1}{z})$ & $\frac{1}{2}(z+\frac{1}{z})$

36) coshz =……. & sinhz =…….

a) $\frac{e^{z}+e^{-z}}{2}$ & $\frac{e^{z}-e^{-z}}{2}$ b) $\frac{e^{z}-e^{-z}}{2}$ & $\frac{e^{z}+e^{-z}}{2}$ c) $\frac{e^{-z}-e^{z}}{2}$ & $\frac{e^{z}+e^{-z}}{2}$ d) None of these.

37) If n is a rational number, then one of the value of $(\cos\theta + i\sin\theta)^n$ is …

a) $\cos\theta + i\sin\theta$ b) $\cos n\theta + i\sin n\theta$ c) $\cos n\theta - i\sin n\theta$ d) None of these.

38) If n is a natural number, then $(\cos\theta - i\sin\theta)^n$ = ……

a) $\cos\theta - i\sin\theta$ b) $\cos n\theta + i\sin n\theta$ c) $\cos n\theta - i\sin n\theta$ d) None of these.

39) If n is a natural number, then $(\cos\theta + i\sin\theta)^{-n}$ = ……

a) $\cos\theta + i\sin\theta$ b) $\cos n\theta + i\sin n\theta$ c) $\cos n\theta - i\sin n\theta$ d) None of these.

40) If n is a natural number, then $(\cos\theta - i\sin\theta)^{-n}$ = ……

a) $\cos\theta - i\sin\theta$ b) $\cos n\theta + i\sin n\theta$ c) $\cos n\theta - i\sin n\theta$ d) None of these.

41) $(\cos 3\theta + i\sin 3\theta)^8(\cos 4\theta - i\sin 4\theta)^{-2}$

a) $\cos 6\theta + i\sin 6\theta$, b) $\cos 7\theta + i\sin 7\theta$ c) $\cos 32\theta + i\sin 32\theta$ d) None of these.

42) If x = cosθ+isinθ, then $x + \frac{1}{x}$ = …….

a) 2cosθ b) cosθ c) 2sinθ d) None of these.

43) If x = cosθ+isinθ, then $x - \frac{1}{x}$ = …….

a) 2cosθ b) cosθ c) 2isinθ d) None of these.

44) If x = cosθ+isinθ, then $x^m + \frac{1}{x^m}$ = …….

a) 2cosmθ b) cosmθ c) 2sinmθ d) None of these.

45) If x = cosθ+isinθ, then $x^m - \frac{1}{x^m}$ = …….

a) 2cosmθ b) cosmθ c) 2isinmθ d) None of these.

46) cosiz = …… & siniz = …….

a) coshz & isinhz b) cosz & sinz c) icoshz & sinhz d) None of these.

47) taniz = …… & cotiz = …….

a) tanhz & icothz b) tanhz & cothz c) itanhz & -icothz d) None of these.

48) coseciz = …… & seciz = …….

a) coshz & isinhz b) -icosechz & sechz c) icoshz & sinhz d) None of these.

49) coshiz = …… & sinhiz = …….

a) cosz & isinz b) cosz & sinz c) icosz & sinz d) None of these.

50) tanhiz = …… & cothiz = …….

a) tanz & icotz b) tanz & cotz c) itanz & -icotz d) None of these.

UNIT-2: FUNCTIONS OF COMPLEX VARIABLES

==

Distance between two complex numbers: Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ be any two complex numbers in the complex plane, then the distance between z_1 and z_2 is given by $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Equation of a circle in a complex plane:

If $\delta > 0$ be any real number, then $|z - z_0| = \delta$ is represents a circle with centre at z_0 and radius δ where z_0 is the fixed complex number and z any point on the circle.

Neighbourhood of a point: Let z_0 be a fixed point in the complex plane C and $\delta > 0$ be a real number, then the set $\{z \,/\, |z - z_0| < \delta\}$ is called the δ −neighborhood of z_0 and is denoted by $N_\delta(z_0)$.

Deleted neighbourhood of a point: Let z_0 be a fixed point in the complex plane C and $\delta > 0$ be a real number, then the set $\{z \,/\, 0 < |z - z_0| < \delta\}$ is called the $\delta -$ deleted neighborhood of z_0 and is denoted by $N'_\delta(z_0)$.

Interior point: Let $S \subset C$, A point $z_0 \in S$ is called and interior point of S if there exist $\delta > 0$ such that $N_\delta(z_0) \subset S$.

Open set: A set $S \subset C$ is said to be open set if every point of S is an interior point of S.

Boundary point: A point z_0 is called a boundary point of the set S if every neighborhood of z_0 contains at least one point belonging to S and one point not belonging to S.

Exterior Point: A point which is neither an interior point nor a boundary point of the set S is called an exterior point of S.

Boundary of a set: The set of all boundary points of S is called a boundary of the set S

Bounded set: A set S in the z-plane is called a bounded set if there exist the positive constant M such that $|z| \le M$, for every $z \in S$.

Limit point: A point $z_0 \in C$ is called a limit point of the set S if every deleted neighbourhood of z_0 contains at least one point of S.

Closed set: A set S is called a closed set if it contains all its limit points.

Connected set: A set S is called a connected set if any two points of S can be joined by a continuous curve all of whose points belongs to S.

Domain or Region: An open connected set in C is called an open domain or open region.

Closed Domain: If boundary points of S are also included in an open domain, it is called closed domain.

Function of a complex variable: A rule f which associates with each z in S, a unique complex number w, is called a complex valued function of a complex variable z defined on S. w is called an image of z under f and we write $w = f(z)$.

Limit of a Function: If for small $\varepsilon > 0$, there exist $\delta > 0$ depends on ε such that $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. Then l is said to be limit of a complex function f(z) as $z \to z_0$. Denoted by $\lim_{z \to z_0} f(z) = l$.

Algebra of Limits:

If $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$ then

i) $\lim_{z \to z_0} [f(z) \pm g(z)] = l \pm m$

ii) $\lim_{z \to z_0} [f(z)g(z)] = lm$

iii) $\lim_{z \to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{l}{m}$ provided $m \neq 0$

==

Ex.: Evaluate $\lim_{z \to 1-i} [x + i(2x + y)]$

Sol. Consider $\lim_{z \to 1-i} [x + i(2x + y)]$

$= \lim_{(x,y) \to (1,-1)} [x + i(2x + y)]$

$= 1 + i(2 - 1)$

$= 1 + i$

==

Ex.: Evaluate $\lim_{z \to (2+3i)} [3x + i(2x - 4y)]$

Sol. Consider $\lim_{z \to (2+3i)} [3x + i(2x - 4y)]$

$= \lim_{(x,y) \to (2,3)} [3x + i(2x - 4y)]$

$= 6 + i(4 - 12)]$

$= 6 - 8i$

==

Ex.: Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exists.

Sol. Consider $\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x+iy \to 0} \frac{x-iy}{x+iy}$

Path along x-axis i.e. $y = 0$, we have,

$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = 1 \because x \neq 0$

and path along y-axis i.e. $x = 0$, we have

$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{-iy}{iy} = -1 \quad \because y \neq 0$

For two different paths, we get two different limits

$\therefore \lim_{z \to 0} \frac{\bar{z}}{z}$ does not exists is proved.

==

Ex.: Evaluate $\lim\limits_{z\to i}\frac{z^5-i}{z+1}$

Sol. Consider $\lim\limits_{z\to i}\frac{z^5-i}{z+1} = \frac{i^5-i}{i+1} = \frac{i-i}{1+i} = 0$

==

Ex.: Evaluate $\lim\limits_{z\to i}\frac{z^2+1}{z-i}$ (Oct.2019)

Sol. Consider $\lim\limits_{z\to i}\frac{z^2+1}{z-i} = \lim\limits_{z\to i}\frac{z^2-i^2}{z-i}$

$= \lim\limits_{z\to i}\frac{(z-i)(z+i)}{z-i}$

$= \lim\limits_{z\to i}(z+i) \quad \because z-i\neq 0$

$= (i+i)$

$= 2i$

==

Ex.: Evaluate $\lim\limits_{z\to 1+i}\frac{z^4+4}{z^2-2i}$

Sol. Consider $\lim\limits_{z\to 1+i}\frac{z^4+4}{z^2-2i}$

$= \lim\limits_{z\to 1+i}\frac{(z^2)^2-(2i)^2}{z^2-2i}$

$= \lim\limits_{z\to 1+i}\frac{(z^2-2i)(z^2+2i)}{z^2-2i}$

$= \lim\limits_{z\to 1+i}(z^2+2i) \quad \because z^2-2i\neq 0$

$= (1+i)^2+2i$

$= 1+2i-1+2i$

$= 4i$

==

Ex.: Evaluate $\lim\limits_{z\to 1+i}\frac{z^4+4}{z-1-i}$ (Oct.2019)

Sol. Consider $\lim\limits_{z\to 1+i}\frac{z^4+4}{z-1-i}$

$= \lim\limits_{z\to 1+i}\frac{(z^2)^2-(2i)^2}{z-1-i}$

$= \lim\limits_{z\to 1+i}\frac{(z^2-2i)(z^2+2i)}{z-1-i}$

$= \lim\limits_{z\to 1+i}\frac{[z^2-(1+i)^2](z^2+2i)}{z-1-i}$

$= \lim\limits_{z\to 1+i}\frac{(z-1-i)(z+1+i)(z^2+2i)}{z-1-i}$

$= \lim\limits_{z\to 1+i}(z+1+i)(z^2+2i) \quad \because z-1-i\neq 0$

$= (1+i+1+i)[(1+i)^2+2i)]$

$= 2(1+i)[1+2i-1+2i]$

$= 8i(1+i)$

$= -8+8i$

$= -8(1-i)$

==

Ex.: Evaluate $\lim\limits_{z\to 1+i}\frac{(z^4+4)\,(1+i-z)}{z^2-2iz+2i-2z}$

Sol. Consider $\lim\limits_{z\to 1+i}\frac{(z^4+4)\,(1+i-z)}{z^2-2iz+2i-2z}$

$$= \lim_{z\to 1+i}\frac{-[(z^2)^2-(2i)^2](z-1-i)}{(z-1-i)^2}$$

$$= \lim_{z\to 1+i}\frac{-(z^2-2i)(z^2+2i)}{z-1-i} \qquad \because z-1-i\neq 0$$

$$= \lim_{z\to 1+i}\frac{-[z^2-(1+i)^2](z^2+2i)}{z-1-i}$$

$$= \lim_{z\to 1+i}\frac{-(z-1-i)(z+1+i)(z^2+2i)}{z-1-i}$$

$$= \lim_{z\to 1+i} -(z+1+i)(z^2+2i) \quad \because z-1-i\neq 0$$

$$= -(1+i+1+i)[(1+i)^2+2i)]$$

$$= -2(1+i)[1+2i-1+2i]$$

$$= -8i(1+i)$$

$$= 8-8i$$

$$= 8(1-i)$$

==

Ex.: Evaluate $\lim\limits_{z\to e^{i\pi/3}}\frac{(z-e^{\frac{i\pi}{3}})z}{z^3+1}$

Sol. Consider $\lim\limits_{z\to e^{i\pi/3}}\frac{(z-e^{\frac{i\pi}{3}})z}{z^3+1}$

$$= \lim_{z\to e^{i\pi/3}}\frac{(z-e^{\frac{i\pi}{3}})z}{z^3-(e^{i\pi/3})^3}$$

$$= \lim_{z\to e^{i\pi/3}}\frac{(z-e^{\frac{i\pi}{3}})z}{(z-e^{i\pi/3})\,[z^2+ze^{i\pi/3}+(e^{i\pi/3})^2]}$$

$$= \lim_{z\to e^{i\pi/3}}\frac{z}{z^2+ze^{i\pi/3}+(e^{i\pi/3})^2} \qquad \because z-e^{i\pi/3}\neq 0$$

$$= \frac{e^{i\pi/3}}{e^{i2\pi/3}+e^{i2\pi/3}+e^{i2\pi/3}}$$

$$= \frac{e^{i\pi/3}}{3e^{i2\pi/3}}$$

$$= \frac{1}{3}\,e^{-i\pi/3}$$

$$= \frac{1}{3}\,(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})$$

$$= \frac{1}{3}\,(\frac{1}{2}-i\frac{\sqrt{3}}{2})$$

$$= \frac{1}{6}\,(1-i\sqrt{3})$$

==

Continuity of a function at a point: A complex function f (z) is said to be continuous at a point z = z_0 if f (z_0) is defined, $\lim_{z \to z_0} f(z)$ is exist and $\lim_{z \to z_0} f(z) = f(z_0)$.

Continuity of a function at a point: A complex function f (z) is said to be continuous at a point z = z_0 if for small ε > 0, there exist δ > 0 depends on ε such that $|f(z) - f(z_0)| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Removable discontinuity:

A complex function f (z) is said to have removable discontinuity at z = z_0 if $\lim_{z \to z_0} f(z) \neq f(z_0)$ and the discontinuity can be removed by giving the value to f (z_0) as $\lim_{z \to z_0} f(z)$.

Continuity of a function on a set: A complex function f (z) is said to be continuous on a set S if it is continuous at each point of S.

==

Ex. Discuss the continuity of the function $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$

$= 3+4i$ if z = 2i at z = 2i

Sol. Let $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$

$= 3+4i$ if z = 2i

Here f (2i) = 3+4i …… (1)

Now $\lim_{z \to 2i} f(z)$

$$= \lim_{z \to 2i} \frac{z^2+4}{z-2i}$$

$$= \lim_{z \to 2i} \frac{z^2-(2i)^2}{z-2i}$$

$$= \lim_{z \to 2i} \frac{(z-2i)(z+2i)}{z-2i}$$

$$= \lim_{z \to 2i} (z+2i) \quad \because z - 2i \neq 0$$

$$= 2i + 2i$$

$$= 4i$$

∴ $\lim_{z \to 2i} f(z)$ is exist and $\lim_{z \to 2i} f(z) \neq f(2i)$ by (1)

∴ f (z) is not continuous at z = 2i.

==

Ex. If $(z) = \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$, $z \neq i$ is continuous at $z = i$, then find the value of $f(i)$.

Sol. Let $f(z) = \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$, $z \neq i$ is continuous at $z = i$

∴ $\lim_{z \to i} f(z) = f(i)$

∴ $f(i) = \lim_{z \to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$

$$= \lim_{z \to i} \frac{3z^4+3z^2-2z^3-2z+5z^2+5}{z-i}$$

$$= \lim_{z \to i} \frac{3z^2(z^2+1)-2z(z^2+1)+5(z^2+1)}{z-i}$$

$$= \lim_{z \to i} \frac{(z^2+1)(3z^2-2z+5)}{z-i}$$

$$= \lim_{z \to i} \frac{(z-i)(z+i)(3z^2-2z+5)}{z-i}$$

$$= \lim_{z \to i} (z+i)(3z^2-2z+5) \qquad \because z-i \neq 0$$

$$= 2i(-3-2i+5)$$

$$= 2i(-2i+2)$$

$$= 4i(-i+1)$$

$\therefore$ f (i) = $4(1+i)$

==

Ex. If $(z) = \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$, if $z \neq i$ and f(i) = 2 + 3i. Examine f(z) for continuity at $z = i$,

Sol. Let f (z) = $\frac{3z^4-2z^3+8z^2-2z+5}{z-i}$, if $z \neq i$ and f(i) = 2 + 3i …… (1)

Consider $\lim_{z \to i} f(z)$

$$= \lim_{z \to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$$

$$= \lim_{z \to i} \frac{3z^4+3z^2-2z^3-2z+5z^2+5}{z-i}$$

$$= \lim_{z \to i} \frac{3z^2(z^2+1)-2z(z^2+1)+5(z^2+1)}{z-i}$$

$$= \lim_{z \to i} \frac{(z^2+1)((3z^2-2z+5)}{z-i}$$

$$= \lim_{z \to i} \frac{(z-i)(z+i)(3z^2-2z+5)}{z-i}$$

$$= \lim_{z \to i} (z+i)(3z^2-2z+5) \qquad \because z-i \neq 0$$

$$= 2i(-3-2i+5)$$

$$= 2i(-2i+2)$$

$$= 4i(-i+1)$$

$\therefore \lim_{z \to i} f(z) = 4(1+i) \neq f(i)$ by (1).

$\therefore$ f(z) is not continuous at z = i.

==

Derivative at a point:

A complex function f (z) is said to be derivable at point z = z_0 if

$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ or $\lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$ exists and is denoted by $f'(z_0)$.

Algebra of Derivatives: If f (z) and g(z) are differentiable at z, then

i) $\frac{d}{dz}[f(z)\pm g(z)] = \frac{d}{dz}f(z)\pm \frac{d}{dz}g(z)$

ii) $\frac{d}{dz}[f(z)g(z)] = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$

iii) $\frac{d}{dz}[kf(z)] = k\frac{d}{dz}f(z)$ where k is any constant.

iv) $\frac{d}{dz}[\frac{f(z)}{g(z)}] = \frac{g(z)\frac{d}{dz}f(z)-f(z)\frac{d}{dz}g(z)}{[g(z)]^2}$ if $g(z) \neq 0$

v) Chain Rule: If t = f(w) and w = g(z) then $\frac{dt}{dz} = \frac{dt}{dw}\frac{dw}{dz}$

==

Theorem: Every differentiable complex function is continuous.

Proof. Let f (z) is any complex function differentiable at point z = z_0.

$\therefore\ f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ ………(1) is exists.

Consider $\lim_{z\to z_0}[f(z) - f(z_0)] = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\times(z-z_0)$

$= \lim_{z\to z_0}\frac{f(x)-f(z_0)}{z-a}\times\lim_{z\to z_0}(z-z_0)$

$= f'(z_0)\times 0$

$\therefore\ \lim_{z\to z_0} f(z) - f(z_0) = 0$

$\therefore\ \lim_{z\to z_0} f(z) = f(z_0)$

i.e. f (z) is continuous at point z = z_0.

Hence every differentiable function is continuous is proved.

==

Remark: Every continuous function may not be differentiable.

==

Ex.: Show that the function f (z) = $\bar{z}$ is continuous at every point in the z-plane, but not differentiable.

Proof. Let f (z) = $\bar{z}$

$\therefore f(z_0) = \overline{z_0}$

and $\lim_{z\to z_0} f(z) = \lim_{z\to z_0}\bar{z} = \overline{z_0} = f(z_0)$

$\therefore$ f (z) is continuous at every point in z-plane

Now consider

$$f'(z_0) = \lim_{\delta z\to 0}\frac{f(z_0+\delta z)-f(z_0)}{\delta z}$$

$$= \lim_{\delta z\to 0}\frac{\overline{z_0+\delta z}-\overline{z_0}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\overline{z_0}+\overline{\delta z}-\overline{z_0}}{\delta z}$$

$$\therefore f'(z_0) = \lim_{\delta z \to 0} \frac{\overline{\delta z}}{\delta z}$$

Let $\delta z \to 0$ along x-axis, then $\overline{\delta z} = \delta z$

$$\therefore f'(z_0) = \lim_{\delta z \to 0} \frac{\overline{\delta z}}{\delta z} = \lim_{\delta z \to 0} \frac{\delta z}{\delta z} = 1$$

Let $\delta z \to 0$ along y-axis, then $\overline{\delta z} = -\delta z$

$$\therefore f'(z_0) = \lim_{\delta z \to 0} \frac{\overline{\delta z}}{\delta z} = \lim_{\delta z \to 0} \frac{-\delta z}{\delta z} = -1$$

Along two different paths, we get two different limits.

$\therefore f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$ does not exists.

Hence f (z) is continuous at every point in z-plane but is not differentiable is proved.

==

Ex.: Let f (z) = $z^2 + 5z + c$, where c is any arbitrary constant (real or complex). Find $f'(z_0)$ by the definition of the derivative.

Solution. Let f (z) = $z^2 + 5z + c$

$$\therefore f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z_0+\delta z)^2 + 5(z_0+\delta z) + c - {z_0}^2 - 5z_0 - c}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{2z_0\delta z+\delta z^2+5\delta z}{\delta z}$$

$$= \lim_{\delta z \to 0} (2z_0 + \delta z + 5) \quad \because \delta z \neq 0$$

$$= 2z_0 + 0 + 5$$

$$\therefore f'(z_0) = 2z_0 + 5$$

==

Analytic function: A function f(z) of complex variable z is said to be analytic function at a point z_0 if f '(z) exists at each point z in some neighbourhood of z_0.

Remark: 1) A function f(z) is said to be analytic in a domain D if it is analytic at each point of D.

2) An **analytic function** is also called **regular** or **holomorphic** function.

3) If f(z) = u + iv analytic function, then u and v are harmonic conjugates of each other.

4) If f(z) = u + iv analytic function, then $u_x = v_y$ and $u_y = -v_x$ are called Cauhy Riemann Equations or C. R. Equations

5) If f(z) is an analytic function, then $\overline{f(z)}$ is independent of z.

==

Ex.: Show that $f(z) = |z|^2$ is not analytic at any point $z \neq 0$

Proof. Let $f(z) = |z|^2$

$$\therefore f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{|z+\delta z|^2 - |z|^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z+\delta z)(\overline{z+\delta z}) - z\bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z+\delta z)(\bar{z} + \overline{\delta z}) - z\bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{z\bar{z} + z\overline{\delta z} + \delta z\bar{z} + \delta z\overline{\delta z} - z\bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{z\overline{\delta z} + \delta z\bar{z} + \delta z\overline{\delta z}}{\delta z}$$

$$= \lim_{\delta z \to 0} [z\frac{\overline{\delta z}}{\delta z} + \bar{z} + \overline{\delta z}]$$

$$= \lim_{\delta z \to 0} [z\frac{\overline{\delta z}}{\delta z} + \bar{z}] + \bar{0}$$

$$= \begin{cases} z + \bar{z} & \text{along real axis } \overline{\delta z} = \delta z. \\ -z + \bar{z} & \text{along imaginary axis } \overline{\delta z} = -\delta z. \end{cases}$$

We observe that $f'(z)$ exist at $z = 0$ only.

Hence f(z) is not analytic anywhere in the complex plane.

==

Cauchy-Riemann Equations: For $f(z) = u + iv$, $u_x = v_y$ and $u_y = -v_x$

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are called C. R. equations.

==

Necessary condition for analytic function: A complex function $f(z) = u+iv$ is analytic at a point $z = x+iy$ of its domain D is that at (x, y) the first order partial derivatives of u and v w.r.t. x and y exists and satisfies the C. R. equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Proof: Let a complex function $f(z) = u(x, y)+iv(x, y)$ is analytic at a point $z = x+iy$ of its domain D. Then $f'(z)$ is exists and $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ …… (1)

i.e. limit is same along any path as $\delta z \to 0$.

i) Let $\delta z \to 0$ along real axis i.e. along $\delta y = 0$, we have

$$f'(z) = \lim_{\delta x \to 0} \frac{f(x+iy+\delta x) - f(x+iy)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - u(x,y) - iv(x,y)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x,y)-u(x,y) + iv(x+\delta x,y)-iv(x,y)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x,y)-u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x,y)-v(x,y)}{\delta x}$$

$\therefore f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ ....(2) by definition of partial derivatives.

ii) Let $\delta z \to 0$ along imaginary axis i.e. along $\delta x = 0$, we have

$$f'(z) = \lim_{\delta y \to 0} \frac{f(x+iy+i\delta y)-f(x+iy)}{i\delta y}$$

$$= \lim_{\delta y \to 0} \frac{u(x,y+\delta y)+iv(x,y+\delta y)-u(x,y) - iv(x,y)}{i\delta y}$$

$$= \lim_{\delta y \to 0} \frac{u(x,y+\delta y)-u(x,y) + iv(x,y+\delta y)-iv(x,y)}{i\delta y}$$

$$= \frac{1}{i} \lim_{\delta y \to 0} \frac{u(x+\delta x,y)-u(x,y)}{\delta y} + \lim_{\delta x \to 0} \frac{v(x+\delta x,y)-v(x,y)}{\delta y}$$

$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ by definition of partial derivatives.

$\therefore f'(z)= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ ...... (3)

From (2) and (3), we get,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary parts, we get,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Hence proved.

===

Sufficient condition for analytic function: Let f(z) = u+iv = u(x, y)+iv(x, y). If the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists and are continuous at a point (x, y) in the domain D and they satisfy the C. R. equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x, y) then f(z) is analytic at a point z = x+iy

Proof: Let f(z) = u+iv = u(x, y)+iv(x, y).

As the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists and are continuous at a point (x, y) in the domain D.

$\therefore$ u(x,y) and v(x, y) are diffrentiable at point (x, y).

$\therefore \delta u = u(x+\delta x,y+\delta y)-u(x,y) = \frac{\partial u}{\partial x}\delta x+ \frac{\partial u}{\partial y}\delta y + \alpha_1\delta x+ \beta_1\delta y$

$\therefore \delta v = v(x+\delta x,y+\delta y)-v(x,y) = \frac{\partial v}{\partial x}\delta x+ \frac{\partial v}{\partial y}\delta y+ \alpha_2\delta x+ \beta_2\delta y$

Where $\alpha_1, \beta_1, \alpha_2, \beta_2 \to 0$ as $\delta x, \delta y \to 0$

Now $\delta f(z) = f(z+\delta z)-f(z) = \delta u + i\delta v$

$= \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \alpha_1\delta x + \beta_1\delta y\right) + i\left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \alpha_2\delta x + \beta_2\delta y\right)$

$= \left(\frac{\partial u}{\partial x}\delta x - \frac{\partial v}{\partial x}\delta y + \alpha_1\delta x + \beta_1\delta y\right) + i\left(\frac{\partial v}{\partial x}\delta x + \frac{\partial u}{\partial x}\delta y + \alpha_2\delta x + \beta_2\delta y\right)$

by C-R equations.

$= \frac{\partial u}{\partial x}(\delta x + i\delta y) + i\frac{\partial v}{\partial x}(\delta x + i\delta y) + (\alpha_1 + i\alpha_2)\delta x + (\beta_1 + i\beta_2)\delta y$

$= \frac{\partial u}{\partial x}\delta z + i\frac{\partial v}{\partial x}\delta z + \alpha\delta x + \beta\delta y$ where $\delta z = \delta x + i\delta y$, $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$

$\frac{\delta f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \alpha\frac{\delta x}{\delta z} + \beta\frac{\delta y}{\delta z}$

As $|\delta x| \le |\delta z|$ and $|\delta y| \le |\delta z| \Rightarrow \left|\frac{\delta x}{\delta z}\right| \le 1$ and $\left|\frac{\delta y}{\delta z}\right| \le 1$

Now $\delta x, \delta y \to 0$ i.e. $\delta z \to 0 \Rightarrow \alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2 \to 0$

$\Rightarrow \lim_{\delta z \to 0} \alpha\frac{\delta x}{\delta z} = 0$ and $\lim_{\delta z \to 0} \beta\frac{\delta y}{\delta z} = 0$

$\therefore \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\therefore f'(z) = u_x + iv_x$

$\therefore f(z)$ is differentiable at $z = x + iy$.

Hence proved.

==

Ex.: Show that the function defined by $f(z) = \sqrt{|xy|}$ where $z \ne 0$ and $f(0) = 0$, is not analytic at $z = 0$ even though the C-R equations are satisfied at $z = 0$ i.e. at origin.

Proof. Let $f(z) = \sqrt{|xy|} = u + iv$ where $z \ne 0$ and $f(0) = 0$

$\therefore u(x, y) = \sqrt{|xy|}$, $v(x, y) = o$ and $u(0, 0) = 0$, $v(0, 0) = 0$

$\therefore u_x(0, 0) = \lim_{h\to 0}\frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$

$u_y(0, 0) = \lim_{k\to 0}\frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$

As $v(x, y) = o$ for all x, y

$\therefore v_x(0, 0) = 0, v_y(0, 0) = 0$

$\therefore u_x(0, 0) = 0 = v_y(0, 0)$ and $u_y(0, 0) = 0 = -v_x(0, 0)$

Thus four partial derivatives u_x, u_y, v_x, v_y exists and satisfies C-R equations.

Now consider

$f'(0) = \lim_{z \to 0}\frac{f(z) - f(0)}{z - 0}$

$= \lim_{(x,y) \to (0,0)} \frac{\sqrt{|xy|} - 0}{x + iy}$

Along real axis i.e. $y = 0$, we have,

$f'(0) = \lim_{x \to 0}\frac{\sqrt{|0|} - 0}{x + i0} = 0$

Along the st. line $y = x$, we have,

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x^2|}-0}{x+ix}$$
$$= \lim_{x\to 0} \frac{x}{x(1+i)}$$
$$= \lim_{x\to 0} \frac{1}{(1+i)} \qquad \because x \neq 0$$
$$= \frac{1}{(1+i)}$$

For two different paths, we get two different limits.

$\therefore f'(0)= \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exists i.e. f(z) not differentiable at z = 0.

Hence f(z) is not analytic at z = 0 even though the C-R equations are satisfied at z = 0 is proved.

===

Ex.: If f(z) is analytic function with real part u is constant, then show that f(z) is a constant function.

Proof. Let f (z) = u + iv is analytic function with real part u is constant.

$\therefore$ u and v are satisfies C – R equations

$u_x = v_y$ and $u_y = -v_x$ …… (1)

and u = c, where c is constant.

$\therefore u_x = 0$ and $u_y = 0$

$\therefore f'(z) = u_x + iv_x = u_x - iu_y = 0 - i0 = 0.$ by (1) $v_x = -u_y$

$\therefore$ f(z) is a constant function is proved.

===

Ex.: If f(z) is analytic function with imaginary part v is constant, then show that f(z) is a constant function.

Proof. Let f (z) = u + iv is analytic function with imaginary part v is constant.

$\therefore$ u and v are satisfies C-R equations

$u_x = v_y$ and $u_y = -v_x$ …… (1)

and v = c, where c is constant.

$\therefore v_x = 0$ and $v_y = 0$

$\therefore f'(z) = u_x + iv_x = v_y + iv_x = 0 + i0 = 0.$ by (1) $u_x = v_y$

$\therefore$ f(z) is a constant function is proved.

===

Ex.: If f(z) and $\overline{f(z)}$ are analytic functions of z, then show that f(z) is a constant function.

Proof. Let f (z) = u + iv and $\overline{f(z)}$ = u - iv are analytic functions of z.

$\therefore u_x = v_y$ and $u_y = -v_x$ …… (1)

Also $u_x = -v_y$ and $u_y = -(-v)_x = v_x$…… (2) $\quad \because \overline{f(z)}$ = u - iv is analytic

Adding the corresponding equations (1) and (2), we get,

$2u_x = 0$ and $2u_y = 0$

$\therefore u_x = 0$ and $u_y = 0$

$\therefore f'(z) = u_x + iv_x = u_x - iu_y = 0 - i0 = 0.$

$\therefore$ f(z) is a constant function is proved.

==

Ex.: If f(z) is analytic function with constant modulus, then show that f(z) is a constant function.

Proof. Let f (z) = u + iv is analytic function with constant modulus.

$\therefore$ u and v are satisfies C-R equations

$u_x = v_y$ and $u_y = -v_x$ …… (1)

and $|f(z)| = \sqrt{u^2+v^2}$ is constant say k.

i.e. $\sqrt{u^2+v^2} = k$

$\therefore u^2 + v^2 = k^2$ …… (2)

Differentiating equation (2) partially w.r.t. x and y, we get,

$2uu_x + 2vv_x = 0$ i.e. $uu_x - vu_y = 0$ …… (3) by (1) $v_x = -u_y$

and $2uu_y + 2vv_y = 0$ i.e. $uu_y + vu_x = 0$ …… (4) by (1) $v_y = u_x$

Consider u(3) + v(4), we get,

$u^2u_x - uvu_y + vuu_y + v^2u_x = 0$

i.e. $(u^2+ v^2)u_x = 0$

Similarly u(4)-v(3) gives $(u^2+ v^2)u_y = 0$.

If $u^2+ v^2 = 0$, then u = v = 0 and hence f(z) = 0 is constant function.

But if $u^2+ v^2 \neq 0$, then $u_x = 0$ and $u_y = 0$

$\therefore f'(z) = u_x + iv_x = u_x - iu_y = 0 - i0 = 0$.

$\therefore$ f(z) is a constant function is proved.

==

Ex.: Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$.

Proof. Let z = x + iy, then $\bar{z}$ = x - iy .

$\therefore z + \bar{z} = 2x$ and $z - \bar{z} = 2iy$

$\therefore x = \frac{1}{2}(z + \bar{z}) = 2x - iy$ and $y = \frac{1}{2i}(z - \bar{z})$

$\therefore \frac{\partial x}{\partial z} = \frac{1}{2}, \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$ and $\frac{\partial y}{\partial z} = \frac{1}{2i}, \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}$

Now by chain rule $\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$ and $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$

$\therefore \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$

$\therefore 2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ and $2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$

Taking product, we get,

$4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$\therefore 4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Hence proved.

==

Ex.: If f(z) is an analytic function of z, then show that

i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$ ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\{R[f(z)]\}^2 = 2|f'(z)|^2$

Proof. Let f(z) is an analytic function of z, then $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$ …… (1)

i) Consider

$$\begin{aligned}
\text{LHS.} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 \\
&= \left(4\frac{\partial^2}{\partial z\,\partial \bar{z}}\right)[f(z)f(\bar{z})] \qquad \text{by (1) and } |f(z)|^2 = f(z)\overline{f(z)} = f(z)f(\bar{z}) \\
&= 4\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial \bar{z}}[f(z)f(\bar{z})]\right\} \\
&= 4\frac{\partial}{\partial z}\{f(z)f'(\bar{z})]\} \\
&= 4\,\{f'(z)\overline{f'(z)}\,\} \\
&= 4|f'(z)|^2 \\
&= \text{RHS.}
\end{aligned}$$

Hence proved.

i) Consider

$$\begin{aligned}
\text{LHS.} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\{R[f(z)]\}^2 \\
&= \left(4\frac{\partial^2}{\partial z\,\partial \bar{z}}\right)\left\{\frac{1}{2}[f(z) + \overline{f(z)}]\right\}^2 \qquad \text{by (1) and } R[f(z)] = \frac{1}{2}[f(z) + \overline{f(z)}] \\
&= \frac{\partial}{\partial z}\left\{\frac{\partial}{\partial \bar{z}}[f(z) + f(\bar{z})]^2\right\} \\
&= 2\frac{\partial}{\partial z}\{[f(z) + f(\bar{z})]f'(\bar{z})\} \\
&= 2\{f'(z)\overline{f'(z)}\} \\
&= 2|f'(z)|^2 \\
&= \text{RHS.}
\end{aligned}$$

Hence proved.

==

Laplace Differential Equation: Let Φ(x, y) be a real valued function of real variables x and y, then the differential equation $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ i.e. $\nabla^2\Phi = 0$ is called Laplace differential equation.

Harmonic function: A real valued function Φ(x, y) of real variables x and y is called a harmonic function if it satisfies Laplace differential equation $\nabla^2\Phi = 0$.

Laplace Operator: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called Laplace del operator.

==

Theorem: The real and imaginary parts of an analytic function satisfy Laplace differential equations.

or

Show that the real and imaginary part of an analytic function are harmonic.

Proof. Let f (z) = u + iv is an analytic function.

$\therefore$ u and v satifies C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y}$ and $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}$

Adding we get,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0 \qquad \because \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$

i.e. $\nabla^2 u = 0$. Thus u satisfies Laplace differential equation.

Again from (1), we get

$\therefore \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x\,\partial y}$ and $\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y\,\partial x}$

Adding we get,

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y\,\partial x} = 0 \qquad \because \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

i.e. $\nabla^2 v = 0$. Thus v satisfies Laplace differential equation.

$\therefore$ The real and imaginary part of an analytic function are harmonic.

Hence proved.

==

Ex.: Show that the real and imaginary part of the function e^z satisfy C-R equations and they are harmonic.

Proof. Let f (z) = $e^z = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y = u + iv$

be a given function with real and imaginary parts are

$u = e^x\cos y$ and $v = e^x\sin y$

Differentiating partially w.r.t. x and y, we get

$\therefore u_x = e^x\cos y$, $u_y = -e^x\sin y$, $v_x = e^x\sin y$ and $v_y = e^x\cos y$

We observe that $u_x = v_y$ and $u_y = -v_x$

Thus, u and v satisfies C-R equations.

Now $u_{xx} = e^x\cos y$, $u_{yy} = -e^x\cos y$, $v_{xx} = e^x\sin y$ and $v_{yy} = -e^x\sin y$

$\therefore u_{xx} + u_{yy} = e^x\cos y - e^x\cos y = 0$ and $v_{xx} + v_{yy} = e^x\sin y - e^x\sin y = 0$

i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$

i.e. u and v satisfies Laplace differential equation

$\therefore$ u and v are satisfies C-R equations and they are harmonic.

Hence proved.

==

Construction of Analytic function:

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Method-I: Case-i) Suppose u i.e. real part of analytic function is given:

We have to find v such that f(z) = u + iv is analytic function.

As f(z) = u + iv is analytic function

∴ u and v are satisfies C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

By total differentiation

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

∴ $dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ by (1) ∵ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are obtained from given u.

∴ dv = Mdx + Ndy …… (2) where $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

Now $\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

∴ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = -(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}) = -\nabla^2 u = 0$ ∵ u is harmonic.

∴ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation (2) is exact and it's G. S. is given by

$v = \int_{y-const.} Mdx + \int(\text{terms of N not containing x})dy + c$,

where c is constant of integration.

Using this v and given u, we get an analytic function f(z) = u + iv.

Case-ii) Suppose v i.e. imaginary part of analytic function is given:

We have to find u such that f(z) = u + iv is analytic function.

As f(z) = u + iv is analytic function

∴ u and v are satisfies C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

By total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

∴ $du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$ by (1) ∵ $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are obtained from given v.

∴ du = Mdx + Ndy …… (2) where $M = \frac{\partial v}{\partial y}$ and $N = -\frac{\partial v}{\partial x}$

Now $\frac{\partial M}{\partial y} = \frac{\partial^2 v}{\partial y^2}$ and $\frac{\partial N}{\partial x} = -\frac{\partial^2 v}{\partial x^2}$

∴ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = \nabla^2 v = 0$ ∵ v is harmonic.

∴ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence equation (2) is exact and it's G. S. is given by

$u = \int_{y-const.} Mdx + \int(\text{terms of N not containing x})dy + c$,

where c is constant of integration.

Using this u and given v, we get an analytic function f(z) = u + iv.

==

Ex.: Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.

Proof. Let $u = x^3 - 3xy^2$ be a given function.

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2$ and $\frac{\partial u}{\partial y} = -6xy$ …… (1)

$\therefore \frac{\partial^2 u}{\partial x^2} = 6x$ and $\frac{\partial^2 u}{\partial y^2} = -6x$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$ i.e. $\nabla^2 u = 0$

Hence u is harmonic function is proved.

Now to find an analytic function f(z) = u + iv, we to find v,

As f(z) = u + iv is an analytic function

$\therefore$ u and v are satisfies C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

To find v consider

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$\therefore dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ by (1)

$\therefore dv = 6xy\, dx + (3x^2 - 3y^2)\, dy$ which is an exact equation.

$\therefore$ It's G. S. is

$v = \int_{y-\text{const.}} (6xy)dx + \int (-3y^2)\, dy + c'$

i.e. $v = 3x^2y - y^3 + c'$.

$\therefore$ By using this v and given u, an analytic function is

$f(z) = u + iv = (x^3 - 3xy^2) + i(2x^2y - y^3 + c')$

$\therefore f(z) = z^3 + ic'$ obtained by putting x = z and y = 0.

$\therefore f(z) = z^3 + c$ where c = ic'

Which is the required analytic function in z.

==

Ex.: Show that $\frac{1}{2}\log(x^2 + y^2)$ satisfies Laplace equation. Finds its harmonic conjugates.

Proof. Let $u = \frac{1}{2}\log(x^2 + y^2)$ is an analytic function of z, then

$\therefore \frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{2x}{x^2+y^2}\right) = \frac{x}{x^2+y^2}$ and $\frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{2y}{x^2+y^2}\right) = \frac{y}{x^2+y^2}$ …… (1)

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ and $\frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$ i.e. $\nabla^2 u = 0$

Hence u satisfies Laplace equation is proved.

Now to find harmonic conjugate of u,

Consider

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$\therefore dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ by using C-R equations $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}$

$\therefore dv = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ which is an exact equation.

$\therefore$ It's G. S. is

$$v = \int_{y-\text{const.}} \left(-\frac{y}{x^2+y^2}\right)dx + \int 0dy + c$$

i.e. $v = -\tan^{-1}\left(\frac{x}{y}\right) + c$ is the harmonic conjugate of u.

==

Ex.: Determine the analytic function $f(z) = u + iv$ if $u = x^2 - y^2$ and $f(0) = 1$

Solution. Let $f(z) = u + iv$ is an analytic function.

$\therefore$ u and v are satisfies C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots\ldots (1)$$

As $u = x^2 - y^2$ is given

$\therefore \frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = -2y$...... (2)

Now to find an analytic function $f(z) = u + iv$, we have to find v.

Consider

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$\therefore dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ by using C-R equations $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & $\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}$

$\therefore dv = (2y)\, dx + (2x)dy$ which is an exact equation.

$\therefore$ It's G. S. is

$$v = \int_{y-\text{const.}} (2y)dx + \int (0)dy + c$$

i.e. $v = 2xy + c$.

$\therefore$ By using this v and given u, an analytic function is

$f(z) = u + iv = (x^2 - y^2) + i(2xy + c)$

$\therefore f(z) = z^2 + ic$ obtained by putting $x = z$ and $y = 0$.

Now $f(0) = 1$ gives $0 + ic = 1$ i.e. $ic = 1$

$\therefore f(z) = z^2 + 1$

Which is the required analytic function in z.

==

Ex.: Find an analytic function $f(z) = u + iv$ and express it in terms of z, if $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Solution. Let $f(z) = u + iv$ is an analytic function.

$\therefore$ u and v are satisfies C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

As $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is given

$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ and $\frac{\partial u}{\partial y} = -6xy - 6y$…… (2)

Now to find an analytic function f(z) = u + iv, we have to find v.

Consider

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$\therefore dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ by (1)

$\therefore dv = (6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)dy$ by (2)

which is an exact equation.

$\therefore$ It's G. S. is

$v = \int_{y-const.}(6xy + 6y)dx + \int(-3y^2)dy + c'$

i.e. $v = 3x^2y + 6xy - y^3 + c'$.

$\therefore$ By using this v and given u, an analytic function is

$f(z) = u + iv = (x^3 - 3xy^2 + 3x^2 - 3y^2 + 1) + i(3x^2y + 6xy - y^3 + c')$

$\therefore f(z) = z^3 + 3z^2 + c$ obtained by putting x = z and y = 0 and taking 1+ic' = c

Which is the required analytic function in z.

==

Ex.: Find an analytic function f(z) = u + iv whose real part is given by $u = e^x(x\cos y - y\sin y)$

Solution. Let f(z) = u + iv is an analytic function.

$\therefore$ u and v are satisfies C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

As $u = e^x(x\cos y - y\sin y)$ is given

$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y = e^x(x\cos y - y\sin y + \cos y)$

and $\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = -e^x(x\sin y + \sin y + y\cos y)$…… (2)

Now to find an analytic function f(z) = u + iv, we have to find v.

Consider

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

$\therefore dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$ by (1)

$\therefore dv = e^x(x\sin y + \sin y + y\cos y)\,dx + e^x(x\cos y - y\sin y + \cos y)dy$ by (2)

which is an exact equation.

$\therefore$ It's G. S. is

$v = \int_{y-const.}^{\cdot} e^x(x\sin y + \sin y + y\cos y)dx + \int (0)dy + c'$

i.e. $v = e^x(x\sin y + y\cos y) + c$ using $\int e^x[f(x) + f'(x)]\, dx = e^x f(x) + c'$

∴ By using this v and given u, an analytic function is

$f(z) = u + iv = e^x(x\cos y - y\sin y) + i[e^x(x\sin y + y\cos y) + c']$

∴ $f(z) = e^z\,(z + c)$ obtained by putting x = z and y = 0 and taking ic' = c

Which is the required analytic function in z.

==

Ex.: Find an analytic function f(z) = u + iv, if $v = e^{-y}\sin x$ and f(0) = 1

Solution. Let f(z) = u + iv is an analytic function.

∴ u and v are satisfies C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ …… (1)

As $v = e^{-y}\sin x$ is given

∴ $\frac{\partial v}{\partial x} = e^{-y}\cos x$ and $\frac{\partial u}{\partial y} = -e^{-y}\sin x$ …… (2)

Now to find an analytic function f(z) = u + iv, we have to find u.

Consider

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

∴ $du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$ by (1)

∴ $dv = -e^{-y}\sin x\, dx - e^{-y}\cos x\, dy$ by (2)

which is an exact equation.

∴ It's G. S. is

$u = \int_{y-const.}^{\cdot}(-e^{-y}\sin x)dx + \int (0)dy + c$

i.e. $u = e^{-y}\cos x + c$.

∴ By using this v and given u, an analytic function is

$f(z) = u + iv = (e^{-y}\cos x + c) + i(e^{-y}\sin x)$

∴ $f(z) = \cos z + c + i\sin z = e^{iz} + c$ obtained by putting x = z and y = 0

Now f(0) = 1 gives 1 = 1 + c i.e. c = 0

∴ $f(z) = e^{iz}$

Which is the required analytic function in z.

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Milne Thomson Method:

Case-i) Suppose u i.e. real part of analytic function f(z) = u + iv is given:

As f(z) = u + iv is analytic function

∴ u and v are satisfies C-R equations $u_x = v_y$ and $u_y = -v_x$ …… (1)

Since u is given, u_x and u_y are calculated.

Now $f'(z) = u_x + iv_x = u_x(x, y) - iu_y(x, y)$ by (1) $v_x = -u_y$

Say $f'(z) = u_1(x, y) - iu_2(x, y)$ where $u_1(x, y) = u_x(x, y)$ and $u_2(x, y) = u_y(x, y)$

$\therefore f'(z) = u_1(z, 0) - iu_2(z, 0)$ by putting $x = z$ and $y = 0$.

Integrating both sides w. r. t. z, we get

$f(z) = \int [u_1(z,0) - iu_2(z,0)]dz + c$

be the required an analytic function.

Case-ii) Suppose v i.e. real part of an analytic function $f(z) = u + iv$ is given:

As $f(z) = u + iv$ is analytic function

$\therefore$ u and v are satisfies C-R equations $u_x = v_y$ and $u_y = -v_x$ …… (1)

Since v is given, v_x and v_y are calculated.

Now $f'(z) = u_x + iv_x = v_y(x, y) + iv_x(x, y)$ by (1) $u_x = v_y$

Say $f'(z) = v_2(x, y) + iv_1(x, y)$ where $v_1(x, y) = v_x(x, y)$ and $v_2(x, y) = v_y(x, y)$

$\therefore f'(z) = v_2(z, 0) + iv_1(z, 0)$ by putting $x = z$ and $y = 0$.

Integrating both sides w. r. t. z, we get

$f(z) = \int [v_2(z,0) + iv_1(z,0)]dz + c$

be the required an analytic function.

==

Ex.: Find an analytic function $f(z) = u + iv$ whose real part is $u = e^{-2xy}\sin(x^2 - y^2)$

Solution. Let $u = e^{-2xy}\sin(x^2 - y^2)$

$\therefore u_x = -2ye^{-2xy}\sin(x^2 - y^2) + 2xe^{-2xy}\cos(x^2 - y^2)$

and $u_y = -2xe^{-2xy}\sin(x^2 - y^2) - 2ye^{-2xy}\cos(x^2 - y^2)$

$\therefore u_1(z, 0) = u_x(z, 0) = 0 + 2z\cos(z^2 - 0) = 2z\cos z^2$

and $u_2(z, 0) = u_y(z, 0) = -2z\sin(z^2 - 0) - 0 = -2z\sin z^2$

By Milne Thomson Method, we get,

$f(z) = \int [u_1(z,0) - iu_2(z,0)]dz + c$

$= \int [2z\cos z^2 + i2z\sin z^2]dz + c$

$= \int [\cos z^2 + i\sin z^2](2zdz) + c$

$= \frac{1}{i}\int e^{iz^2}(2izdz) + c$

$= -ie^{iz^2} + c$

Which is the required analytic function.

==

Ex.: Find an analytic function $f(z) = u + iv$ whose imaginary part is $v = e^x(x\sin y + y\cos y)$ using Milne Thomson Method.

Solution. Let $v = e^x(x\sin y + y\cos y)$

$\therefore v_x = e^x(x\sin y + y\cos y) + e^x\sin y = e^x(x\sin y + y\cos y + \sin y)$

and $v_y = e^x(x\cos y + \cos y - y\sin y)$

$\therefore v_1(z, 0) = v_x(z, 0) = 0$

and $v_2(z, 0) = v_y(z, 0) = e^z(z + 1)$

By Milne Thomson Method, we get,

$f(z) = \int [v_2(z, 0) + iv_1(z, 0)]dz + c$

$= \int [e^z (z + 1) + 0]dz + c$

$= \int e^z (z + 1)dz + c$

$= ze^z + c$

Which is the required analytic function.

===

MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) If $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ be any two complex numbers in the complex plane, then the distance between z_1 and z_2 is ……

a) $|z_1 + z_2|$ b) $|z_1 - z_2|$ c) $|z_1 z_2|$ d) None of these

2) If $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ be any two complex numbers, then $|z_1 - z_2|$ = ……

a) $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ b) $\sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2}$

c) $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$ d) None of these

3) If $\delta > 0$ be any real number, then the equation of a circle with centre at z_0 and radius δ is ……

a) $|z_0| = \delta$ b) $|z + z_0| = \delta$ c) $|z - z_0| = \delta$ d) None of these

4) Let z_0 be a fixed point in the complex plane C and $\delta > 0$ be a real number, then the set …… is called the δ −neighborhood $N_\delta(z_0)$ of z_0.

a) $\{z \,/\, |z + z_0| < \delta\}$ b) $\{z \,/\, |z - z_0| < \delta\}$

c) $\{z \,/\, |z - z_0| > \delta\}$ d) None of these

5) Let z_0 be a fixed point in the complex plane C and $\delta > 0$ be a real number, then the set …… is called a deleted neighborhood $N'_\delta(z_0)$ of z_0.

a) $\{z \,/\, 0 < |z - z_0| < \delta\}$ b) $\{z \,/\, 0 < |z + z_0| < \delta\}$

c) $\{z \,/\, |z - z_0| > \delta\}$ d) None of these

6) Let $S \subset C$, if there exist $\delta > 0$ such that $N_\delta(z_0) \subset S$, then $z_0 \in S$ is called …… point of S.

a) an interior b) an exterior c) a boundary d) None of these

7) A set $S \subset C$ is said to be open set if every point of S is …… point of S.

a) an interior b) an exterior c) a boundary d) None of these

8) If every neighborhood of z_0 contains at least one point belonging to S and one point not belonging to S, then point z_0 is called …… point of the set S.

a) an interior b) an exterior c) a boundary d) None of these

9) If there exist a positive constant M such that $|z| \le M$, for every $z \in S$, then the set S in the z-plane is called a …… set

a) open b) bounded c) closed d) None of these

10) A point $z_0 \in C$ is called a limit point of the set S if every deleted neighbourhood of z_0 contains …… point of S.

a) at least one b) all c) no d) None of these

11) A set S is called a closed set if it contains all its ……. points.

a) interior b) boundary c) limit d) None of these

12) If the limit, $\lim_{z \to z_0} f(z)$ exists, then it is ……

a) 0 b) unique c) ∞ d) None of these

13) For $f(z) = u(x, y) + iv(x, y)$, if $\lim_{z \to z_0} f(z) = a + ib$, then $\lim_{(x,y) \to (x_0, y_0)} u(x, y) =$ ……

a) a b) b c) a + ib d) None of these

14) For $f(z) = u(x, y) + iv(x, y)$, if $\lim_{z \to z_0} f(z) = a + ib$, then $\lim_{(x,y) \to (x_0, y_0)} v(x, y) =$ ……

a) a b) b c) a + ib d) None of these

15) If $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$, then $\lim_{z \to z_0} [f(z) \pm g(z)] =$ ……

a) $l \pm m$ b) l c) m d) $l.m$

16) If $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$, then $\lim_{z \to z_0} [f(z). g(z)] =$ ……

a) $l \pm m$ b) l c) m d) $l.m$

17) If $\lim_{z \to z_0} f(z) = l$ and $\lim_{z \to z_0} g(z) = m$, then $\lim_{z \to z_0} [\frac{f(z)}{g(z)}] = \frac{l}{m}$ if ……

a) $l \neq 0$ b) $m \neq 0$ c) $\frac{l}{m} \neq 0$ d) None of these

18) $\lim_{z \to (1-i)} [x+i(2x+y)] =$……..

a) 1-i b) -1+i c) 1+i d) -1-i

19) $\lim_{z \to (2+3i)} [3x+i(2x-4y)] =$……..

a) 6-8i b) -6-8i c) 6+8i d) -6+8i

20) $\lim_{z \to i} \frac{z^5 - i}{z+i} =$ ……

a) 1 b) 0 c) 5 d) None of these

21) If $\lim_{z \to i} \frac{z^8 - 1}{z+i} = \alpha$, then the value of α is……

a) i b) 0 c) -i d) 1

22) $\lim_{z \to i} z^5 - i =$ ……

a) 1 b) 5 c) 0 d) 4

23) $\lim_{z \to i} \frac{z+i}{z^3}$

a) -2 b) 2 c) -2i d) -4i

24) $\lim_{z \to 1+i} \frac{z^4+4}{z^2-2i} =$ …….

a) 2i b) 4i c) -2i d) -4i

25) If $\lim\limits_{z \to 1+i} \frac{z^4+4}{z^2-2i}$ = A, then the value of A is……

a) 4i b) 0 c) -4i d) 1

26) $\lim\limits_{z \to 1+i} \frac{z^4+4}{z-1-i}$

a) 8-8i b) -8+8i c) -8-8i d) 8+8i

27) $\lim\limits_{z \to 0} \frac{\bar{z}}{z}$ is……

a) 1 b)-1 c) 0 d) does not exist

28) A complex function f(z) is continuous at z = z_0 if ………

a) $\lim\limits_{z \to z_0} f(z) = f(z_0)$ b) $\lim\limits_{z \to z_0} f(z) \neq f(z_0)$

c) only $\lim\limits_{z \to z_0} f(z)$ is exists d) None of these

29) If $\lim\limits_{z \to 4i} \frac{z^2+16}{z-4i}$, z = 4i is continuous at z = 4i, then f(4i) is……

a) 4i b) 0 c) 8i d) 1

30) If f(z) is differential at z_0, then it is continuous at z_0 is ……

a) true statement b) false statement

c) both true and false d) None of these

31) If f(z) is continuous at z_0, then it is differential at z_0 is ……

a) true statement b) false statement

c) both true and false d) None of these

32) If f(z) is continuous at z_0, then it may not differential at z_0 is ……

a) true statement b) false statement

c) both true and false d) None of these

33) The real part of e^z is ………

a) $e^x \cos y$ b) $e^x \sin y$ c) $e^x \cos x$ d) None of these

34) The imaginary part of e^z is ………

a) $e^x \cos y$ b) $e^x \sin y$ c) $e^x \cos x$ d) None of these

35) If f(z) = u + iv then $\overline{f(z)}$ = ………

a) -u + iv b) u – iv c) -u - iv d) None of these

36) If $\lim\limits_{z \to z_0} f(z) = a + ib$ then $\lim\limits_{z \to z_0} \overline{f(z)}$ = ………

a) a - ib b) -a + ib c) -a - ib d) None of these

37) A complex function f (z) is said to be derivable at point z = z_0 if ……. exists and is denoted by $f'(z_0)$.

a) $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z+z_0}$ b) $\lim\limits_{z \to z_0} \frac{f(z)+f(z_0)}{z+z_0}$ c) $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ d) None of these

38) If f (z) and g(z) are differentiable at z, then $\frac{d}{dz}[f(z) \pm g(z)]$ = ……

a) $\frac{d}{dz} f(z) . \frac{d}{dz} g(z)$ b) $\frac{d}{dz} f(z) \pm \frac{d}{dz} g(z)$ c) $\frac{d}{dz} f(z) \mp \frac{d}{dz} g(z)$ d) None of these

39) If f (z) and g(z) are differentiable at z, then $\frac{d}{dz}$ [f(z).g(z)] = ……

a) $\frac{d}{dz}$ f(z). $\frac{d}{dz}$g(z)
b) $f(z)\frac{d}{dz}g(z) - g(z)\frac{d}{dz}f(z)$
c) $f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$
d) None of these

40) If f (z) and g(z) are differentiable at z, then $\frac{d}{dz}[\frac{f(z)}{g(z)}]$ = ……if $g(z) \neq 0$

a) $\frac{g(z)\frac{d}{dz}f(z)-f(z)\frac{d}{dz}g(z)}{[g(z)]^2}$
b) $\frac{df(z)}{dg(z)}$
c) $f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$
d) None of these

41) The function f (z) = $\bar{z}$ is ……. at every point in the z-plane.

a) continuous but not differentiable
b) differentiable
c) not continuous
d) None of these

42) If f (z) = z^2 + 5z + c, then f '(z_0) = …….

a) $2z_0 + 5 + c$
b) $2z_0 + 5$
c) $2z_0 - 5$
d) None of these

43) If a function f(z) is differential at every point of neighbourhood of z_0, then f(z) is …… at point z_0.

a) analytic
b) not analytic
c) harmonic
d) None of these

44) A function which is differential at every point of region is said to be ……in that region.

a) not analytic
b) analytic
c) harmonic
d) None of these

45) If f(z) is analytic at z_0, then it is not differential at z_0 is ……

a) false statement
b) true statement
c) both true and false
d) None of these

46) f(z) = $|z|^2$ is …… at any point $z \neq 0$.

a) differentiable
b) analytic
c) not analytic
d) None of these

47) If a complex function f(z) = u+iv is analytic at a point z = x+iy of its domain D, then at (x, y) the first order partial derivatives of u and v w.r.t. x and y exists and …… the C. R. equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

a) satisfies
b) not satisfies
c) may or may not satisfies
d) None of these

48) Let f(z) = u+iv = u(x, y)+iv(x, y). If the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists and are continuous at a point (x, y) in the domain D and they satisfy the C. R. equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x, y), then f(z) is ……… at a point z = x+iy

a) differentiable
b) analytic
c) not analytic
d) None of these

49) If f (z) = $\sqrt{|xy|}$ where $z \neq 0$ and f(0) = 0, then f(z) is …… at z = 0 even though the C-R equations are satisfied at z = 0.

a) differentiable
b) analytic
c) not analytic
d) None of these

50) If real part u of an analytic function f(z) = u + iv is constant, then f(z) is a ……… function.

a) non constant b) constant c) harmonic d) None of these

51) If imaginary part v of an analytic function f(z) = u + iv is constant, then f(z) is a …… function.

a) zero b) constant c) harmonic d) None of these

52) If f(z) is analytic function with constant modulus, then f(z) is a ……… function.

a) non constant b) constant c) harmonic d) None of these

53) If f(z) and $\overline{f(z)}$ are analytic function of z then f(z) is a ………function.

a) non constant b) constant c) harmonic d) None of these

54) An analytic function is also called ………function.

a) regular or holomorphic b) constant c) harmonic d) None of these

55) The function f (z) = $\bar{z}$ is not analytic function……….

a) may or may not true b) true statement
c) false statement d) Neither continuous nor differentiable

56) The function f (z) = e^z is ……….

a) analytic for all b) not analytic
c) not continuous d) Neither continuous nor differentiable

57) For an analytic function f (z) = u+iv, Cauchy Riemann equations are……

a) $u_x= -v_y$ & $u_y= v_x$ b) $u_x= -v_y$ & $u_y= -v_x$ c) $u_x= v_y$ & $u_y= -v_x$ d) None of these

58) If a real part u of an analytic function f (z) = u+iv is given, then f '(z) = ……

a) $u_x(x, y) + iu_y(x, y)$ b) $u_x(x, y) - iu_y(x, y)$
c) $u_x(x, y) - u_y(x, y)$ d) None of these

59) If an imaginary part v of an analytic function f (z) = u+iv is given, then f '(z) = ……

a) $v_y(x, y) + iv_x(x, y)$ b) $v_y(x, y) – iv_x(x, y)$
c) $v_x(x, y) - iv_y(x, y)$ d) None of these

60) If f(z) = u + iv is analytic function with u = $x^2 + y$, then v_y = ……

a) 2y b) 2x c) x d) y

61) If f(z) = u + iv is analytic function with v = $2x^3 + y^2$, then u_y = ……

a) 6y b) 6x c) -6x d) -6y

62) If u = x, then an analytic function f(z) = u + iv is ……

a) x+iy+c b) x+iy c) x-iy d) None of these

63) Let ∅ = ∅(x, y) be a function of two real variables x and y, then Laplace differential equation is given by …….

a) $\frac{\partial^2 \emptyset}{\partial x^2} - \frac{\partial^2 \emptyset}{\partial y^2} = 0$ b) $\frac{\partial^2 \emptyset}{\partial x^2} + 3\frac{\partial^2 \emptyset}{\partial y^2} = 0$ c) $\frac{\partial^2 \emptyset}{\partial x^2} - 2\frac{\partial^2 \emptyset}{\partial y^2} = 0$ d) $\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0$

64) If $\emptyset = \emptyset(x, y)$, then $\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$ is called ……

a) Laplace differential equation b) C. R. equation c) linear d) None of these

65) If $\emptyset(x, y) = x + y$, then $\frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$ ……

a) true statement b) false statement c) neither true nor false d) None of these

66) Laplace differential equation of real valued function $\emptyset$ (x, y) is …….

a) $\nabla^2\emptyset = 0$ b) $\nabla^2\emptyset = 1$ c) $\nabla^2\emptyset = -1$ d) None of these

67) If u = u(x, y) satisfy Laplace differential equation $u_{xx} + u_{yy} = 0$, then u is called ……. function.

a) analytic b) not analytic c) harmonic d) None of these

68) The real and imaginary parts of an analytic function …… Laplace differential equation.

a) satisfy b) does not satisfy c) may or may not satisfy d) None of these

69) An analytic function f (z) = u + iv, such that u and v must satisfy Laplace differential equation, then u and v are ……

a) analytic b) non-analytic c) harmonic d) None of these

70) If a real part u of an analytic function f (z) = u + iv is given, then by Milne-Thomson Method f (z) = ………

a) $\int u_1(z, o)dz + i\int u_2(z, o)dz + c$ b) $\int u_1(z, o)dz - i\int u_2(z, o)dz + c$

c) $\int u_2(z, o)dz - i\int u_1(z, o)dz + c$ d) None of these

71) If $u = x^2 - y^2$ is a real part of an analytic function f (z) = u + iv, then by Milne-Thomson Method f (z) = ………

a) $z^2 + c$ b) $iz^2 + c$ c) $-z^2 + c$ d) None of these

72) If an imaginary part v of an analytic function f (z) = u + iv is given, then by Milne-Thomson Method f (z) = ………

a) $\int v_1(z, o)dz + i\int v_2(z, o)dz + c$ b) $\int v_1(z, o)dz - i\int v_2(z, o)dz + c$,

c) $\int v_2(z, o)dz + i\int v_1(z, o)dz +$ d) None of these

73) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} =$ ………..

a) $4\frac{\partial^2}{\partial z\,\partial \bar{z}}$ b) 0 c) $-4\frac{\partial^2}{\partial z\,\partial \bar{z}}$ d) None of these

74) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 =$ ………..

a) $|f'(z)|^2$ b) $2|f'(z)|^2$ c) $4|f'(z)|^2$ d) None of these

75) $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\{R[f(z)]\}^2 =$ ………..

a) $|f'(z)|^2$ b) $2|f'(z)|^2$ c) $4|f'(z)|^2$ d) None of these

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UNIT-3: COMPLEX INTEGRATION

Closed Curve: A curve z = f(t), $a \le t \le b$, is said to be closed curve if its initial and final point coincides.

Simple Curve: A curve z = f(t), $a \le t \le b$, is said to be simple curve if it does not intersect itself anywhere.

Jordan Arc: A simple curve is called Jordan arc.

Simple Closed Curve: A curve z = f(t), $a \le t \le b$, is said to be simple closed curve if it does not intersect itself anywhere except initial and final point.

Closed Jordan Curve: A simple closed curve is called closed Jordan curve.

Smooth or Regular Curve: A curve $z = f(t) = \phi(t) + i\psi(t)$, $a \le t \le b$, is said to be smooth or regular curve if ϕ and ψ have continuous derivatives which does not vanish simultaneously for any value of t in [a, b].

Contour: A continuous chain of a finite number of smooth curve is called a contour.

Length of Contour: Length of contour $f(t) = \phi(t) + i\psi(t)$, $a \le t \le b$ is given by

$L = \int_a^b \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}dt$

Remark: Geometrically, a simple closed curve C is a circle or square or rectangle, and it divides the plane into two regions.

Jordan Curve Theorem: Any closed Jordan curve C separate the plane into two regions having C as common boundary.

Remark: Out of two regions interior of C is bounded and the other outer region is unbounded.

Simply Connected Region: A region R in the complex plane is called simply connected if any simple closed curve which lies inside R can be shrunk to a point without leaving R.

Multiply Connected Region: A region which is not simply connected is called multiply connected.

Line Integral: If f(z) is continuous inside a region R, then line integral of f(z) along a curve C which lies in R is $\int_C f(z)dz$.

Properties of Line Integral:

i) If – C is the curve traversed opposite that of C, then $\int_{-C} f(z)dz = -\int_C f(z)dz$

ii) If $C = C_1 + C_2 + C_3 + \ldots\ldots + C_n$, then $\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$

Ex. Evaluate $\int_C (x^2 + y^2 - xyi)dz$, where C is the line segment: z = 0 to z = 1 + i

Solution: Parametric equation of the line segment C: z = 0 to z = 1 + i is x = t, y = t,

so that z = x + iy = t + it = (1+i)t, $0 \le t \le 1$.

$\therefore$ $f(z) = x^2 + y^2 - xyi = t^2 + t^2 - t^2 i = (2 - i)\, t^2$ and dz = (1+i)dt

$\therefore \int_C f(z)dz = \int_{t=0}^{1} (2 - i)\, t^2\, (1 + i)dt$

$= (2 + 2i - i + 1)\, [\frac{t^3}{3}]_0^1$

$= (3 + i)[\frac{1}{3} - 0]$

$= 1 + \frac{1}{3} i$

==

Ex. If $f(z) = y - x - 3x^2 i$, then evaluate $\int_C f(z)dz$, where C is the straight line segment from z = 0 to z = 1 + i

Solution: Parametric equation of the line segment C: z = 0 to z = 1 + i is x = t, y = t,

so that z = x + iy = t + it = (1+i)t, $0 \le t \le 1$.

$\therefore$ $f(z) = y - x - 3x^2 i = t - t - 3t^2 i = -3t^2 i$ and dz = (1+i)dt

$\therefore \int_C f(z)dz = \int_{t=0}^{1} (-3t^2 i)\, (1 + i)dt$

$= -i(1 + i)\, [t^3]_0^1$

$= (-i + 1)[1 - 0]$

$= 1 - i$

==

Ex. If $f(z) = y - x - 3x^2 i$, then evaluate $\int_C f(z)dz$, where C consist of two straight line segments one from z = 0 to z = i and then from z = i to z = 1 + i

Solution: Let $C = C_1 + C_2$, where C_1 is the straight line segments from z = 0 to z = i and C_2 is the straight line segments from z = i to z = 1 + i

$\therefore \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ …… (1)

Parametric equation of the line segment C_1: z = 0 to z = i is x = 0, y = t,

so that z = x + iy = 0 + it = ti, $0 \le t \le 1$.

$\therefore$ $f(z) = y - x - 3x^2 i = t - 0 - 0i = t$ and dz = idt

$\therefore \int_C f(z).dz = \int_{t=0}^{1} t i dt$

$= i[\frac{t^2}{2}]_0^1$

$= i[\frac{1}{2} - 0]$

$= \frac{1}{2} i$

Again parametric equation of the line segment C_2: z = i to z = 1 + i is x = t, y =1 so that z = x + iy = t + i , $0 \leq t \leq 1$.

$\therefore f(z) = y - x - 3x^2 i = 1 - t - 3t^2 i$ and dz = dt

$\therefore \int_C f(z)dz = \int_{t=0}^{1} (1 - t - 3t^2 i)dt$

$= [t - \frac{t^2}{2} - t^3 i]_0^1$

$= [1 - \frac{1}{2} - i - 0]$

$= \frac{1}{2} - i$

Putting in (1), we get,

$\int_C f(z)dz = \frac{1}{2} i + \frac{1}{2} - i = \frac{1}{2} (1 - i)$

==

Ex. Evaluate $\int_C zdz$, where C is the arc of the parabola $y^2 = 4ax$ from (0, 0) to (a, 2a)

Solution: Let C is the arc of the parabola $y^2 = 4ax$ from (0, 0) to (a, 2a)

$\therefore$ Parametric equation of C is $x = at^2$, $y = 2at$,

so that $z = x + iy = at^2 + 2ati = a(t^2 + 2ti)$, $0 \leq t \leq 1$.

$\therefore f(z) = z = a(t^2 + 2ti)$ and $dz = a(2t + 2i)dt = 2a(t + i)dt$

$\therefore \int_C f(z)dz = \int_{t=0}^{1} a(t^2 + 2ti)\, 2a(t + i)dt$

$= 2a^2 \int_{t=0}^{1} (t^2 + 2ti)\, (t + i)dt$

$= 2a^2 \int_{t=0}^{1} (t^3 + t^2 i + 2t^2 i - 2t)\, dt$

$= 2a^2 \int_{t=0}^{1} (t^3 - 2t + 3t^2 i)\, dt$

$= 2a^2 [\frac{t^4}{4} - t^2 + t^3 i]_0^1$

$= 2a^2 [\frac{1}{4} - 1 + i - 0]$

$= 2a^2 [-\frac{3}{4} + i]$

$= -\frac{1}{2} a^2(3 - 4i)$

==

Ex. Evaluate $\int_C (z - a)^n dz$, where C is the circle: $|z - a| = r$ and n is positive or negative integer. (Oct.2019)

Solution: Let C is the circle: $|z - a| = r$ and n is positive or negative integer.

∴ Parametric equation of C is $z = a + re^{i\theta}$, where $0 \le \theta \le 2\pi$

∴ $f(z) = (z - a)^n = (re^{i\theta})^n = r^n e^{ni\theta}$ and $dz = re^{i\theta} i d\theta$

$$\therefore \int_C f(z)dz = \int_{t=0}^{2\pi} (r^n e^{ni\theta})\, re^{i\theta} i d\theta$$

$$= ir^{n+1} \int_{t=0}^{1} (e^{(n+1)i\theta})\, d\theta$$

$$= ir^{n+1} \left[\frac{e^{(n+1)i\theta}}{(n+1)i}\right]_0^{2\pi} \qquad \text{if } n \neq -1$$

$$= \frac{r^{(n+1)}}{(n+1)} [e^{(n+1)i\theta}]_0^{2\pi}]$$

$$= \frac{ir^{(n+1)}}{(n+1)} [e^{2(n+1)i\pi} - 1]$$

$$= \frac{ir^{(n+1)}}{(n+1)} (1 - 1)$$

$$= 0 \qquad \text{if } n \neq -1$$

If n =-1, then $f(z) = (z - a)^{-1} = \frac{1}{z-a} = \frac{1}{re^{i\theta}}$

$$\therefore \int_C f(z)dz = \int_{t=0}^{2\pi} \left(\frac{1}{re^{i\theta}}\right) re^{i\theta} i d\theta$$

$$= i[\theta]_0^{2\pi}$$

$$= i[2\pi - 0]$$

$$= 2\pi i$$

i.e. $\int_C (z - a)^n dz = 0$ for any integer n except $n \neq -1$.

and $\int_C \frac{1}{z-a} dz = 2\pi i$

===

Ex. Show that the integral of $\frac{1}{z}$ along a semicircular arc from -1 to 1, has the value $-\pi i$ or πi according as the arc lies above or below the real axis.

Proof: i) If C_1 is a semicircular arc from -1 to 1 lies above the real axis.

∴ Parametric equation of C_1 is $z = e^{i\theta}$, where θ varies from π to 0.

∴ $f(z) = \frac{1}{z} = \frac{1}{e^{i\theta}}$ and $dz = e^{i\theta} i d\theta$

$$\therefore \int_C f(z)dz = \int_{t=\pi}^{0} \left(\frac{1}{e^{i\theta}}\right) e^{i\theta} i d\theta$$

$$= i[\theta]_\pi^0$$

$$= i[0 - \pi]$$

$= -\pi i$

Parametric equation of C_2 is $z = e^{i\theta}$, where θ varies from π to 2π.

$\therefore f(z) = \frac{1}{z} = \frac{1}{e^{i\theta}}$ and $dz = e^{i\theta} i d\theta$

$\therefore \int_C f(z)dz = \int_{t=\pi}^{2\pi} \left(\frac{1}{e^{i\theta}}\right) e^{i\theta} i d\theta$

$= i[\theta]_{\pi}^{2\pi}$

$= i[2\pi - \pi]$

$= \pi i$

Hence proved.

==

Cauchy's Integral Theorem: If f(z) is analytic on and within a simple closed contour C, then $\int_C f(z)dz = 0$.

Remark: Cauchy's Integral Theorem is also called Cauchy-Gaursat's Theorem or Cauchy-Theorem.

==

Corollary-1: If f(z) is analytic in a simply connected region R, then $\int_a^b f(z)dz$ is independent of path of the integration in R joining the points a and b.

Proof: Let f(z) is analytic in a simply connected region R. Let A(a) and B(b) be two points representing the complex numbers a and b respectively within R.

Let C_1 and C_2 be two arcs in R joining A(a) and B(b). Now C = APBQA is a simple closed curve in R.

$\therefore$ f(z) is analytic on and within a simple closed contour C, then by Cauchy's Integral Theorem $\int_{C = APBQA} f(z)dz = 0$.

i.e. $\int_{APB} f(z)dz + \int_{BQA} f(z)dz = 0$.

i.e. $\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0$

$\therefore \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$

$\therefore \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

This shows that $\int_a^b f(z)dz$ is independent of path joining A(a) and B(b) within R.

==

Ex. Verify Cauchy's Integral Theorem for $f(z) = z + 1$ around the contour $|z| = 1$.

Proof: Here the contour C is the circle $|z| = 1$, which is simple closed curve.

As $f(z) = z + 1$ is analytic everywhere in the complex plane, hence it is analytic inside and on C.

∴ By Cauchy’s Integral Theorem, $\int_C f(z)dz = 0$.

i.e. $\int_C (z+1)dz = 0$ …… (1)

Now parametric equation of C is $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

∴ $dz = e^{i\theta} i d\theta$

∴ $\int_C f(z)dz = \int_{|z|=1} (z+1)dz$

$= \int_0^{2\pi} (e^{i\theta} + 1)\, e^{i\theta} i d\theta$

$= \int_0^{2\pi} (e^{2i\theta} + e^{i\theta})\, i d\theta$

$= i\left[\frac{e^{2i\theta}}{2i} + \frac{e^{i\theta}}{i}\right]_0^{2\pi}$

$= \left[\frac{e^{2i\theta}}{2} + e^{i\theta}\right]_0^{2\pi}$

$= \left[\frac{e^{4\pi i}}{2} + e^{2\pi i}\right] - \left[\frac{e^{0}}{2} + e^{0}\right]$

$= \frac{1}{2} + 1 - \frac{1}{2} - 1$

∴ $\int_C f(z)dz = 0$

Hence Cauchy’s theorem is verified.

===

Ex. Verify Cauchy’s Integral Theorem for $f(z) = z^2$ around the circle $|z| = 1$.

Proof: Here the closed contour C is the circle $|z| = 1$, which is simple closed curve.

As $f(z) = z^2$ is analytic everywhere in the complex plane, hence it is analytic inside and on C.

y Cauchy’s Integral Theorem, $\int_C f(z)dz = 0$.

i.e. $\int_C z^2 dz = 0$ …… (1)

Now parametric equation of C is $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

∴ $dz = e^{i\theta} i d\theta$

∴ $\int_C f(z)dz = \int_{|z|=1} z^2 dz$

$= \int_0^{2\pi} (e^{i\theta})^2\, e^{i\theta} i d\theta$

$= \int_0^{2\pi} (e^{3i\theta})\, i d\theta$

$= i\left[\frac{e^{3i\theta}}{3i}\right]_0^{2\pi}$

$= \left(\frac{e^{6\pi i}}{3} - \frac{e^{0}}{3}\right)$

$= \frac{1}{3} - \frac{1}{3}$

$\therefore \int_C f(z)dz = 0$

Hence Cauchy's theorem is verified.

==

Ex. Use Cauchy Goursat Theorem to obtain the value $\int_C e^z dz$, where C is the circle $|z| = 1$ and hence deduce that i) $\int_0^{2\pi} e^{\cos\theta}\sin(\theta + \sin\theta)\, d\theta = 0$

ii) $\int_0^{2\pi} e^{\cos\theta}\cos(\theta + \sin\theta)\, d\theta = 0$

Proof: Take $f(z) = e^z$ which is analytic everywhere in the complex plane, hence it is analytic inside and on the circle C: $|z| = 1$

$\therefore$ By Cauchy's Integral Theorem, $\int_C f(z)dz = 0$.

i.e. $\int_C e^z dz = 0$ …… (1)

Now parametric equation of C is $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

$\therefore dz = e^{i\theta} i d\theta$

$\therefore \int_C f(z)dz = \int_{|z|=1} e^z dz$

$= \int_0^{2\pi} e^{e^{i\theta}}\ e^{i\theta} i d\theta$

$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\ e^{i\theta} i d\theta$

$= i\int_0^{2\pi} e^{\cos\theta + i(\theta+\sin\theta)}\, d\theta$

$= i\int_0^{2\pi} e^{\cos\theta}\ e^{i(\theta+\sin\theta)} d\theta$

$= i\int_0^{2\pi} e^{\cos\theta}\,[\cos(\theta+\sin\theta) + i\sin(\theta+\sin\theta)]\, d\theta$

$= i\int_0^{2\pi} e^{\cos\theta}\cos(\theta+\sin\theta) - \int_0^{2\pi} e^{\cos\theta}\sin(\theta+\sin\theta)\, d\theta$

But $\int_C f(z)dz = 0$

$\therefore i\int_0^{2\pi} e^{\cos\theta}\cos(\theta+\sin\theta) - \int_0^{2\pi} e^{\cos\theta}\sin(\theta+\sin\theta)\, d\theta = 0 = 0 + i0$

Equating real and imaginary parts, we get,

i) $\int_0^{2\pi} e^{\cos\theta}\sin(\theta + \sin\theta)\, d\theta = 0$ and ii) $\int_0^{2\pi} e^{\cos\theta}\cos(\theta + \sin\theta)\, d\theta = 0$

Hence proved.

==

Ex. Verify Cauchy's Theorem for $f(z) = z$ around a closed curve C, where C is the rectangle bounded by the lines: $x = 0$, $x = 1$, $y = 0$, $y = 1$.

Proof: Here C the rectangle bounded by the lines: x = 0, x = 1, y = 0, y = 1.

i.e. C = OABDO, where O(0, 0), A(1, 0), B(1, 1) and D(0, 1)

$\therefore \int_C f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz + \int_{BD} f(z)dz + \int_{DO} f(z)dz$

$\therefore \int_C zdz = \int_{OA} zdz + \int_{AB} zdz + \int_{BD} zdz + \int_{DO} zdz$ …… (1)

As f(z) = z is analytic everywhere in the complex plane, hence it is analytic on and inside rectangle C.

$\therefore$ By Cauchy's Theorem, $\int_C f(z)dz = 0$.

i.e. $\int_C zdz = 0$ …… (2)

i) Along OA, y = 0 and x varies from 0 to 1.

$\therefore$ z = x + iy = x and dz = dx

$\therefore \int_{OA} zdz = \int_0^1 xdx = [\frac{x^2}{2}]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$

ii) Along AB, x = 1 and y varies from 0 to 1.

$\therefore$ z = x + iy = 1 + iy and dz = idy

$\therefore \int_{AB} zdz = \int_0^1 (1 + iy)idy = i[y + \frac{y^2}{2} i]_0^1 = i(1+ \frac{1}{2} i - 0) = -\frac{1}{2} + i$

iii) Along BD, y = 1 and x varies from 1 to 0.

$\therefore$ z = x + iy = x + i and dz = dx

$\therefore \int_{BD} zdz = \int_1^0 (x + i)dx = [\frac{x^2}{2} + ix]_1^0 = 0 - \frac{1}{2} - i = - \frac{1}{2} - i$

iv) Along DO, x = 0 and y varies from 1 to 0.

$\therefore$ z = x + iy = iy and dz = idy

$\therefore \int_{DO} zdz = \int_1^0 (iy)idy = -[\frac{y^2}{2}]_1^0 = -[0 - \frac{1}{2}] = \frac{1}{2}$

Putting these values in (1), we get,

$\int_C zdz = \int_{OA} zdz + \int_{AB} zdz + \int_{BD} zdz + \int_{DO} zdz = \frac{1}{2} - \frac{1}{2} + i - \frac{1}{2} - i + \frac{1}{2}$

$\therefore \int_C f(z)dz \ = 0$

Hence Cauchy's theorem is verified.

==

Remark: If f(z) is analytic in a region bounded by two simple closed curves C_1 and C_2 and also on C_1 and C_2, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Cauchy's theorem for a system of contours: Let $C_1, C_2, C_3, \ldots\ldots, C_n$ be a system of closed Jordon contours such that $C_1, C_2, C_3, \ldots\ldots, C_n$ are all lie inside C and outside to each other. Let R be a region from C obtained by excluding interiors of each of the curves C_k of C. If f(z) is analytic in R and on each of the contours $C, C_1, C_2, \ldots\ldots, C_n$, then $\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$

where each contour is traversed in the positive (anticlockwise) sense.

==

Cauchy's Integral Formula for f(a): If f(z) is analytic inside and on a simple closed contour C of a simply connected region R and point z = a lies inside C,

then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$ i.e. $\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$

Proof: Let f(z) is analytic inside and on a simple closed contour C and point z = a lies inside C.

$\therefore \frac{f(z)}{z-a}$ is analytic inside C except at the point z = a.

We draw a circle C_1 with centre at z = a and radius δ such that it lies completely inside C.

Now $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1 as well as on C and C_1.

Hence by corollary of Cauchy's Integral Theorem, we have

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz$$

$$= \int_{C_1} \frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)-f(a)}{z-a}dz + f(a)\int_{C_1} \frac{1}{z-a}dz \;\ldots\ldots (1)$$

Parametric equation of circle C_1: $|z-a| = \delta$ is $z-a = \delta e^{i\theta}$, $0 \le \theta \le 2\pi$.

$\therefore dz = \delta e^{i\theta} i d\theta$

$$\therefore \int_{C_1} \frac{1}{z-a}dz = \int_{t=0}^{2\pi}\left(\frac{1}{\delta e^{i\theta}}\right)\delta e^{i\theta} i d\theta$$

$$= i[\theta]_0^{2\pi}$$

$$= i[2\pi - 0]$$

$$= 2\pi i$$

Substituting in (1), we get,

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)-f(a)}{z-a}dz + 2\pi i f(a)$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)-f(a)}{z-a}dz + f(a) \;\ldots\ldots (2)$$

Now consider $\left|\frac{1}{2\pi i}\int_{C_1} \frac{f(z)-f(a)}{z-a}dz\right| \le \frac{1}{2\pi}\int_{C_1} \frac{|f(z)-f(a)|}{|z-a|}|dz| \because |i| = 1$

As every analytic function is differentiable and hence continuous.

$\therefore$ f(z) is continuous at z = a and hence for $\varepsilon > 0$, $\exists\ \delta > 0$ depend on ε, such that $|f(z) - f(a)| < \varepsilon$ whenever $|z - a| < \delta$.

$\therefore \left|\frac{1}{2\pi i} \int_{C_1} \frac{f(z)-f(a)}{z-a} dz\right| < \frac{1}{2\pi} \int_{C_1} \frac{\varepsilon}{\delta} |dz| = \frac{\varepsilon}{2\pi\delta} \int_{C_1} |dz|$

$\therefore \left|\frac{1}{2\pi i} \int_{C_1} \frac{f(z)-f(a)}{z-a} dz\right| < \frac{\varepsilon}{2\pi\delta}\ (2\pi\delta) \qquad \because \int_{C_1} |dz| = \text{Length of } C_1 = 2\pi\delta$

$\therefore \left|\frac{1}{2\pi i} \int_{C_1} \frac{f(z)-f(a)}{z-a} dz\right| < \varepsilon$

For $\varepsilon \to 0$, we have,

$\left|\frac{1}{2\pi i} \int_{C_1} \frac{f(z)-f(a)}{z-a} dz\right| = 0$

i.e. $\frac{1}{2\pi i} \int_{C_1} \frac{f(z)-f(a)}{z-a} dz = 0$

$\therefore$ equation (2) reduces to

$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz = f(a)$

==

Cauchy's Integral Formula for f '(a): If f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then $f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$

Proof: Choose h such that a + h is also lies inside C.

$\therefore$ By Cauchy's integral theorem, we have

$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$ and $f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a-h} dz$

$$\therefore f(a+h) - f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a-h} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{1}{z-a-h} - \frac{1}{z-a}\right] f(z) dz$$

$$= \frac{1}{2\pi i} \int_C \left[\frac{h}{(z-a-h)(z-a)}\right] f(z) dz$$

$$\therefore \frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i} \int_C \left[\frac{1}{(z-a-h)(z-a)}\right] f(z) dz$$

Taking limit as $h \to 0$, we get,

$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$

==

Cauchy's Integral Formula for $f^n(a)$: If f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz, \ n \in N$$

==

Ex. Evaluate by Cauchy integral formula $\int_C \frac{e^z}{z-2} dz$, where C is the circle $|z-2| = 1$

Solution: Take $f(z) = e^z$ which is analytic everywhere in the complex plane, hence it is analytic inside and on the circle C: $|z-2| = 1$ & the point $z = 2$ lies inside C.

∴ By Cauchy's integral formula f(a), we have,

$f(2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-2} dz$

$\therefore \int_C \frac{f(z)}{z-2} dz = 2\pi i f(2)$

$\therefore \int_C \frac{e^z}{z-2} dz = 2\pi i e^2$

==

Ex. Evaluate by Cauchy integral formula $\int_C \frac{3z-1}{z^2-2z-3} dz$, where C is the circle $|z| = 4$

Solution: First resolve the integrand into partial fractions as

$\frac{3z-1}{z^2-2z-3} = \frac{3z-1}{(z-3)(z+1)} = \frac{1}{z+1} + \frac{2}{z-3}$

$\therefore \int_C \frac{3z-1}{z^2-2z-3} dz = \int_C \frac{1}{z+1} dz + 2\int_C \frac{1}{z-3} dz$

$\therefore \int_C \frac{3z-1}{z^2-2z-3} dz = \int_C \frac{f(z)}{z+1} dz + 2\int_C \frac{f(z)}{z-3} dz$ …… (1)

Where $f(z) = 1$ is analytic everywhere in the complex plane, hence it is analytic inside and on the circle C: $|z| = 4$ and the points $z = 3$ and $z = -1$ both lies inside C.

∴ By Cauchy's integral formula,

$f(3) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-3} dz$

$\therefore \int_C \frac{f(z)}{z-3} dz = 2\pi i f(3)$

$\therefore \int_C \frac{1}{z-3} dz = 2\pi i \qquad \because f(3) = 1$

& $f(-1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z+1} dz$

$\therefore \int_C \frac{f(z)}{z+1} dz = 2\pi i f(-1)$

$\therefore \int_C \frac{1}{z+1} dz = 2\pi i \qquad \because f(-1) = 1$

Putting these values in (1), we get,

$\therefore \int_C \frac{3z-1}{z^2-2z-3} dz = 2\pi i + 2(2\pi i) = 6\pi i$

==

Ex. Using Cauchy's Integral formula, evaluate $\int_C \frac{dz}{z^3(z+4)} dz$, where C is the circle $|z| = 2$

Solution: We observe that $\frac{1}{z^3(z+4)}$ is not analytic at $z = 0$ and $z = -4$, out of these only the point $z = 0$ lies inside circle C: $|z| = 2$.

$\therefore$ We take $f(z) = \frac{1}{(z+4)}$ which is analytic inside and on the circle C: $|z| = 2$ and the point $z = 0$ lies inside C.

$\therefore$ By Cauchy's integral formula for $f''(a)$,

$$f''(0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-0)^3}dz$$

$$\therefore \int_C \frac{f(z)}{z^3}dz = \pi i\, f''(0)$$

As $f(z) = \frac{1}{(z+4)}$ $\therefore f'(z) = \frac{-1}{(z+4)^2}$ & $f''(z) = \frac{2}{(z+4)^3}$ $\therefore f''(0) = \frac{2}{64} = \frac{1}{32}$

$$\therefore \int_C \frac{1}{z^3(z+4)}dz = \frac{\pi i}{32}$$

==

Ex. Evaluate $\int_{|z|=2} \frac{e^{2z}}{(z-1)^4}dz$, Using Cauchy's Integral formula.

Solution: We take $f(z) = e^{2z}$ which is analytic inside and on the circle C: $|z| = 2$ and the point $z = 1$ lies inside C.

$\therefore$ By Cauchy's integral formula for $f'''(a)$,

$$f'''(1) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-1)^4}dz$$

$$\therefore \int_C \frac{f(z)}{(z-1)^4}dz = \frac{1}{3}\pi i\, f'''(0)$$

As $f(z) = e^{2z}$ $\therefore f'(z) = 2e^{2z}$, $f''(z) = 4e^{2z}$ & $f'''(z) = 8e^{2z}$ $\therefore f'''(1) = 8e^2$

$$\therefore \int_{|z|=2} \frac{e^{2z}}{(z-1)^4}dz = \frac{8}{3}\pi e^2 i$$

==

Ex. Evaluate $\int_{|z|=1} \frac{e^z}{z}dz$ and deduce that

i) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\, d\theta = 2\pi$ and ii) $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\, d\theta = 0$

Solution: Take $f(z) = e^z$ which is analytic everywhere in the complex plane, hence it is analytic inside and on the circle C: $|z| = 1$ & the point $z = 0$ lies inside C.

$\therefore$ By Cauchy's integral formula f(a), we have,

$$f(0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}dz$$

$$\therefore \int_C \frac{f(z)}{z}dz = 2\pi i f(0)$$

$$\therefore \int_C \frac{e^z}{z}dz = 2\pi i e^0 = 2\pi i \text{ ...... (1)}$$

Now parametric equation of C is $z = e^{i\theta}$, $0 \le \theta \le 2\pi$.

$\therefore dz = e^{i\theta} i\, d\theta$

$$\therefore \int_C \frac{f(z)}{z} dz = \int_{|z|=1} \frac{e^z}{z} dz$$

$$= \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}} e^{i\theta} i d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\, i d\theta$$

$$= i \int_0^{2\pi} e^{\cos\theta} e^{i\sin\theta}\, d\theta$$

$$= i \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) + i\sin(\sin\theta)\, d\theta$$

$$= i \int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) - \int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta$$

But $\int_C \frac{f(z)}{z} dz = 2\pi i$

$\therefore i \int_0^{2\pi} e^{\cos\theta} \cos(\theta + \sin\theta) - \int_0^{2\pi} e^{\cos\theta} \sin(\theta + \sin\theta)\, d\theta = 2\pi i = 0 + 2\pi i$

Equating imaginary and real parts, we get,

i) $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\, d\theta = 0$ and ii) $\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\, d\theta = 0$

Hence proved.

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Complex Sequence: An infinite sequence $\{z_n\}$ in which each term z_n is a complex number is called complex sequence.

Convergence of Sequence: A sequence $\{z_n\}$ is called convergent sequence if it has a limit otherwise it is called divergent sequence.

e.g. For $|z| < 1$, the geometric sequence $z, z^2, \ldots\ldots, z^n, \ldots\ldots$ is convergent to 0.

Complex Series: A series $\sum_{n=1}^{\infty} z_n$ in which each term z_n is a complex number is called complex series.

Partial Sum of Series: $S_n = \sum_{k=1}^{n} z_k$ is called partial sum of series $\sum_{n=1}^{\infty} z_n$.

Convergence of Series: If sequence of partial sums $\{S_n\}$ is convergent then series is convergent and if sequence of partial sums $\{S_n\}$ is divergent then series is divergent.

Absolutely Convergent Series: A series $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Power Series: A series of the form $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_0, a_1, a_2, \ldots, a_n, \ldots$ and z_0 are complex numbers is called a power series.

Remark: i) Every absolutely convergent series is convergent.

ii) Geometric series $\sum_{n=0}^{\infty} z^n$ or $\sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \ldots + z^n + \ldots$ is convergent if $|z| < 1$ and divergent if $|z| \geq 1$

iii) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for z = $z_1(\neq 0)$, then it is absolutely convergent for any value of z such that $|z| < |z_1|$

iv) If a power series $\sum_{n=0}^{\infty} a_n z^n$ diverges for z = z_1, then it is diverges for any value of z such that $|z| > |z_1|$

==

Taylor's Series: If f(z) is analytic in a region R and z_0 lies in R, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^n(z_0)}{n!}(z - z_0)^n + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n$$

is called Taylor's series.

Maclaurin's Series: If f(z) is analytic in a region R and 0 lies in R, then

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \ldots + \frac{f^n(0)}{n!}z^n + \ldots = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$$

is called Maclaurin's series.

Laurent's Series: If f(z) is analytic on two concentric circles C_1 and C_2 with centre at z = a and also the ring shaped region bounded by these two circles, then for any point z in this region $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$

is called Laurent's series.

Where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$, for n = 0, 1, 2, …… and

$$b_n = \frac{1}{2\pi i}\int_{C_2} (z-a)^{n-1} f(z)dz \quad \text{for } n = 1, 2, 3, \ldots\ldots$$

Remark: i) In Laurent's series $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called analytic part and $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is called principal part of series.

ii) The ring shaped region bounded by two circles C_1 and C_2 is called annulus.

Results: Taylor's /Maclaurin's series expansions of some standard functions are as:

i) $\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + \ldots + (-1)^n z^n + \ldots = \sum_{n=0}^{\infty} (-1)^n z^n$ for $|z| < 1$

ii) $\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$

iii) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$

iv) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \ldots$

v) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + \frac{(-1)^n z^{2n}}{(2n)!} + \ldots$

vi) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + \frac{z^{2n+1}}{(2n+1)!} + \ldots$

vii) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + \frac{z^{2n}}{(2n)!} + \ldots$

==

Ex. Expand in Taylor's series: a) $\frac{1}{(z-2)}$ for $|z| < 2$ b) $\frac{1}{(z-1)(z-2)}$ for $|z| < 1$

Solution: a) $|z| < 2 \Rightarrow |\frac{z}{2}| < 1$

$\therefore \frac{1}{(z-2)} = -\frac{1}{2}\left[\frac{1}{(1-\frac{z}{2})}\right]$

$= -\frac{1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n$ by Taylor's series expansion

$\therefore \frac{1}{(z-2)} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$ for $|z| < 2$

b) First we resolve $\frac{1}{(z-1)(z-2)}$ into partial fractions as

$\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

Now $|z| < 1 \Rightarrow |z| < 2 \Rightarrow |\frac{z}{2}| < 1$

$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

$= \frac{1}{(1-z)} - \frac{1}{2(1-\frac{z}{2})}$

$= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}(\frac{z}{2})^n$

by Taylor's series expansion

$= \sum_{n=0}^{\infty}(1 - \frac{1}{2^{n+1}})z^n$ for $|z| < 1$

==

Ex. Prove that $\frac{1}{4z-z^2} = \sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$, where $0 < |z| < 4$.

Proof : $0 < |z| < 4 \Rightarrow |\frac{z}{4}| < 1$

Consider L.H.S. $= \frac{1}{4z-z^2}$

$= \frac{1}{4z(1-\frac{z}{4})}$

$= \frac{1}{4z}\sum_{n=0}^{\infty}(\frac{z}{4})^n$ by Taylor's series expansion

$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$

= R.H.S.

Hence proved.

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Ex. Find the expansion of $f(z) = \frac{1}{(z^2+1)(z^2+2)}$ in powers of z, when $|z| < 1$

Solution: $|z| < 1 \Rightarrow |z^2| < 1 \Rightarrow |z^2| < 2 \Rightarrow |\frac{z^2}{2}| < 1$

Now $f(z) = \frac{1}{(z^2+1)(z^2+2)} = \frac{1}{(z^2+1)} - \frac{1}{(z^2+2)}$

$= \frac{1}{(1+z^2)} - \frac{1}{2(1+\frac{z^2}{2})}$

$= \sum_{n=0}^{\infty}(-1)^n (z^2)^n - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n (\frac{z^2}{2})^n$

by Taylor's series expansion

$= \sum_{n=0}^{\infty}(-1)^n z^{2n} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{2^n}$

$\therefore f(z) = \sum_{n=0}^{\infty}(-1)^n (1 - \frac{1}{2^{n+1}})z^{2n}$

be the required expansion, when $|z| < 1$

==

Ex. Expand $f(z) = \frac{1}{(z-2)}$ in Laurent's series valid for $|z| > 2$

Solution: $|z| > 2 \Rightarrow 2 < |z| \Rightarrow |\frac{2}{z}| < 1$

$\therefore f(z) = \frac{1}{(z-2)}$

$= \frac{1}{z}\left[\frac{1}{(1-\frac{2}{z})}\right]$

$= \frac{1}{z}\sum_{n=0}^{\infty}(\frac{2}{z})^n$ by Laurent's series expansion

$\therefore \frac{1}{(z-2)} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$ for $|z| > 2$

==

Ex. Obtain the expansion of $f(z) = \frac{z^2-1}{(z+2)(z+3)}$, in the powers of z in the region

(i) $|z| < 2$ (ii) $2 < |z| < 3$ (iii) $|z| > 3$.

Solution: First we express $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ into partial fractions as follows

$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{A}{(z+2)} + \frac{B}{(z+3)}$ …… (1)

i.e. $z^2 - 1 = (z+2)(z+3) + A(z+3) + B(z+2)$ ……(2)

Putting z = -2 in (2), we get,

$4 - 1 = 0 + A + 0 \quad \therefore A = 3$

Again putting z = -3 in (2), we get,

$9 - 1 = 0 + 0 - B \quad \therefore B = -8$

From (1), we have,

$(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$

(i) $|z| < 2 \Rightarrow |z| < 3 \Rightarrow |\frac{z}{2}| < 1$ & $|\frac{z}{3}| < 1$

$\therefore (z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} = 1 + \frac{3}{2}\frac{1}{(1+\frac{z}{2})} - \frac{8}{3}\frac{1}{(1+\frac{z}{3})}$

$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$

by Taylor's series expansion

$= 1 + 3\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$

(ii) $2 < |z| < 3 \Rightarrow 2 < |z|$ & $|z| < 3 \Rightarrow |\frac{2}{z}| < 1$ & $|\frac{z}{3}| < 1$

$\therefore (z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} = 1 + \frac{3}{z}\frac{1}{(1+\frac{2}{z})} - \frac{8}{3}\frac{1}{(1+\frac{z}{3})}$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$

by Taylor's series expansion

$= 1 + 3\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$

(iii) $|z| > 3 \Rightarrow |z| > 2 \Rightarrow |\frac{3}{z}| < 1$ & $|\frac{2}{z}| < 1$

$\therefore (z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)} = 1 + \frac{3}{z}\frac{1}{(1+\frac{2}{z})} - \frac{8}{z}\frac{1}{(1+\frac{3}{z})}$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$

by Taylor's series expansion

$= 1 + 3\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{z^{n+1}}$

==

Ex. Obtain Laurents expansion of $(z) = \frac{z^2-6}{z^2-5z+6}$ valid in the region $2 < |z| < 3$

Solution: First we express $(z) = \frac{z^2-6}{z^2-5z+6} = \frac{z^2-6}{(z-2)(z-3)}$ into partial fractions as follows

$\frac{z^2-6}{(z-2)(z-3)} = 1 + \frac{A}{(z-2)} + \frac{B}{(z-3)}$ …… (1)

i.e. $z^2 - 6 = (z-2)(z-3) + A(z-3) + B(z-2)$ ……(2)

Putting z = 2 in (2), we get,

$4 - 6 = 0 + A(-1) + 0 \quad \therefore -A = -2 \quad \therefore A = 2$

Again putting z = 3 in (2), we get,

9 – 6 = 0 + 0 + B(1) $\quad\therefore$ B = 3

From (1), we have,

$(z) = 1 + \frac{2}{(z-2)} + \frac{3}{(z-3)}$

Now $2 < |z| < 3 \Longrightarrow 2 < |z|$ & $|z| < 3 \Longrightarrow |\frac{2}{z}| < 1$ & $|\frac{z}{3}| < 1$

$\therefore (z) = 1 + \frac{2}{(z-2)} + \frac{3}{(z-3)}$

$= 1 + \frac{2}{z(1-\frac{2}{z})} - \frac{1}{(1-\frac{z}{3})}$

$= 1 + \frac{2}{z}\sum_{n=0}^{\infty}(\frac{2}{z})^n - \sum_{n=0}^{\infty}(\frac{z}{3})^n$

by Taylor's series expansion

$= 1 + \sum_{n=0}^{\infty}\frac{2^{n+1}}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{3^n}$ for $2 < |z| < 3$

===

Ex. Obtain the expansion of $(z) = \frac{z^2-4}{z^2+5z+4}$ in the powers of z for

(i) $|z| < 1$ (ii) $1 < |z| < 4$ (iii) $|z| > 4$.

Solution: First we express $(z) = \frac{z^2-4}{z^2+5z+4} = \frac{z^2-4}{(z+1)(z+4)}$ into partial fractions as follows

$\frac{z^2-4}{(z+1)(z+4)} = 1 + \frac{A}{(z+1)} + \frac{B}{(z+4)}$ …… (1)

i.e. $z^2 - 4 = (z+1)(z+4) + A(z+4) + B(z+1)$ ……(2)

Putting $z = -1$ in (2), we get,

1 – 4 = 0 + A(3) + 0 $\quad\therefore$ A = −1

Again putting $z = -4$ in (2), we get,

16 – 4 = 0 + 0 + B(−3) $\quad\therefore -3B = 12 \therefore B = -4$

From (1), we have,

$(z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$

(i) $|z| < 1 \Longrightarrow |z| < 4 \Longrightarrow |\frac{z}{4}| < 1$

$\therefore (z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$

$= 1 - \frac{1}{(1+z)} - \frac{1}{(1+\frac{z}{4})}$

$= 1 - \sum_{n=0}^{\infty}(-1)^n z^n - \sum_{n=0}^{\infty}(-1)^n (\frac{z}{4})^n$

by Taylor's series expansion

$$= 1 - \sum_{n=0}^{\infty}(-1)^n z^n - \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{4^n}$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \left(1 + \frac{1}{4^n}\right)z^n \qquad \text{for } |z| < 1$$

(ii) $1 < |z| < 4 \Longrightarrow 1 < |z| \;\&\; |z| < 4 \Longrightarrow \left|\frac{1}{z}\right| < 1 \;\&\; \left|\frac{z}{4}\right| < 1$

$$\therefore (z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$$

$$= 1 - \frac{1}{z(1+\frac{1}{z})} - \frac{1}{(1+\frac{z}{4})}$$

$$= 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}(-1)^n \left(\frac{z}{4}\right)^n$$

by Taylor's series expansion

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{4^n}$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{z^{n+1}} + \frac{z^n}{4^n}\right) \qquad \text{for } 1 < |z| < 4$$

(iii) $|z| > 4 \Longrightarrow |z| > 1 \Longrightarrow \left|\frac{4}{z}\right| < 1 \;\&\; \left|\frac{1}{z}\right| < 1$

$$\therefore (z) = 1 - \frac{1}{(z+1)} - \frac{4}{(z+4)}$$

$$= 1 - \frac{1}{z(1+\frac{1}{z})} - \frac{4}{z(1+\frac{4}{z})}$$

$$= 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{z}\right)^n - \frac{4}{z}\sum_{n=0}^{\infty}(-1)^n \left(\frac{4}{z}\right)^n$$

by Taylor's series expansion

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{n+1}} - \frac{4}{z}\sum_{n=0}^{\infty}(-1)^n \frac{4^n}{z^n}$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}(-1)^n \frac{4^{n+1}}{z^{n+1}}$$

$$= 1 - \sum_{n=0}^{\infty}(-1)^n \left(1 + 4^{n+1}\right)\frac{1}{z^{n+1}} \qquad \text{for } |z| > 4$$

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MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) A curve z = f(t), a ≤ t ≤ b, is said to be ……. curve if its initial and final point coincides.

a) open   b) closed   c) continuous   d) None of these

2) A curve z = f(t), a ≤ t ≤ b, is said to be …… curve if it does not intersect itself anywhere

a) simple   b) continuous   c) multiple   d) None of these

3) A simple curve is also called …… arc

a) Euler's b) Lagrange's c) Jordan d) None of these

4) A curve z = f(t), a ≤ t ≤ b, is said to be ……. if it does not intersect itself anywhere except initial and final point.

a) simple closed curve b) simple curve

c) multiple curve d) None of these

5) A simple closed curve is also called closed …… curve

a) Euler's b) Lagrange's c) Jordan d) None of these

6) If ϕ and ψ have continuous derivatives which does not vanish simultaneously for any value of t in [a, b], then a curve z = f(t) = ϕ(t) + iψ(t), a ≤ t ≤ b, is said to be or regular curve.

a) simple b) smooth c) multiple d) None of these

7) Smooth curve is also called …… curve.

a) regular b) simple c) multiple d) None of these

8) A continuous chain of a finite number of smooth curves is called a ……

a) arc b) circle c) contour d) None of these

9) Length of contour f(t) = ϕ(t) + iψ(t), a ≤ t ≤ b is given by L = ……

a) $\int_a^b \sqrt{[\phi(t)]^2 + [\psi(t)]^2}\, dt$ b) $\int_a^b \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt$

c) $\int_a^b \sqrt{[[\phi'(t)]^2 + [\psi(t)]^2}\, dt$ d) $\int_a^b \sqrt{[\phi(t)]^2 + [\psi'(t)]^2}\, dt$

10) A simple closed curve C divides the plane into …… regions.

a) two b) three c) four d) five

11) Any closed Jordan curve C separate the plane into two regions having C as common boundary. Is the statement of …… theorem.

a) Lagrange's b) Jordan Curve c) Cauchy's d) Euler's

12) If any simple closed curve which lies inside R can be shrunk to a point without leaving R, then a region R in the complex plane is called ……

a) simply connected b) simple

c) multiply connected d) None of these

13) If a region R is not simply connected, then it said to be ……

a) simply connected b) simple

c) multiply connected d) None of these

14) If f(z) is continuous inside a region R and curve C lies in R, then $\int_C f(z)dz$ is

called ……

a) line integral b) surface integral c) volume integral d) None of these

15) If – C is the curve traversed opposite that of C, then $\int_{-C} f(z)dz =$

a) $\int_C f(z)dz$ b) $-\int_C f(z)dz$ c) $-\int_{-C} f(z)dz$ d) None of these

16) If C = $C_1 + C_2 + C_3 + \ldots\ldots + C_n$, then $\int_C f(z)dz = \ldots\ldots$

a) $\sum_{k=1}^{n}\int_{C_k} f(z)dz$ b) $\int_{C_k} f(z)dz$ c) $\int_{C_1} f(z)dz$ d) None of these

17) Parametric equation of the line segment joining the points z = 0 to z = 1 + i is ……

a) x = t, y = t b) x = 0, y = t c) x = t, y = 0 d) None of these

18) Parametric equation of the line segment joining the points z = 0 to z = i is ……

a) x = t, y = t b) x = 0, y = t c) x = t, y = 0 d) None of these

19) Parametric equation of the line segment joining the points z = i to z = 1 + i is ……

a) x = t, y = t b) x = 0, y = t c) x = t, y = 1 d) None of these

20) Parametric equation of the parabola $y^2 = 4ax$ is ……

a) $x = at^2, y = at$, b) $x = at^2, y = 2at$ c) $x = 2at, y = at^2$, d) None of these

21) Parametric equation of the parabola $x^2 = 4ay$ is ……

a) $x = at^2, y = at$, b) $x = at^2, y = 2at$ c) $x = 2at, y = at^2$ d) None of these

22) If n is any integer except $n \neq -1$ and C is the circle $|z - a| = r$, then $\int_C (z-a)^n dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) $4\pi i$ d) πi

23) If C is the circle $|z - a| = r$, then $\int_C \frac{1}{z-a} dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) $4\pi i$ d) πi

24) If C is the semicircular arc from -1 to 1lies above the real axis, then $\int_C \frac{1}{z-a} dz = \ldots$

a) 0 b) $2\pi i$ c) $4\pi i$ d) $-\pi i$

25) If C is the semicircular arc from -1 to 1lies below the real axis, then $\int_C \frac{1}{z-a} dz = \ldots$

a) 0 b) $2\pi i$ c) πi d) $-\pi i$

26) By Cauchy's Integral theorem, if f(z) is analytic on and inside a simple closed contour C, then $\int_C f(z)dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) $-2\pi i$ d) πi

27) Cauchy's Integral theorem or Cauchy's theorem is also called …….

a) Cauchy's Integral formula b) Cauchy's Goursat theorem

c) Cauchy's Residue theorem d) None of these

28) If C is the circle |z| = 1, then $\int_C e^z dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) πi d) $-\pi i$

29) If C is the circle |z| = 1, then $\int_C z dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) πi d) $-\pi i$

30) If C is the circle |z| = 1, then $\int_C z^2 dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) πi d) $-\pi i$

31) If C is the circle |z| = 1, then $\int_C (z+1) dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) πi d) $-\pi i$

32) If C is the rectangle bounded by the lines: x = 0, x = 1, y = 0, y = 1, then $\int_C z dz = \ldots\ldots$

a) 0 b) $2\pi i$ c) πi d) $-\pi i$

33) If f(z) is analytic in a region bounded by two simple closed curves C_1 and C_2 and also on C_1 and C_2, then $\int_{C_1} f(z)dz = \ldots\ldots$

a) 0 b) $\int_{C_2} f(z)dz$ c) $-\int_{C_2} f(z)dz$ d) None of these

34) Let $C_1, C_2, C_3, \ldots\ldots, C_n$ be a system of closed Jordon contours traversed in the positive (anticlockwise) sense such that $C_1, C_2, C_3, \ldots\ldots, C_n$ are all lie inside C and outside to each other. Let R be a region obtained by excluding from the interiors of C, each of the curves C_k together with their interiors. If f(z) is analytic in R and on each of the contours $C, C_1, C_2, \ldots\ldots, C_n$, then $\int_C f(z)dz = \ldots\ldots$

a) $\int_{C_1} f(z)dz$ b) $\int_{C_2} f(z)dz$ c) $\int_{C_k} f(z)dz$ d) $\sum_{k=1}^{n} \int_{C_k} f(z)dz$

35) By Cauchy's Integral Formula for f(a), if f(z) is analytic on and inside a simple closed contour C and a is any point inside C, then f(a) = ……

a) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$ b) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$ c) $\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$ d) None of these

36) If f(z) is analytic on and inside a simple closed contour C and a is any point inside C, then $\int_C \frac{f(z)}{(z-a)}dz = \ldots\ldots$

a) f(a) b) $2\pi i f'(a)$ c) $2\pi i f(a)$ d) None of these

37) By Cauchy's Integral Formula for f '(a), if f(z) is analytic on and inside a simple closed contour C and a is any point inside C, then f '(a) = ……

a) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$ b) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$ c) $\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$ d) None of these

38) If f(z) is analytic on and inside a simple closed contour C and a is any point inside

C, then $\int_C \frac{f(z)}{(z-a)^2} dz = \ldots\ldots$

a) f(a) b) $2\pi i f'(a)$ c) $2\pi i f(a)$ d) None of these

39) By Cauchy's Integral Formula for $f^{(n)}(a)$, if f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then $f^{(n)}(a) = \ldots\ldots, n \in N$

a) $\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}} dz$ b) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n}} dz$ c) $\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n-1}} dz$ d) None of these

40) By Cauchy's Integral Formula, if f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then $f^{(5)}(a) = \ldots\ldots, n \in N$

a) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^6} dz$ b) $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^5} dz$ c) $\frac{60}{\pi i}\int_C \frac{f(z)}{(z-a)^6} dz$ d) None of these

41) If f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \ldots\ldots, n \in N$

a) $\frac{2\pi i}{n!} f^{(n)}(a)$ b) $\frac{n!}{2\pi i} f^{(n)}(a)$ c) $2\pi i f^{(n)}(a)$ d) None of these

42) If f(z) is analytic inside and on a simple closed contour C and a is any point inside C, then $\int_C \frac{f(z)}{(z-a)^{n}} dz = \ldots\ldots, n \in N$

a) $\frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$ b) $\frac{n!}{2\pi i} f^{(n)}(a)$ c) $2\pi i f^{(n)}(a)$ d) None of these

43) If C is the circle $|z-2| = 1$, then by Cauchy integral formula $\int_C \frac{e^z}{z-2} dz = \ldots\ldots$

a) 0 b) $2\pi i e^2$ c) $2\pi i$ d) None of these

44) If C is the circle $|z| = 1$, then by Cauchy integral formula $\int_C \frac{z+2}{z} dz = \ldots\ldots$

a) 0 b) $4\pi i$ c) $2\pi i$ d) None of these

45) By Cauchy integral formula $\int_{|z|=1} \frac{e^z}{z} dz = \ldots\ldots$

a) 0 b) $2\pi i e^2$ c) $2\pi i$ d) None of these

46) An infinite sequence $\{z_n\}$ in which each term z_n is a complex number is called

a) complex sequence b) complex series

c) absolute sequence d) None of these

47) If a sequence $\{z_n\}$ has a limit as $n \to \infty$, then is called ……sequence.

a) convergent b) divergent

c) may be convergent or divergent d) None of these

48) If a sequence $\{z_n\}$ has no limit as $n \to \infty$, then is called ……sequence.

a) convergent b) divergent

c) may be convergent or divergent d) None of these

49) For $|z| < 1$, the geometric sequence $z, z^2, \ldots\ldots, z^n, \ldots\ldots$ is convergent to ……

a) 1 b) -1 c) 0 d) None of these

50) A series $\sum_{n=1}^{\infty} z_n$ in which each term z_n is a complex number is called ……

a) complex sequence　　b) complex series

c) absolute sequence　　d) None of these

51) For a series $\sum_{n=1}^{\infty} z_n$, $S_n = \sum_{k=1}^{n} z_k$ is called ……

a) partial sum　b) finite series　c) finite sequence　d) None of these

52) If sequence of partial sums $\{S_n\}$ is convergent then series is ……

a) convergent　　b) divergent

c) may be convergent or divergent　　d) None of these

53) If sequence of partial sums $\{S_n\}$ is divergent then series is ……

a) convergent　　b) divergent

c) may be convergent or divergent　　d) None of these

54) A series of the form $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ where $a_0, a_1, a_2, \ldots, a_n, \ldots$ and z_0 are complex numbers is called a……

a) power series　b) Taylors series　c) Maclaurin's series　d) None of these

55) Statement "Every absolutely convergent series is convergent" is ……

a) true　b) false　c) may be true or false　d) None of these

56) Geometric series $\sum_{n=0}^{\infty} z^n$ or $\sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \ldots + z^n + \ldots$ is convergent if ……

a) $|z| > 1$　b) $|z| = 1$　c) $|z| < 1$　d) None of these

57) Geometric series $\sum_{n=0}^{\infty} z^n$ or $\sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \ldots + z^n + \ldots$ is divergent if ……

a) $|z| \geq 1$　b) $|z| = 0$　c) $|z| < 1$　d) None of these

58) If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = z_1 (\neq 0)$, then it is absolutely convergent for any value of z such that ……

a) $|z| = |z_1|$　b) $|z| < |z_1|$　c) $|z| > |z_1|$　d) None of these

59) If a power series $\sum_{n=0}^{\infty} a_n z^n$ diverges for $z = z_1 (\neq 0)$, then it is absolutely divergent for any value of z such that ……

a) $|z| = |z_1|$　b) $|z| < |z_1|$　c) $|z| > |z_1|$　d) None of these

60) If f(z) is analytic in a region R and z_0 lies in R, then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^n(z_0)}{n!}(z - z_0)^n + \ldots$$

i.e. $f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n$ is called ……

a) Taylor's series　　b) Maclaurin's series

c) Laurent's series　　d) None of these

61) If f(z) is analytic in a region R and z_0 lies in R, then

$f(z) = f(0) + f'(0) z + \frac{f''(0)}{2!} z^2 + \ldots + \frac{f^n(0)}{n!} z^n + \ldots = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$ is called ……

a) Taylor's series b) Maclaurin's series

c) Laurent's series d) None of these

62) If f(z) is analytic on two concentric circles C_1 and C_2 with centre at z = a and also the ring shaped region bounded by these two circles, then for any point z in this region $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is called ……

Where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$, for n = 0, 1, 2, …… and

$b_n = \frac{1}{2\pi i} \int_{C_2} (z-a)^{n-1} f(z) dz$ for n = 1, 2, 3, ……

a) Taylor's series b) Maclaurin's series

c) Laurent's series d) None of these

63) For $|z| < 1$, $1 - z + z^2 - z^3 + \ldots + (-1)^n z^n + \ldots = \sum_{n=0}^{\infty} (-1)^n z^n$ is an expansion of ……

a) $\frac{1}{1+z}$ b) $\frac{1}{1-z}$ c) e^z d) None of these

64) For $|z| < 1$, $1 + z + z^2 + z^3 + \ldots + z^n + \ldots = \sum_{n=0}^{\infty} z^n$ is an expansion of …..

a) $\frac{1}{1+z}$ b) $\frac{1}{1-z}$ c) e^z d) None of these

65) $1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$ is an expansion of …..

a) $\frac{1}{1+z}$ b) $\frac{1}{1-z}$ c) e^z d) None of these

66) $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + \frac{(-1)^n z^{2n+1}}{(2n+1)!} + \ldots$ is an expansion of …..

a) cosz b) sinz c) coshz d) sinhz

67) $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + \frac{(-1)^n z^{2n}}{(2n)!} + \ldots$ is an expansion of …..

a) cosz b) sinz c) coshz d) sinhz

68) $z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + \frac{z^{2n+1}}{(2n+1)!} + \ldots$ is an expansion of …..

a) cosz b) sinz c) coshz d) sinhz

69) $1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + \frac{z^{2n}}{(2n)!} + \ldots$ is an expansion of …..

a) cosz b) sinz c) coshz d) sinhz

70) For $|z| < 2$, Taylor's series expansion of $\frac{1}{(z-2)}$ is ……

a) $-\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ b) $-\sum_{n=0}^{\infty} \frac{z^n}{2^n}$ c) $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ d) None of these

71) For $0 < |z| < 4$, $\frac{1}{4z - z^2} = \ldots\ldots$

a) $\sum_{n=0}^{\infty} \frac{z^{n+1}}{4^{n+1}}$ b) $\sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$ c) $\sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n-1}}$ d) None of these

72) For $|z| > 2$, Laurent's series expansion of $\frac{1}{(z-2)}$ is ……

a) $\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$ b) $-\sum_{n=0}^{\infty} \frac{2^n}{z^n}$ c) $-\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$ d) None of these

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UNIT-4: CALCULUS OF RESIDUES

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Zero: A value of z which satisfies f(z) = 0 is called zero of an analytic function f(z).

Remark: i) If $f(z) = (z-a)^m\Phi(z)$ with $\Phi(a) \neq 0$, then z = a is called zero of order m.

ii) A zero of order one is called simple zero.

iii) A zero of order two is called double zero.

e.g. i) $f(z) = (z-2)^3(z+1)^2(z^2+1)^4$ has zero of order 3 at z = 2, double zero at z = -1 and zeros of orders 4 at $z = \pm i$.

ii) $f(z) = \frac{z^2+4}{z^3+2z^2+z}$ has simple zeros at points at z = 2i and z = -2i.

iii) $f(z) = \left(\frac{z+1}{z^2+1}\right)^2$ has double zero at z = -1.

Singular Point: A point z = a is called singular point or singularity of a function f(z) if f(z) is not analytic z = a.

e.g. i) $f(z) = \frac{1}{z}$ has singular point at z = 0.

ii) $f(z) = \frac{1}{z(z-i)}$ has singular points at z = 0 and z = i.

iii) $f(z) = \frac{z+1}{z^2(z^2+1)}$ has singular points at z = 0 and $z = \pm i$

Isolated Singularity: Singularity z = a is called isolated singularity of a function f(z) if f(z) is analytic in a deleted neighborhood of z = a but not analytic at z = a.

Removable Singularity: Singularity z = a is called removable singularity of a function f(z) if principal part of f(z) contain no terms.

Note: A function f(z) has removable singularity at z = a if $\lim_{z\to a} f(z)$ is exists

e.g. $f(z) = \frac{\sin z}{z}$ has removable singularity at z = 0 $\because \lim_{z\to 0}\frac{\sin z}{z} = 1$ is exists.

Essential Singularity: Singularity z = a is called essential singularity of a function f(z) if principal part of f(z) contains an infinite number of terms.

e.g. $f(z) = e^{\frac{1}{z}}$ has essential singularity at z = 0 $\because$ $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \ldots\ldots$ contain infinite number of terms of negative powers of z.

Pole: A singular point of a function f(z) is called a pole of function f(z).

Remark: i) If $f(z) = \frac{\Phi(z)}{(z-a)^m}$ and $\Phi(z)$ is analytic at z = a, then singular point z = a is called pole of order m.

ii) A pole of order one is called simple pole.

iii) A pole of order two is called double pole.

iv) A pole of order three is called triple pole.

Residue: The coefficient b_1 of $\frac{1}{z-a}$ in the Laurent's series expansion of f(z) is called residue of a function f(z). Denoted by $\underset{z=a}{\text{Res}}\, f(z) = b_1$

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Remark: i) If z = a is a simple pole, then $\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a}[(z-a)f(z)]$

ii) If z = a is a double pole, then $\underset{z=a}{\text{Res}}\, f(z) = \lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$

iii) If z = a is a pole of order m, then

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$$

==

Ex. Find the sum of residue of $f(z) = \frac{e^z}{z^2+a^2}$ at its poles.

Solution: Given function $f(z) = \frac{e^z}{z^2+a^2} = \frac{e^z}{(z-ai)(z+ai)}$ has simple poles at z = ai and z = -ai.

$\therefore \underset{z=ai}{\text{Res}}\, f(z) = \lim_{z\to ai}(z-ai)\, f(z)$

$= \lim_{z\to ai}[\frac{e^z}{(z+ai)}]$

$= \frac{e^{ai}}{2ai}$

Similarly $\underset{z=-ai}{\text{Res}}\, f(z) = \frac{e^{-ai}}{-2ai}$

$\therefore$ The sum of residues $= \underset{z=ai}{\text{Res}}\, f(z) + \underset{z=-ai}{\text{Res}}\, f(z)$

$= \frac{e^{ai}}{2ai} - \frac{e^{-ai}}{2ai}$

$= \frac{1}{a}(\frac{e^{ai}-e^{-ai}}{2i})$

$= \frac{\sin a}{a}$

==

Ex. Find poles and residues at poles of $f(z) = \frac{1}{z(z-1)^2}$. Also find the sum of these residues.

Solution: Given function $f(z) = \frac{1}{z(z-1)^2}$ has simple pole at z = 0 and double pole at z = 1.

Now $\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0}[(z-0)f(z)]$

$= \lim_{z\to 0}[\frac{1}{(z-1)^2}]$

$= \frac{1}{(-1)^2}$

$= 1$

& $\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$= \lim_{z\to 1}\frac{d}{dz}[\frac{1}{z}]$

$= \lim_{z\to 1}[\frac{-1}{z^2}]$

$= \frac{-1}{(1)^2}$

$= -1$

$\therefore$ Sum of residues $= \underset{z=0}{\text{Res}}\, f(z) + \underset{z=1}{\text{Res}}\, f(z) = 1 + (-1) = 0$

Ex. Find the residues of $(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$ at its poles.

Solution: Given function $f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$ has simple poles at z = 1, 2 and 3.

∴ Residues of f(z) at these poles are as follows:

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} [(z-1)f(z)]$$
$$= \lim_{z\to 1} \left[\frac{z^2}{(z-2)(z-3)}\right]$$
$$= \frac{(1)^2}{(-1)(-2)}$$
$$= \frac{1}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} [(z-2)f(z)]$$
$$= \lim_{z\to 2} \left[\frac{z^2}{(z-1)(z-3)}\right]$$
$$= \frac{(2)^2}{(1)(-1)}$$
$$= -4$$

$$\& \operatorname*{Res}_{z=3} f(z) = \lim_{z\to 3} [(z-3)f(z)]$$
$$= \lim_{z\to 3} \left[\frac{z^2}{(z-1)(z-2)}\right]$$
$$= \frac{(3)^2}{(2)(1)}$$
$$= \frac{9}{2}$$

==

Ex. Find the residues of $f(z) = \frac{1}{(z^2+1)^3}$ at z = i.

Solution: Given function $f(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$ has poles of order 3 at z = i and -i.

∴ Residue of f(z) at the pole z = i is

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{(3-1)!} \lim_{z\to i} \frac{d^2}{dz^2}[(z-i)^3 f(z)]$$
$$= \frac{1}{2} \lim_{z\to i} \frac{d}{dz}\left\{\frac{d}{dz}\left[\frac{1}{(z+i)^3}\right]\right\}$$
$$= \frac{1}{2} \lim_{z\to i} \frac{d}{dz}\left[\frac{-3}{(z+i)^4}\right]$$
$$= \frac{1}{2} \lim_{z\to i} \left[\frac{12}{(z+i)^5}\right]$$
$$= \frac{6}{(2i)^5}$$
$$= \frac{6}{32i}$$
$$= \frac{-3i}{16}$$

Ex. Compute residues at double pole of $f(z) = \frac{z^2+2z+3}{(z-i)^2(z+4)}$.

Solution: Given function $f(z) = \frac{z^2+2z+3}{(z-i)^2(z+4)}$ has double pole at z = i and simple pole at z = -4.

∴ Residues of f(z) at the double pole is

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}[(z-i)^2 f(z)]$$

$$= \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}[\frac{z^2+2z+3}{(z+4)}]$$

$$= \lim_{z\to i}[\frac{(z+4)(2z+2)-(z^2+2z+3)(1)}{(z+4)^2}]$$

$$= \lim_{z\to i}[\frac{z^2+8z+5}{(z+4)^2}]$$

$$= \frac{-1+8i+5}{(i+4)^2}$$

$$= \frac{4+8i}{15+8i}\times\frac{15-8i}{15-8i}$$

$$= \frac{60-32i+120i+64}{225+64}$$

$$= \frac{124+88i}{289}$$

==

Ex. Find the residue of $f(z) = \frac{ze^z}{(z-1)^3}$ at its pole.

Solution: Given function $f(z) = \frac{ze^z}{(z-1)^3}$ has pole of order 3 at z = 1

∴ Residue of f(z) at this pole is

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(3-1)!}\lim_{z\to 1}\frac{d^2}{dz^2}[(z-1)^3 f(z)]$$

$$= \frac{1}{2!}\lim_{z\to 1}\frac{d}{dz}\{\frac{d}{dz}[ze^z]\}$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}(ze^z+e^z)$$

$$= \frac{1}{2}\lim_{z\to 1}(ze^z+e^z+e^z)$$

$$= \frac{3e}{2}$$

==

Ex. Find the residue of $f(z) = \frac{z^2+2z}{(z+1)^2(z+4)}$ at its poles.

Solution: Given function $f(z) = \frac{z^2+2z}{(z+1)^2(z+4)}$ has double pole at z = -1 and simple pole at z = -4.

$$\therefore \operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1}\frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1}\frac{d}{dz}[\frac{z^2+2z}{(z+4)}]$$

$$= \lim_{z \to -1} \left[\frac{(z+4)(2z+2)-(z^2+2z)(1)}{(z+4)^2}\right]$$

$$= \lim_{z \to -1} \left[\frac{z^2+8z+8}{(z+4)^2}\right]$$

$$= \frac{1-8+8}{(3)^2}$$

$$= \frac{1}{9}$$

$$\& \operatorname*{Res}_{z=-4} f(z) = \lim_{z \to -4} [(z+4)f(z)]$$

$$= \lim_{z \to -4} \left[\frac{z^2+2z}{(z+1)^2}\right]$$

$$= \frac{16-8}{(-3)^2}$$

$$= \frac{8}{9}$$

===

Cauchy's Residue theorem: If f(z) is analytic on and inside a closed contour C except at a finite number of singular points, say n, then $\int_C f(z)dz = 2\pi i \sum R$, where $\sum R$ denote the sum of the residues at its poles inside C.

Proof: Let $a_1, a_2, a_3, \ldots\ldots, a_n$ be the n singular points (poles) of f(z) inside C.

Let $C_1, C_2, C_3, \ldots\ldots, C_n$ be the circles with centres at $a_1, a_2, a_3, \ldots\ldots, a_n$ respectively such that they lie completely inside C and outside each other.

Then by Cauchy's theorem for a system of contours, we have,

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz \ \ldots\ldots (1)$$

Now, $\operatorname*{Res}_{z=a_k} f(z) = b_1 =$ Coefficient of $\frac{1}{z-a_k} = \frac{1}{2\pi i}\int_{C_k} f(z)dz$

$$\therefore \int_{C_k} f(z)dz = 2\pi i \operatorname*{Res}_{z=a_k} f(z)$$

Putting in equation (1), we get,

$$\int_C f(z)dz = \sum_{k=1}^{n} 2\pi i \operatorname*{Res}_{z=a_k} f(z)$$

i.e. $\int_C f(z)dz = 2\pi i \sum R$,

where $\sum R$ denote the sum of the residues at its poles inside C.

===

Ex. Evaluate by Cauchy's residue theorem: $\int_C \frac{5z-2}{z(z-1)}dz$, where C is the circle $|z| = 2$

Solution: Given integrant $f(z) = \frac{5z-2}{z(z-1)}$ has simple poles at z = 0 and z = 1.

Both these poles lies inside circle C: $|z| = 2$ and f(z) is analytic on and inside C except these poles.

∴ By Cauchy's Residue Theorem,

$$\int_C f(z)dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right] \ldots\ldots (1)$$

Now $\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}[(z-0)f(z)]$

$= \lim_{z\to 0}[\frac{5z-2}{(z-1)}]$

$= \frac{-2}{-1}$

$= 2$

& $\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}[(z-1)f(z)]$

$= \lim_{z\to 1}[\frac{5z-2}{z}]$

$= \frac{5-2}{1}$

$= 3$

Putting in (1), we get,

$\int_C f(z)dz = 2\pi i\,[2+3]$

$\therefore \int_C \frac{5z-2}{z(z-1)}dz = 10\pi i$

==

Ex. Evaluate $\int_{|z|=3} \frac{e^z}{z(z-1)^2} dz$ by Cauchy's residue

Solution: Given integrant $f(z) = \frac{e^z}{z(z-1)^2}$ has simple pole at $z = 0$ and double pole at $z = 1$. Both these poles lies inside circle C: $|z| = 3$ and f(z) is analytic on and inside C except these poles.

∴ By Cauchy's Residue Theorem,

$\int_C f(z)dz = 2\pi i\,[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)]$ …… (1)

Now $\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}[(z-0)f(z)]$

$= \lim_{z\to 0}[\frac{e^z}{(z-1)^2}]$

$= \frac{1}{(-1)^2}$

$= 1$

& $\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}\frac{d}{dz}[(z-1)^2 f(z)]$

$= \lim_{z\to 1}\frac{d}{dz}[\frac{e^z}{z}]$

$= \lim_{z\to 1}[\frac{ze^z - e^z(1)}{z^2}]$

$= \frac{e-e}{(1)^2}$

$= 0$

Putting in (1), we get,

$\int_C f(z)dz = 2\pi i\,[1+0]$

$\therefore \int_{|z|=3} \frac{e^z}{z(z-1)^2}dz = 2\pi i$

==

Ex. Evaluate $\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz$ by Cauchy's residue theorem, where C is
(i) The circle $|z-2| = 2$ (ii) The circle $|z| = 4$

Solution: Given integrant $f(z) = \frac{3z^2+2}{(z-1)(z^2+9)} = \frac{3z^2+2}{(z-1)(z-3i)(z+3i)}$ has simple poles at z = 1, z = 3i and z = -3i.

Now $\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} [(z-1)f(z)]$

$$= \lim_{z\to 1} \left[\frac{3z^2+2}{z^2+9}\right] = \frac{5}{10} = \frac{1}{2}$$

& $\operatorname{Res}_{z=3i} f(z) = \lim_{z\to 3i} [(z-3i)f(z)]$

$$= \lim_{z\to 3i} \left[\frac{3z^2+2}{(z-1)(z+3i)}\right]$$
$$= \frac{-27+2}{(3i-1)(6i)}$$
$$= \frac{-25}{6(-3-i)}$$
$$= \frac{25}{6(3+i)} \times \frac{(3-i)}{(3-i)}$$
$$= \frac{25(3-i)}{6(9+1)}$$
$$= \frac{5}{12}(3-i)$$
$$= \frac{5}{4} - \frac{5}{12} i$$

Similarly, $\operatorname{Res}_{z=-3i} f(z) = \frac{5}{4} + \frac{5}{12} i$

i) Let C is the circle $|z-2| = 2$, then only the pole z = 1 lies inside circle C and f(z) is analytic on and inside C except this pole.

∴ By Cauchy's Residue Theorem,

$$\int_C f(z)dz = 2\pi i \left[\operatorname{Res}_{z=1} f(z)\right]$$

$$\int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz = 2\pi i \left[\frac{1}{2}\right] = \pi i$$

ii) Let C is the circle $|z| = 4$, then all the poles z = 1, z = 3i and z = -3i lies inside circle C and f(z) is analytic on and inside C except these poles.

∴ By Cauchy's Residue Theorem,

$$\int_C f(z)dz = 2\pi i \left[\operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=3i} f(z) + \operatorname{Res}_{z=-3i} f(z)\right]$$
$$= 2\pi i \left[\frac{1}{2} + \frac{5}{4} - \frac{5}{12} i + \frac{5}{4} + \frac{5}{12} i\right]$$
$$= 2\pi i(3)$$

$$\therefore \int_C \frac{3z^2+2}{(z-1)(z^2+9)} dz = 6\pi i$$

===

7) Evaluate $\int_{|z|=2} \frac{dz}{z^3(z+4)}$ by Cauchy's residue theorem.

Solution: Given function $f(z) = \frac{1}{z^3(z+4)}$ has pole of order 3 at z = 0 and simple pole at z = -4. Out of these only the pole z = 0 lies inside the circle C: $|z| = 2$ and f(z) is analytic on and inside C except this pole.

∴ By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i[\operatorname*{Res}_{z=0} f(z)]$$

$$\therefore \int_{|z|=2} \frac{dz}{z^3(z+4)} = 2\pi i\{\frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}[(z-0)^3 f(z)]\}$$

$$= \pi i \lim_{z\to 0}\frac{d}{dz}\{\frac{d}{dz}[\frac{1}{(z+4)}]\}$$

$$= \pi i \lim_{z\to 0}\frac{d}{dz}[\frac{-1}{(z+4)^2}]$$

$$= \pi i \lim_{z\to 0}[\frac{2}{(z+4)^3}]$$

$$= \pi i [\frac{2}{(4)^3}]$$

$$= \frac{\pi i}{32}$$

==

Contour integrations of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$:

Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

In this case we substitute

$z = e^{i\theta}$, then we have $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}(z+\frac{1}{z})$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}(z-\frac{1}{z})$

and $dz = ie^{i\theta}d\theta = izd\theta$ i.e. $d\theta = \frac{dz}{iz}$ where $0 \le \theta \le 2\pi$.

By Cauchy's residue theorem, we have

$$\therefore I = \int_C f(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z}))\frac{dz}{iz}$$

$= \int_C f(z)\,dz$, where C is the unit circle: $|z|=1$.

= 2πi[sum of the residues of f(z) at the poles which lies inside C: $|z|=1$]

==

Ex. Use the contour integration to evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

Put $z = e^{i\theta}$ ∴ $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$, where $0 \le \theta \le 2\pi$

$\therefore I = \int_C \frac{1}{5+\frac{3}{2}(z+\frac{1}{z})}\frac{dz}{iz}$ where C is the unit circle $|z| = 1$

$$= \int_C \frac{-2i}{5+\frac{3}{2}(z+\frac{1}{z})}\frac{dz}{2z}$$

$= \int_C \frac{-2i}{10z+3z^2+3}\, dz$

$\therefore I = \int_C f(z)dz$

where $f(z) = \frac{-2i}{3z^2+10z+3} = \frac{-2i}{(3z+1)(z+3)}$ has simple poles at $z = \frac{-1}{3}$ and z = -3.

Out of these only the pole $z = \frac{-1}{3}$ lies inside the unit circle C: $|z| = 1$ and f(z) is analytic on and inside C except this pole.

$\therefore$ By Cauchy's residue theorem,

$\int_C f(z)dz = 2\pi i[\operatorname*{Res}_{z=\frac{-1}{3}} f(z)]$

$\therefore I = 2\pi i \lim_{z\to\frac{-1}{3}}[(z+\frac{1}{3})f(z)]$

$= \frac{2}{3}\pi i \lim_{z\to\frac{-1}{3}}[(3z+1)f(z)]$

$= \frac{2}{3}\pi i \lim_{z\to\frac{-1}{3}}[\frac{-2i}{(z+3)}]$

$= \frac{2}{3}\pi i[\frac{-2i}{(\frac{-1}{3}+3)}]$

$= \frac{4\pi}{(-1+9)}$

$\therefore \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$

==

Ex. Use the contour integration to evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Put $z = e^{i\theta}$ $\therefore d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, where $0 \leq \theta \leq 2\pi$

$\therefore I = \int_C \frac{1}{5+\frac{4}{2i}(z-\frac{1}{z})} \frac{dz}{iz}$ where C is the unit circle $|z| = 1$

$= \int_C \frac{1}{5iz+2z(z-\frac{1}{z})}\, dz$

$= \int_C \frac{1}{5iz+2z^2-2}\, dz$

$\therefore I = \int_C f(z)dz$

where $f(z) = \frac{1}{2z^2+5iz-2} = \frac{1}{(2z+i)(z+2i)}$ has simple poles at $z = \frac{-i}{2}$ and z = -2i.

Out of these only the pole $z = \frac{-i}{2}$ lies inside the unit circle C: $|z| = 1$ and f(z) is analytic on and inside C except this pole.

$\therefore$ By Cauchy's residue theorem,

$\int_C f(z)dz = 2\pi i[\operatorname*{Res}_{z=\frac{-i}{2}} f(z)]$

$$\therefore I = 2\pi i \lim_{z\to\frac{-i}{2}} [(z+\frac{i}{2})f(z)]$$

$$= \pi i \lim_{z\to\frac{-i}{2}} [(2z+i)f(z)]$$

$$= \pi i \lim_{z\to\frac{-i}{2}} [\frac{1}{(z+2i)}]$$

$$= \pi i [\frac{1}{(\frac{-i}{2}+2i)}]$$

$$= \frac{2\pi}{(-1+4)}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$$

==

Ex. Use the contour integration to evaluate $\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$.

Solution: Let $I = \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$

Put $z = e^{i\theta}$ $\therefore d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$, where $0 \le \theta \le 2\pi$

$\therefore I = \int_C \frac{1}{3+(z+\frac{1}{z})} \frac{dz}{iz}$ where C is the unit circle $|z| = 1$

$$= \int_C \frac{-i}{3+(z+\frac{1}{z})} \frac{dz}{z}$$

$$= \int_C \frac{-i}{3z+z^2+1} dz$$

$\therefore I = \int_C f(z)dz$

where $f(z) = \frac{-i}{z^2+3z+1} = \frac{-i}{(z+\frac{3}{2})^2-\frac{5}{4}} = \frac{-i}{\left(z+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(z+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}$

has simple poles at $z = -\frac{3}{2}+\frac{\sqrt{5}}{2}$ and $z = -\frac{3}{2}-\frac{\sqrt{5}}{2}$.

Out of these only the pole $z = -\frac{3}{2}+\frac{\sqrt{5}}{2}$ lies inside the unit circle C: $|z| = 1$ and f(z) is analytic on and inside C except this pole.

$\therefore$ By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \left[\operatorname*{Res}_{z=-\frac{3}{2}+\frac{\sqrt{5}}{2}} f(z)\right]$$

$$\therefore I = 2\pi i \lim_{z\to-\frac{3}{2}+\frac{\sqrt{5}}{2}} [(z+\frac{3}{2}-\frac{\sqrt{5}}{2})f(z)]$$

$$= 2\pi i \lim_{z\to-\frac{3}{2}+\frac{\sqrt{5}}{2}} [\frac{-i}{\left(z+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}]$$

$$= 2\pi i [\frac{-i}{\left(-\frac{3}{2}+\frac{\sqrt{5}}{2}+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}]$$

$$= \frac{2\pi}{\sqrt{5}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{2\pi}{\sqrt{5}}$$

Contour integrations of the type $\int_{-\infty}^{\infty} f(x)\, dx$:

If $f(x) = \frac{P(x)}{Q(x)}$ is a rational polynomial, with

i) P(x) and Q(x) are polynomials in x,

ii) degree of Q(x) - degree of P(x) ≥ 2,

iii) Q(x) = 0 has no real roots, then

$\therefore \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i$ [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

==

Ex. Show that $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi$

Proof: Let $I = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$

Then, here P(x) = 1 and $Q(x) = x^2 + 1$ and $f(x) = \frac{P(x)}{Q(x)}$.

i) P(x) and Q(x) are polynomials in x.

ii) degree of Q(x) - degree of P(x) = 2 – 0 = 2 ≥ 2

iii) Q(x) = 0 gives $x^2 + 1 = 0$ i.e. $(x - i)(x + i) = 0$

$\therefore \pm i$ are the roots of Q(x) = 0 i.e. Q(x) = 0 has no real roots.

$\therefore I = 2\pi i$ [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

$\therefore I = 2\pi i[\operatorname{Res}_{z=i} f(z)]$ …… (1)

Now $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$

$\therefore \operatorname{Res}_{z=i} f(z) = \lim_{z\to i} [(z - i)f(z)]$

$= \lim_{z\to i}[\frac{1}{(z+i)}]$

$= \frac{1}{2i}$

Putting in (1), we get,

$I = 2\pi i\, [\frac{1}{2i}]$

$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi$ Hence proved.

==

Ex. Evaluate by contour integration $\int_{-\infty}^{\infty} \frac{1}{x^4+13x^2+36} dx$.

Solution: Let $I = \int_{-\infty}^{\infty} \frac{1}{x^4+13x^2+36} dx$

Then, here P(x) = 1 and $Q(x) = x^4 + 13x^2 + 36$ and $f(x) = \frac{P(x)}{Q(x)}$.

i) P(x) and Q(x) are polynomials in x.

ii) degree of Q(x) - degree of P(x) = 4 – 0 = 4 ≥ 2

iii) Q(x) = 0 gives $x^4 + 13x^2 + 36 = 0$ i.e. $(x^2 + 4)(x^2 + 9) = 0$

∴ $\pm 2i$ and $\pm 3i$ are the roots of Q(x) = 0 i.e. Q(x) = 0 has no real roots.

∴ I = 2πi [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

∴ $I = 2\pi i[\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=3i} f(z)]$ …… (1)

Now $f(z) = \frac{1}{z^4+13z^2+36} = \frac{1}{(z^2+4)(z^2+9)} = \frac{1}{(z-2i)(z+2i)(z-3i)(z+3i)}$

$$\therefore \operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} [(z-2i)f(z)]$$

$$= \lim_{z\to 2i} [\frac{1}{(z+2i)(z^2+9)}]$$

$$= \frac{1}{4i(-4+9)}$$

$$= \frac{1}{20i}$$

$$\& \operatorname*{Res}_{z=3i} f(z) = \lim_{z\to 3i} [(z-3i)f(z)]$$

$$= \lim_{z\to 3i} [\frac{1}{(z+3i)(z^2+4)}]$$

$$= \frac{1}{6i(-9+4)}$$

$$= \frac{-1}{30i}$$

Putting in (1), we get,

$I = 2\pi i [\frac{1}{20i} - \frac{1}{30i}] = \pi [\frac{1}{10} - \frac{1}{15}]$

$\therefore \int_{-\infty}^{\infty} \frac{1}{x^4+13x^2+36} dx = \frac{\pi}{30}$

==

Ex. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ by contour integration.

Solution: Let $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ …… (1)

Then, here P(x) = 1 and $Q(x) = (x^2 + a^2)(x^2 + b^2)$ and $f(x) = \frac{P(x)}{Q(x)}$.

i) P(x) and Q(x) are polynomials in x.

ii) degree of Q(x) - degree of P(x) = 4 – 0 = 4 ≥ 2

iii) Q(x) = 0 gives $(x^2 + a^2)(x^2 + b^2) = 0$

∴ $\pm ai$ and $\pm bi$ are the roots of Q(x) = 0 i.e. Q(x) = 0 has no real roots.

∴ I = 2πi [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

∴ $I = 2\pi i [\operatorname*{Res}_{z=ai} f(z) + \operatorname*{Res}_{z=bi} f(z)]$ …… (2)

Now $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)} = \frac{1}{(z-ai)(z+ai)(z-bi)(z+bi)}$

$$\therefore \operatorname*{Res}_{z=ai} f(z) = \lim_{z\to ai} [(z-ai)f(z)]$$

$$= \lim_{z\to ai}\left[\frac{1}{(z+ai)(z^2+b^2)}\right]$$

$$= \frac{1}{2ai(-a^2+b^2)}$$

$$= \frac{-1}{2ai(a^2-b^2)}$$

Similarly, $\underset{z=bi}{\text{Res}}\ f(z) = \frac{-1}{2bi(b^2-a^2)} = \frac{1}{2bi(a^2-b^2)}$

Putting in (2), we get,

$$I = 2\pi i\left[\frac{-1}{2ai(a^2-b^2)} + \frac{1}{2bi(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[-\frac{1}{a}+\frac{1}{b}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left(\frac{a-b}{ab}\right)$$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$$

From (1), we get,

$$\int_0^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\left[\frac{\pi}{ab(a+b)}\right] = \frac{\pi}{2ab(a+b)}$$

==

Ex. Evaluate $\int_0^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)}$ by contour integration.

Solution: Let $I = \int_0^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)(x^2+4)}$ …… (1)

Then, here $P(x) = x^2$ and $Q(x) = (x^2+1)(x^2+4)$ and $f(x) = \frac{P(x)}{Q(x)}$.

i) P(x) and Q(x) are polynomials in x.

ii) degree of Q(x) - degree of P(x) = 4 – 2 = 2 ≥ 2

iii) Q(x) = 0 gives $(x^2+1)(x^2+4) = 0$

$\therefore \pm i$ and $\pm 2i$ are the roots of Q(x) = 0 i.e. Q(x) = 0 has no real roots.

$\therefore$ I = 2πi [The sum residues of f(z) at the poles which lies in the upper half of the z-plane]

$\therefore I = 2\pi i\left[\underset{z=i}{\text{Res}}\ f(z) + \underset{z=2i}{\text{Res}}\ f(z)\right]$ …… (2)

Now $f(z) = \frac{z^2}{(z^2+1)(z^2+4)} = \frac{z^2}{(z-i)(z+i)(z-2i)(z+2i)}$

$\therefore \underset{z=i}{\text{Res}}\ f(z) = \lim_{z\to i}[(z-i)f(z)]$

$$= \lim_{z\to i}\left[\frac{z^2}{(z+i)(z^2+4)}\right]$$

$$= \frac{-1}{2i(-1+4)}$$

$$= \frac{-1}{6i}$$

& $\underset{z=2i}{\text{Res}}\ f(z) = \lim_{z\to 2i}[(z-2i)f(z)]$

$= \lim\limits_{z\to 2i}\left[\frac{z^2}{(z+2i)(z^2+1)}\right]$

$= \frac{-4}{4i(-4+1)}$

$= \frac{1}{3i}$

Putting in (2), we get,

$I = 2\pi i\left[\frac{-1}{6i} + \frac{1}{3i}\right] = \pi\left[-\frac{1}{3} + \frac{2}{3}\right]$

$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{3}$

From (1), we get,

$\int_0^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$

==

MULTIPLE CHOICE QUESTIONS [MCQ'S]

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1) If f (a) = o then the point z = a is said to be………. of a function.
   a) pole b) zero c) singular point d) None of these

2) A zero of order one is called………. zero.
   a) simple b) double c) triple d) None of these

3) A zero of order two is called………. zero.
   a) simple b) double c) triple d) None of these

4) A zero of order 3 is called………. zero.
   a) simple b) double c) triple d) None of these

5) If $f(z) = (z-a)^m g(z)$ and $g(a) \neq 0$, then z = a is a zero of order…
   a) 1 b) 2 c) m d) None of these

6) $f(z) = (z-2)^3(z+1)^2(z^2+1)^4$ has zero at z = 2 of order ……
   a) 3 b) 2 c) 4 d) None of these

7) $f(z) = (z-2)^3(z+1)^2(z^2+1)^4$ has …… zero at z = -1.
   a) simple b) double c) triple d) None of these

8) $f(z) = (z-2)^3(z+1)^2(z^2+1)^4$ has zeros of orders 4 at z = ……
   a) 2 b) -1 c) $\pm i$ d) None of these

9) $f(z) = \frac{z^2+4}{z^3+2z^2+z}$ has zeros at points at z = ……
   a) $\pm 2i$ b) ± 2 c) 0 d) None of these

10) $f(z) = \left(\frac{z+1}{z^2+1}\right)^2$ has double zero at z = ……
   a) 2 b) -1 c) $\pm i$ d) None of these)

11) If f (z) is not analytic at z = a, then the point z = a is said to be………. of a function.
   a) singular point b) zero c) critical point d) None of these

==

12) f(z) = $\frac{1}{z}$ has singular point at z = ……

a) 1 b) 0 c) i d) None of these)

13) f (z) = $\frac{1}{z(z-i)}$ has singular points at z = … & z = …

a) o & i b) 1 & i c) 0 & -i d) None of these

14) f (z) = $\frac{z+1}{z^2(z^2+1)}$ has singular points at z = … & z = …

a) 1 & -1 b) 0 & $\pm$i c) 0 & -1 d) None of these

15) A singular point of a function f (z) is also called……… of a function.

a) zero b) pole c) critical point d) None of these

16) If f (z) = $\frac{g(z)}{(z-a)^m}$ and g(z) is analytic at z = a, then z = a is a pole of order…

a) 1 b) 2 c) m d) None of these

17) A pole of order one is called ………. pole.

a) simple b) double c) triple d) None of these

18) A pole of order two is called ………. pole.

a) simple b) double c) triple d) None of these

19) A pole of order 3 is called ………. pole.

a) simple b) double c) triple d) None of these

20) f (z) = $\frac{1}{(z-5)^3(z-4)^2}$ has poles at z = 5 & 4 of orders …..&…. respectively.

a) 5 & 4 b) 3 & 2 c) 2 & 3 d) None of these

21) f (z) = $\frac{1}{z(z-1)^2}$ has simple pole at z = …. & double pole at z = … respectively.

a) 0 & 1 b) 0 & -1 c) 1 & 0 d) None of these

22) Singularity z = a is called removable singularity of a function f(z), if principal part of f(z) contain …… terms.

a) a finite number of b) an infinite number of

c) no d) None of these

23) If $\lim_{z\to a} f(z)$ is exists, then function f(z) has …… singularity at z = a.

a) removable b) essential c) no d) None of these

24) Function f(z) = $\frac{\sin z}{z}$ has …… singularity at z = 0.

a) removable b) essential c) no d) None of these

25) Singularity z = a is called essential singularity of a function f(z), if principal part of f(z) contain …… terms.

a) a finite number of b) an infinite number of

c) no d) None of these

26) Function f(z) = $e^{\frac{1}{z}}$ has …… singularity at z = 0.

a) a removable b) an essential c) no d) None of these

===

27) The coefficient b_1 of $\frac{1}{z-a}$ in the Laurent's series expansion of f(z) is called …… of a function f(z) at its pole z = a.

a) zero b) pole c) residue d) None of these

28) If z = a is a simple pole of a function f(z), then $\mathrm{Res}_{z=a}$ f(z) = ……

a) $\lim_{z\to a}[(z-a)f(z)]$ b) $\lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$

c) $\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$ d) None of these

29) If z = a is a double pole of a function f(z), then $\mathrm{Res}_{z=a}$ f(z) = ……

a) $\lim_{z\to a}[(z-a)f(z)]$ b) $\lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$

c) $\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$ d) None of these

30) If z = a is a triple pole of a function f(z), then $\mathrm{Res}_{z=a}$ f(z) = ……

a) $\lim_{z\to a}[(z-a)f(z)]$ b) $\lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$

c) $\frac{1}{2}\lim_{z\to a}\frac{d^2}{dz^2}[(z-a)^3 f(z)]$ d) None of these

31) If z = a is a pole of order m of a function f(z), then $\mathrm{Res}_{z=a}$ f(z) = ……

a) $\lim_{z\to a}[(z-a)f(z)]$ b) $\lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$

c) $\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$ d) None of these

32) If z = a is pole of order 6 of a function f(z), then $\mathrm{Res}_{z=a}$ f(z) = ……

a) $\lim_{z\to a}[(z-a)f(z)]$ b) $\lim_{z\to a}\frac{d}{dz}[(z-a)^2 f(z)]$

c) $\frac{1}{5!}\lim_{z\to a}\frac{d^5}{dz^5}[(z-a)^6 f(z)]$ d) None of these

33) If f (z) = $\frac{z^2}{(z-1)(z-2)(z-3)}$, then residue of f(z) at its pole z = 1 is…..

a) 1 b) 2 c) $\frac{1}{2}$ d) $\frac{9}{2}$

34) If f (z) = $\frac{z^2}{(z-1)(z-2)(z-3)}$, then residue of f(z) at its pole z = 2 is…..

a) -4 b) 2 c) $\frac{1}{2}$ d) $\frac{9}{2}$

35) If f (z) = $\frac{z^2}{(z-1)(z-2)(z-3)}$, then residue of f(z) at its pole z = 3 is…..

a) 1 b) 2 c) $\frac{9}{2}$ d) $\frac{1}{2}$

36) If f (z) = $\frac{1}{z(z-1)^2}$, then residue of f(z) at its pole z = 0 is…..

a) 1 b) 0 c) -1 d) 2

37) If f (z) = $\frac{1}{z(z-1)^2}$, then residue of f(z) at its pole z = 1 is…..

a) 1 b) 0 c) -1 d) 2

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38) If f (z) = $\frac{ze^z}{(z-1)^3}$, then residue of f(z) at its pole z = 1 is…..

a) 0 b) $\frac{3e}{2}$ c) 1 d) -1

39) The sum of residues of f (z) = $\frac{e^z}{z^2+a^2}$ at its poles is …..

a) sina b) cosa c) $\frac{\sin a}{a}$ d) tana

40) f (z) = $\frac{z^2+4}{z^3+2z^2+2z}$ has …… pole at z = 0.

a) double b) simple c) triple d) None of these

41) By Cauchy's Residue theorem, if f(z) is analytic on and inside a closed contour C except at a finite number of singular points, say n, then $\int_C f(z)dz$ = ……

a) 0 b) 2πi c) -2πi d) $2\pi i \sum R$

42) If f(z) is analytic on and inside a closed contour C except at a finite number of singular points, say n, then $\int_C f(z)dz = 2\pi i \sum R$,

where $\sum R$ denote the sum of the residues at its poles inside C.

Is a statement of ……

a) Cauchy's theorem b) Cauchy's Integral Formula

c) Cauchy's Residue theorem d) None of these

43) f(z) = $\frac{5z-2}{z(z-1)}$ has simple poles at z = 0 and z = 1, out of these …… pole lies inside circle C: $|z| = 2$

a) z = 0 b) z = 1 c) both z = 0 & z=1 d) z = 5/2

44) f(z) = $\frac{e^z}{z(z-1)^2}$ has simple pole at z = 0 and double pole at z=1, out of these which of the poles lies inside circle C: $|z| = 3$?

a) z = 0 b) z = 1 c) both z = 0 & z=1 d) z = 2

45) If C is the circle |z − 2| = 2, then which of the poles of f(z) = $\frac{3z^2+2}{(z-1)(z^2+9)}$ lies inside C?

a) z = 1 b) z = 3i c) z = -3i d) z = 0

46) If C is the circle |z| = 4, then which of the poles of f(z) = $\frac{3z^2+2}{(z-1)(z^2+9)}$ lies inside C?

a) z = 1 b) z = 3i c) z = -3i d) all of these

47) Parametric equation of a circle |z| = r is ……

a) $z = re^{i\theta}, 0 \le \theta \le 2\pi$ b) $z = re^{i\theta}, 0 \le \theta \le \pi$

c) $z = e^{i\theta}, 0 \le \theta \le 2\pi$ d) $z = e^{i\theta}, 0 \le \theta \le \pi$

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48) Parametric equation of a circle $|z - a| = r$ is ……

a) $z = a - re^{i\theta}, 0 \le \theta \le 2\pi$ b) $z = a + re^{i\theta}, 0 \le \theta \le 2\pi$

c) $z = a + e^{i\theta}, 0 \le \theta \le 2\pi$ d) $z = a - e^{i\theta}, 0 \le \theta \le \pi$

49) If $z = e^{i\theta}$, then $d\theta$ = ……

a) $ie^{i\theta}$ b) $\frac{dz}{iz}$ c) dz d) idz

50) If $z = e^{i\theta}$, then $\cos\theta$ = ……

a) $\frac{1}{2i}(z + \frac{1}{z})$ b) $\frac{1}{2}(z - \frac{1}{z})$ c) $\frac{1}{2}(z + \frac{1}{z})$ d) $\frac{1}{2i}(z - \frac{1}{z})$

51) If $z = e^{i\theta}$, then $\sin\theta$ = ……

a) $\frac{1}{2}(z + \frac{1}{z})$ b) $\frac{1}{2}(z - \frac{1}{z})$ c) $\frac{1}{2i}(z + \frac{1}{z})$ d) $\frac{1}{2i}(z - \frac{1}{z})$

52) By contour integration,

$\therefore \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i$ [The sum residues of f(z) at the poles, those lies in the upper half of the z-plane]

If $f(x) = \frac{P(x)}{Q(x)}$ is a rational polynomial, with ……

a) P(x) and Q(x) are polynomials in x b) degree of Q(x) - degree of P(x) ≥ 2

c) Q(x) = 0 has no real roots d) all of these

53) If f(x) is an even function i.e. f(-x) = f(x), then $\int_{0}^{\infty} f(x)\, dx$ = ……

a) $2\int_{-\infty}^{\infty} f(x)\, dx$ b) $\frac{1}{2}\int_{-\infty}^{\infty} f(x)\, dx$ c) 0 d) None of these

54) If $f(x) = \frac{x^2}{(x^2+1)(x^2+4)}$, then the poles of f(z), those lies in the upper half of the z-plane are ……

a) i and 2i b) $-$i and -2i c) i and -2i d) $-$i and 2i

55) If a > 0, b > 0 and $f(x) = \frac{1}{(x^2+a^2)(x^2+b^2)}$, then the poles of f(z), those lies in the upper half of the z-plane are ……

a) $-$ai and -bi b) ai and bi c) ai and -bi d) $-$ai and bi

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."