Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: S.Y.B.SC SEM.-III

SUBJECT: MTH-301: CALCULUS OF SEVERAL VARIABLES

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MTH -301: CALCULUS OF SEVERAL VARIABLES

Unit- 1: Functions of Two and Three Variables **Marks-15**

1.1 Explicit and Implicit Functions
1.2 Continuity
1.3 Partial Derivatives
1.4 Differentiability
1.5 Necessary and Sufficient Conditions for Differentiability
1.6 Partial Derivatives of Higher Order
1.7 Schwarz's Theorem
1.8 Young's Theorem.

Unit-2: Jacobian, Composite Functions and Mean Value Theorems **Marks-15**

2.1 Jacobian (Only for Two and Three Variable)
2.2 Composite Functions (Chain Rule)
2.3 Homogeneous Functions.
2.4 Euler's Theorem on Homogeneous Functions.
2.5 Mean Value Theorem for Function of Two Variables.

Unit -3: Taylor's Theorem and Extreme Values **Marks-15**

3.1 Taylor's Theorem for Function of Two Variables.
3.2 Maclaurin's Theorem for Function of Two Variables.
3.3 Absolute and Relative Maxima & Minima.
3.4 Necessary Condition for Extrema.
3.5 Critical Point, Saddle Point.
3.6 Sufficient Condition for Extrema.

Unit -4: Double and Triple Integrals **Marks-15**

4.1 Double Integrals by Using Cartesian and Polar Coordinates.
4.2 Change of Order of Integration.
4.3 Area by Double Integral.
4.4 Evaluation of Triple Integral as Repeated Integral.
4.5 Volume by Triple Integral.

Recommended Book:

Mathematical Analysis: S.C. Malik and Savita Arora. Wiley Eastern Ltd, New Delhi. 1992 (Chapter 15: Functions of several variables 1, 1.1, 1.2, 1.3, 1.4, 1.6,2, 3, 3.1, 3.2, 4, 4.1, 5, 5.2, 6, 7.2, 9, 9.1, 10, 10.1, 10.2)

Reference Books –

1. Calculus of Several Variables by Schaum's Outline Series.
2. Mathematical Analysis by T. M. Apostol, Narosa Publishing House, New Delhi, 1985

Learning Outcomes:

Upon successful completion of this course the student will be able to understand:

a) limit and continuity of functions of several variables
b) fundamental concepts of multivariable Calculus.
c) series expansion of functions.
d) extreme points of function and their maximum, minimum values at those points.
e) meaning of definite integral as limit as sums.
f) how to solve double and triple integration and use them to find area by double integration and volume by triple integration.

UNIT- 1: FUNCTIONS OF TWO AND THREE VARIABLES

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Functions of Two Variables:

A relation f : $R^2 \rightarrow R$ is said to be a function of two variables x and y if every point (x, y) in R^2 associates a unique real variable z i.e. f (x, y) = z in R.

Functions of Three Variables:

A relation f : $R^3 \rightarrow R$ is said to be a function of three variables x, y and z if every point (x, y, z) in R^3 associates a unique real variable w i.e. f (x, y, z) = w in R.

Neighbourhood of a point:

A set $\delta N(a, b) = \{(x, y) / \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$ is called δ neighbourhood of a point (a, b) in xy-plane. Which is circle with centre at point (a, b) and radius δ.

Deleted Neighbourhood of a point:

A set $\delta N'(a, b) = \{(x, y) / 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$ is called deleted δ neighbourhood of a point (a, b) in xy-plane.

Limit of a function:

If for a arbitrarily small ε > 0, there exist $\delta > 0$ depends on ε such that $|f(x,y) - l| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ or $0 < |x-a| < \delta$ *and* $0 < |y-b| < \delta$. Then l is said to be limit of f (x, y) as (x, y) → (a, b). Denoted by $\lim_{(x,y)\to(a,b)} f(x, y) = l$ or $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$.

This limit is also called double limit or simultaneous limit.

Algebra of Limits:

If $\lim_{(x,y)\to(a,b)} f(x, y) = l$ and $\lim_{(x,y)\to(a,b)} g(x, y) = m$ then

i) $\lim_{(x,y)\to(a,b)} [f(x, y) \pm g(x,y)] = l \pm m$

ii) $\lim_{(x,y)\to(a,b)} [f(x, y)g(x,y)] = lm$

iii) $\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{l}{m}$ provided $m \neq 0$

iv) $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x\ y)} = \sqrt[n]{l}$

Existence of Limit:

i) Limit is exists, if along any path limit is same.

ii) Limit is not exists, if along different paths we get different limits.

Observation:

i) In general if given function contain trigonometric terms or given function is the rational function which is not homogenous of degree 0 and its denominator is in powers of $x^2 + y^2$ then its limit is exist and it is always 0 as (x, y) →(0, 0). Which is shown by using ε- δ definition and inequalities $|x| < \sqrt{x^2 + y^2}, |y| < \sqrt{x^2 + y^2}$.

ii) To prove limit is not exist, we take two paths, first path is y = 0 and choose second path y = f (x) such that degree of numerator and denominator becomes same and having different limit than path y = 0.

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Ex. Evaluate $\lim_{(x,y)\to(a,b)} \frac{x-a}{y-b}$

Sol. Let L = $\lim_{(x,y)\to(a,b)} \frac{x-a}{y-b}$

For the path x = a, we have

$L = \lim_{y\to b} \frac{a-a}{y-b} = 0$

But we observed that for the path y = b,

$L = \lim_{x\to a} \frac{x-a}{b-b}$ does not exists.

Hence $\lim_{(x,y)\to(a,b)} \frac{x-a}{y-b}$ does not exists.

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Ex. Let f (x, y) = $\frac{x^2 - y^2}{x^2 + y^2}$, when (x, y) ≠ (0, 0). Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$

Sol. Let L = $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

For the path y = 0, we have

$L = \lim_{x\to 0} \frac{x^2-0}{x^2+0} = \lim_{x\to 0} 1 = 1 \because x \neq 0$

For the path x = 0, we have

$L = \lim_{y\to 0} \frac{0-y^2}{0+y^2}$

$= \lim_{y\to 0} (-1) \quad \because y \neq 0$

$= -1$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

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Ex. Let f (x, y) = $\frac{x^3+y^3}{x-y}$, when x ≠ y and f (x, y) = 0 when x = y .

Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$

Sol. Let L = $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$

For the path y = 0, we have

$L = \lim_{x\to 0} \frac{x^3+0}{x-0} = \lim_{x\to 0} x^2 = 0$

For the path y = x−x^3, we have

$L = \lim_{x\to 0} \frac{x^3+(x-x^3)^3}{x-(x-x^3)}$

$= \lim_{x\to 0} \frac{x^3[1+(1-x^2)^3]}{x^3}$

$= \lim_{x\to 0} [1+(1-x^2)^3] \quad \because x \neq 0$

$= 2$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

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Ex. Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y)}{x+y}$

Sol. Let L= $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y)}{x+y}$

For the path y = 0, we have

$L = \lim_{x\to 0} \frac{\sin(x^2+0)}{x+0}$

$= \lim_{x\to 0} x \text{ x } \left(\frac{\sin x^2}{x^2}\right)$

$= 0 \text{ x } 1$

$= 0$

For the path x = 0, we have

$L = \lim_{y\to 0} \frac{\sin(0+y)}{0+y}$

$= \lim_{y\to 0} \frac{\sin y}{y}$

$= 1$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y)}{x+y}$ does not exists.

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Ex. Evaluate $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y)}{x+y}$

Sol. Let L= $\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y)}{x+y}$

For the path y = 0, we have

$L = \lim_{x\to 0} \frac{\tan(x^2+0)}{x+0}$

$= \lim_{x\to 0} x \text{ x } \left(\frac{\tan x^2}{x^2}\right)$

$= 0 \text{ x } 1$

$= 0$

For the path x = 0, we have

$$L = \lim_{y\to 0} \frac{\tan(0+y)}{0+y}$$

$$= \lim_{y\to 0} \frac{\tan y}{y}$$

$= 1$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y)}{x+y}$ does not exists.

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Ex. Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x+y}$

Sol. Let L= $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x+y}$

For the path y = 0, we have

$$L = \lim_{x\to 0} \frac{\sin(x^2+0)}{x+0}$$

$$= \lim_{x\to 0} \frac{x\sin x^2}{x^2}$$

$= 0 \text{ x } 1$

$= 0$

For the path y = $x^2 - x$, we have

$$L = \lim_{x\to 0} \frac{\sin[x^2+(x^2-x)^2]}{x+(x^2-x)}$$

$$= \lim_{x\to 0} \frac{\sin\{x^2[1+(x-1)^2]\}}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin\{x^2[1+(x-1)^2]\}}{x^2[1+(x-1)^2]}[1+(x-1)^2]$$

$= (1) \text{ x } [1+(-1)^2]$

$= 2$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x+y}$ does not exists.

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Ex. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

Sol. Let L= $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

For the path y = 0, we have

$$L = \lim_{x\to 0} \frac{0}{x^2+0} = 0$$

For the path $y^3 = x$, we have

$$L = \lim_{x\to 0} \frac{x^2}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{x^2}{2x^2}$$

$$= \lim_{x\to 0} \frac{1}{2} \quad \because x \neq 0$$

$$= \frac{1}{2}$$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ does not exists.

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Ex. Evaluate the limit, if it exists, for the following function

$$(x,) = \frac{x^2y}{x^4+y^2}, \, if \; x^4+y^2 \neq 0,$$

$$= 0, \, if \; x = y = 0.$$

Sol. Let L= $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

For the path y = 0, we have

$$L = \lim_{x\to 0} \frac{0}{x^4+0} = 0$$

For the path $y = x^2$, we have

$$L = \lim_{x\to 0} \frac{x^4}{x^4+x^4}$$

$$= \lim_{x\to 0} \frac{x^4}{2x^4}$$

$$= \lim_{x\to 0} \frac{1}{2} \quad \because x \neq 0$$

$$= \frac{1}{2}$$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exists.

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Ex. Let f $(x, y) = x \sin\frac{1}{x} + y \sin\frac{1}{y}$, $xy \neq 0$. Show that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

Sol. Let f $(x, y) = x \sin\frac{1}{x} + y \sin\frac{1}{y}$, $xy \neq 0$.

Here we use $\varepsilon - \delta$ definition of limit, to find the limit of given function.

Consider

$$|f(x,y) - 0| = \left|x \sin\frac{1}{x} + y \sin\frac{1}{y}\right|$$
$$\leq \left|x \sin\frac{1}{x}\right| + \left|y \sin\frac{1}{y}\right|$$
$$\leq |x|\left|\sin\frac{1}{x}\right| + |y|\left|\sin\frac{1}{y}\right|$$
$$\leq |x| + |y| \quad \because \left|\sin\frac{1}{x}\right| \leq 1 \text{ and } \left|\sin\frac{1}{y}\right| \leq 1$$
$$\leq 2\sqrt{x^2+y^2} \quad \because |x| \leq \sqrt{x^2+y^2} \text{ and } |y| \leq \sqrt{x^2+y^2}$$

$\therefore |f(x,y) - 0| \leq 2\sqrt{x^2+y^2} < \varepsilon$

Now $2\sqrt{x^2+y^2} < \varepsilon \Rightarrow \sqrt{x^2+y^2} < \frac{\epsilon}{2}$

By taking $\frac{\varepsilon}{2} = \delta$, we get, for $\varepsilon > 0, \exists\ \delta = \frac{\varepsilon}{2} > 0$

such that $|f(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\therefore$ By $\varepsilon - \delta$ definition of limit, we get

$\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0.$

Hence proved.

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Repeated Limits:

Let f (x, y) be any real valued function defined in some deleted neighborhood of point (a, b) then $\lim\limits_{y\to b}[\lim\limits_{x\to a} f(x, y)]$ and $\lim\limits_{x\to a}[\lim\limits_{y\to b} f(x, y)]$ are called repeated limits or iterated limits.

Remark:

i) Repeated limits of any function may or may not be equal.

ii) If repeated limits of a given function are not equal then simultaneous limit of a function does not exist.

iii) If repeated limits of a given function are equal then simultaneous limit of a function may or may not be exist.

iv) If simultaneous limit of a given function exist then repeated limits are equal.

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Ex. Let f (x, y) $= \frac{x^2y^2}{x^4+y^4-x^2y^2}$, if (x, y) $\neq$ (0, 0). Verify that, both repeated limits exist and are equal but simultaneous limit does not exist as (x, y) $\to$ (0, 0).

Sol. Let f (x, y) $= \frac{x^2y^2}{x^4+y^4-x^2y^2}$, if (x, y) $\neq$ (0, 0).

First we find repeated limits of given function as follows.

$$\lim_{y\to 0}[\lim_{x\to 0} f(x, y)] = \lim_{y\to 0}[\lim_{x\to 0} \frac{x^2y^2}{x^4+y^4-x^2y^2}] = \lim_{y\to 0} \frac{0}{y^4} = 0$$

$$\& \lim_{x\to 0}[\lim_{y\to 0} f(x, y)] = \lim_{x\to 0}[\lim_{y\to 0} \frac{x^2y^2}{x^4+y^4-x^2y^2}] = \lim_{x\to 0} \frac{0}{x^4} = 0$$

i.e. both repeated limits exists and are equal.

Now to find simultaneous limit, denote

$$L = \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4-x^2y^2}$$

For the path y = 0, we have

$$L = \lim_{x\to 0} \frac{0}{x^4+0} = 0$$

For the path y = x, we have

$$L = \lim_{x\to 0} \frac{x^4}{x^4+x^4-x^4}$$
$$= \lim_{x\to 0} \frac{x^4}{x^4}$$
$$= \lim_{x\to 0} 1 \quad \because x \neq 0$$
$$= 1$$

For two different paths we get two different limits.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4-x^2y^2} \text{ does not exists.}$$

Hence it is verified that, both repeated limits exist and are equal but simultaneous limit does not exist as $(x, y) \to (0, 0)$.

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Ex. Show that, both repeated limits exists but simultaneous limit does not exist for $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$

Sol. Let $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$

First we find repeated limits of given function as follows.

$$\lim_{y\to 0}[\lim_{x\to 0} f(x, y)] = \lim_{y\to 0}[\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2}]$$
$$= \lim_{y\to 0} \frac{0-y^2}{0+y^2} = \lim_{y\to 0}(-1) = -1 \quad \because y \neq 0$$

$$\& \lim_{x\to 0}[\lim_{y\to 0} f(x, y)] = \lim_{x\to 0}[\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}]$$
$$= \lim_{x\to 0} \frac{x^2-0}{x^2+0} = \lim_{x\to 0} 1 = 1 \quad \because x \neq 0$$

i.e. Repeated limits exists.

Now to find simultaneous limit, denote

$L = \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

For the path y = 0, we have

$L = \lim_{x\to 0} \frac{x^2-0}{x^2+0}$

$= \lim_{x\to 0} 1 \quad \because x \neq 0$

$= 1$

For the path x = 0, we have

$L = \lim_{y\to 0} \frac{0-y^2}{0+y^2}$

$= \lim_{y\to 0} (-1) \quad \because y \neq 0$

$= -1$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exists.

Hence proved that, repeated limits exist but simultaneous limit does not exists.

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Continuity of a function:

A function f (x, y) is said to be continuous at point (a, b) if f (a, b) is defined, $\lim_{(x,y)\to(a,b)} f(x, y)$ is exist and $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Remark:

A function f (x, y) is discontinuous at point (a, b) if f (a, b) is not defined or $\lim_{(x,y)\to(a,b)} f(x, y)$ is not exist or $\lim_{(x,y)\to(a,b)} f(x, y) \neq f(a, b)$.

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Ex. Investigate for continuity the function

$f(x, y) = \frac{x^2 y}{x^4+y^2}, \ if\ (x, y) \neq (0, 0)$

$f(0, 0) = 0.$

Sol. Let $f(x, y) = \frac{x^2 y}{x^4+y^2}, if\ (x, y) \neq (0, 0)$

$f(0, 0) = 0$.........(i)

Let $L = \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$

For the path y = 0, we have

$L = \lim_{x\to 0} \frac{0}{x^4+0} = 0$

For the path $y = x^2$, we have

$L = \lim_{x\to 0} \frac{x^4}{x^4+x^4}$

$= \lim_{x\to 0} \frac{x^4}{2x^4}$

$= \lim_{x\to 0} \frac{1}{2} \quad \because x \neq 0$

$= \frac{1}{2}$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} f(x, y)$ does not exists.

Hence f (x, y) is not continuous at (0, 0).

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Ex. Show that the function

$f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}, \ if\ (x, y) \neq (0, 0)$

$= 0, \ if\ (x, y) = (0, 0)$

is continuous at the origin.

Proof. Let $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}}, \ if\ (x, y) \neq (0, 0)$

$= 0, \ if\ (x, y) = (0, 0)$

i.e. f (0, 0) = 0……….(i)

Consider

$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} = \frac{|x||y|}{\sqrt{x^2+y^2}}$

$\leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \because \quad |x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$

$\therefore |f(x,y) - 0| \leq \sqrt{x^2+y^2} < \varepsilon$

Here $\sqrt{x^2+y^2} < \varepsilon$

$\therefore$ By taking $\varepsilon = \delta$, we get, for $\varepsilon > 0, \exists\ \delta = \varepsilon > 0$

such that $|f(x,y) - 0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\therefore$ By $\varepsilon - \delta$ definition of limit, we get

$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$ by equation (i).

Hence given function is continuous at (0, 0) is proved.

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Ex. Show that the function $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at origin.

Proof. Let $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$

and f (0, 0) = 0……….(i)

Consider

$$|f(x,y)-0| = \left|xy\frac{x^2-y^2}{x^2+y^2}\right|$$
$$= |x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right|$$
$$\le \sqrt{x^2+y^2}\sqrt{x^2+y^2}$$
$$\because \quad |x| \le \sqrt{x^2+y^2},\ |y| \le \sqrt{x^2+y^2}\ \&\left|\frac{x^2-y^2}{x^2+y^2}\right| \le 1$$

$\therefore |f(x,y)-0| \le x^2+y^2 < \varepsilon$

Now $x^2+y^2 < \varepsilon \Rightarrow \sqrt{x^2+y^2} < \sqrt{\varepsilon}$

$\therefore$ By taking $\sqrt{\varepsilon} = \delta$, we get, for $\varepsilon > 0, \exists\ \delta = \sqrt{\varepsilon} > 0$

such that $|f(x,y)-0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\therefore$ By $\varepsilon - \delta$ definition of limit, we get

$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$ by equation (i).

Hence given function is continuous at origin is proved.

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Ex. Let $f(x, y) = y + x\sin(\frac{1}{y})$, if $y \neq 0$ and $f(x, 0) = 0$. Show that f is continuous at (0, 0)

Proof. Let $f(x, y) = y + x\sin(\frac{1}{y})$, if $y \neq 0$ and $f(x, 0) = 0$.

$\therefore f(0, 0) = 0$……..(i) is defined.

Consider

$$|f(x,y)-0| = \left|y + x\sin\frac{1}{y}\right|$$
$$\le |y| + \left|x\sin\frac{1}{y}\right|$$
$$\le |y| + |x|\left|\sin\frac{1}{y}\right|$$
$$\le |x| + |y| \quad \because \quad \left|\sin\frac{1}{y}\right| \le 1$$
$$\le 2\sqrt{x^2+y^2} \quad \because \quad |x| \le \sqrt{x^2+y^2} \text{ and } |y| \le \sqrt{x^2+y^2}$$

$\therefore |f(x,y)-0| \le 2\sqrt{x^2+y^2} < \varepsilon$

Now $2\sqrt{x^2+y^2} < \varepsilon \Rightarrow \sqrt{x^2+y^2} < \frac{\epsilon}{2}$

By taking $\frac{\varepsilon}{2} = \delta$, we get, for $\varepsilon > 0, \exists\ \delta = \frac{\varepsilon}{2} > 0$

such that $|f(x,y)-0| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

$\therefore$ By $\varepsilon - \delta$ definition of limit, we get

$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$ by equation (i).

Hence given function is continuous at (0, 0) is proved.

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Partial Derivative:

If $\lim\limits_{h\to 0} \frac{f(x+h,\ y)-f(x,\ y)}{h}$ is exist, then it said to be partial derivative of f (x, y) w. r. to x and is denoted by $f_x(x, y)$ or $\frac{\partial f}{\partial x}$.

Note: First order partial derivatives of f (x, y) w. r. to x and y at point (a, b) are

$\left[\frac{\partial f}{\partial x}\right]_{(a,\ b)} = f_x(a, b) = \lim\limits_{h\to 0} \frac{f(a+h,\ b)-f(a,\ b)}{h}$ and $\left[\frac{\partial f}{\partial y}\right]_{(a,\ b)} = f_y(a, b) = \lim\limits_{k\to 0} \frac{f(a,\ b+k)-f(a,\ b)}{k}$

==

Ex. If $u = x^3z + xy^2 - 2yz$ then find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ at point (1, 2, 3)

Sol. Given $u = x^3z + xy^2 - 2yz$

Differentiating partially w. r. to x, y and z, we get

$\frac{\partial u}{\partial x} = 3x^2z + y^2 - 0 = 3x^2z + y^2$

$\frac{\partial u}{\partial y} = 2xy - 2z$

& $\frac{\partial u}{\partial z} = x^3 - 2y$

At point (1, 2, 3), we have

$\left[\frac{\partial u}{\partial x}\right]_{(1,\ 2,\ 3)} = 3(1^2)(3) + 2^2 = 9 + 4 = 13$

$\left[\frac{\partial u}{\partial y}\right]_{(1,\ 2,\ 3)} = 2(1)(2) - 2(3) = 4 - 6 = -2$

& $\left[\frac{\partial u}{\partial z}\right]_{(1,\ 2,\ 3)} = 1^3 - 2(2) = 1 - 4 = -3$

==

Ex. Find the first order partial derivative of $u = e^x\sin xy$

Sol. Given $u = e^x\sin xy$

Differentiating partially w. r. to x and y, we get

$\frac{\partial u}{\partial x} = e^x\sin xy + ye^x\cos xy$

& $\frac{\partial u}{\partial y} = xe^x\cos xy$

==

Ex. Find the first order partial derivative of $u = \tan^{-1}\frac{y}{x}$

Sol. Given $u = \tan^{-1}\frac{y}{x}$

Differentiating partially w. r. to x and y, we get

$\frac{\partial u}{\partial x} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$

& $\frac{\partial u}{\partial y} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

==

Ex. Find the first order partial derivative of u = log $(x^2+y^2+z^2)$

Sol. Given u = log $(x^2+y^2+z^2)$

Differentiating partially w. r. to x, y and z, we get

$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x) = \frac{2x}{x^2+y^2+z^2}$

$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2+z^2}(2y) = \frac{2y}{x^2+y^2+z^2}$

& $\frac{\partial u}{\partial z} = \frac{1}{x^2+y^2+z^2}(2z) = \frac{2z}{x^2+y^2+z^2}$

==

Ex. If $u = x^2y + y^2z + z^2x$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$

Proof. Given $u = x^2y + y^2z + z^2x$

Differentiating partially w. r. to x, y and z, we get

$\frac{\partial u}{\partial x} = 2xy + z^2$

$\frac{\partial u}{\partial y} = x^2 + 2yz$

& $\frac{\partial u}{\partial z} = y^2 + 2zx$

Adding we get,

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx = (x + y + z)^2$

Hence proved.

==

Ex. If u = log (tanx + tany + tanz), prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Proof. Given u = log (tanx + tany + tanz)

Differentiating partially w. r. to x, we get

$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x)$

$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x . \sec^2 x}{\tan x + \tan y + \tan z}$

$= \frac{2\sin x \cos x}{\tan x + \tan y + \tan z}\left(\frac{1}{\cos^2 x}\right)$

$$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{2tanx}{\tan x + \tan y + \tan z}$$

Similarly

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2tany}{\tan x + \tan y + \tan z} \ \& \ \sin 2z \frac{\partial u}{\partial z} = \frac{2tanz}{\tan x + \tan y + \tan z}$$

Adding we get,

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z} = 2$$

Hence proved.

==

Ex. Let $f(x, y) = \frac{xy}{x^2 + y^2}$, if (x, y) ≠ (0, 0)

$= 0$, if $(x, y) = (0, 0)$

Show that both the first order partial derivatives exist at (0, 0), but the function is not continuous there at.

Proof. Let $f(x, y) = \frac{xy}{x^2 + y^2}$, if (x, y) ≠ (0, 0)

$= 0$, if $(x, y) = (0, 0)$

i.e. f (0, 0) = 0…….(i) is defined.

First we find partial derivatives at point (0, 0)

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\& \ f_y(0, 0) = \lim_{k\to 0} \frac{f(0,\ 0+k) - f(0,\ 0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

i.e. both partial derivatives exist at point (0, 0).

To find limit of a function denote

$$L = \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

For the path y = 0, we have

$$L = \lim_{x\to 0} \frac{0}{x^2+0} = 0$$

For the path y = x, we have

$$L = \lim_{x\to 0} \frac{x^2}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{1}{2} \quad \because x \neq 0$$

$$= \frac{1}{2}$$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} f(x, y)$ does not exists.

Hence both the first order partial derivatives exist at (0, 0), but the function is not continuous there at is proved.

Partial Derivative of Higher Order:

Let f_x and f_y are first order partial derivatives of f (x, y) which are again functions of x and y. By taking partial derivatives of f_x and f_y w. r. to x and y again and again we get partial derivatives of second and higher orders.

Which are denoted by $f_{xx}, f_{xy}, f_{yx}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}$ etc.

Note: i) Second order partial derivatives of f (x, y) at point (a, b) are

$$f_{xx}(a, b) = (f_x)_x(a, b) = \lim_{h\to 0} \frac{f_x(a+h,\ b)-f_x(a,\ b)}{h}$$

$$f_{xy}(a, b) = (f_y)_x(a, b) = \lim_{h\to 0} \frac{f_y(a+h,\ b)-f_y(a,\ b)}{h}$$

$$f_{yx}(a, b) = (f_x)_y(a, b) = \lim_{k\to 0} \frac{f_x(a,\ b+k)-f_x(a,\ b)}{k}$$

$$f_{yy}(a, b) = (f_y)_y(a, b) = \lim_{k\to 0} \frac{f_y(a,\ b+k)-f_y(a,\ b)}{k}$$

ii) $f_{xy}(a, b)$ and $f_{yx}(a, b)$ may or may not be equal.

Working Rule to find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$:

i) Find f (0, 0), f (h, 0), f (0, k) and f (h, k).

ii) Find $f_x(0, k) = \lim_{h\to 0} \frac{f(0+h,\ k)-f(0,\ k)}{h}$ and $f_x(0, 0)$

$f_y(h, 0) = \lim_{k\to 0} \frac{f(h,\ 0+k)-f(h,\ 0)}{k}$ and $f_y(0, 0)$

iii) Find $f_{xy}(0, 0) = (f_y)_x(0, 0) = \lim_{h\to 0} \frac{f_y(h,\ 0)-f_y(0,\ 0)}{h}$

& $f_{yx}(0, 0) = (f_x)_y(0, 0) = \lim_{k\to 0} \frac{f_x(0,\ k)-f_x(0,\ 0)}{k}$

==

Ex. Let f $(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$, (x, y) ≠ (0, 0) and f (0, 0) = 0.

Prove that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Proof. Let f $(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$, (x, y) ≠ (0, 0) and f (0, 0) = 0

∴ i) f (0, 0) = 0, f (h, 0) = 0, f (0, k) = 0 and f (h, k) = $hk\frac{h^2-k^2}{h^2+k^2}$

ii) $f_x(0, k) = \lim_{h\to 0} \frac{f(0+h,\ k)-f(0,\ k)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\{ hk\frac{h^2-k^2}{h^2+k^2} - 0\}$

$= \lim_{h\to 0} k\frac{h^2-k^2}{h^2+k^2}$

$= k\frac{0-k^2}{0+k^2}$

$i.e.$ $f_x(0, k) = -k$

∴ $f_x(0, 0) = 0$

and $f_y(h, 0) = \lim\limits_{k\to 0} \frac{f(h,\ 0+k)-f(h,\ 0)}{k}$

$= \lim\limits_{k\to 0} \frac{1}{k}\{ hk\frac{h^2-k^2}{h^2+k^2} - 0\}$

$= \lim\limits_{k\to 0} h\frac{h^2-k^2}{h^2+k^2}$

$= h\frac{h^2-0}{h^2+0}$

i.e. $f_y(h, 0) = h$

$\therefore f_y(0, 0) = 0$

iii) Now $f_{xy}(0, 0) = (f_y)_x(0, 0) = \lim\limits_{h\to 0} \frac{f_y(h,\ 0)-f_y(0,\ 0)}{h}$

$= \lim\limits_{h\to 0} \frac{h-0}{h}$

$= \lim\limits_{h\to 0} 1 \quad \because h \neq 0$

$= 1$………(i)

& $f_{yx}(0, 0) = (f_x)_y(0, 0) = \lim\limits_{k\to 0} \frac{f_x(0,\ k)-f_x(0,\ 0)}{k}$

$= \lim\limits_{k\to 0} \frac{-k-0}{k}$

$= \lim\limits_{h\to 0} (-1) \quad \because k \neq 0$

i.e. $f_{yx}(0, 0) = -1$ ………(ii)

By equation (i) and (ii), $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ is proved.

==

Ex. Let $f(x, y) = \frac{x^2y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

Show that $f_{xy}(0, 0) = f_{yx}(0, 0)$.

Proof. Let $f(x, y) = \frac{x^2y^2}{x^2+y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

$\therefore$ i) $f(0, 0) = 0$, $f(h, 0) = 0$, $f(0, k) = 0$ and $f(h, k) = \frac{h^2k^2}{h^2+k^2}$

ii) $f_x(0, k) = \lim\limits_{h\to 0} \frac{f(0+h,\ k)-f(0,\ k)}{h}$

$= \lim\limits_{h\to 0} \frac{1}{h}\{ \frac{h^2k^2}{h^2+k^2} - 0\}$

$= \lim\limits_{h\to 0} \frac{hk^2}{h^2+k^2}$

$= \frac{0}{0+k^2}$

i.e. $f_x(0, k) = 0$

$\therefore f_x(0, 0) = 0$

and $f_y(h, 0) = \lim\limits_{k\to 0} \frac{f(h,\ 0+k)-f(h,\ 0)}{k}$

$= \lim\limits_{k\to 0} \frac{1}{k}\{ \frac{h^2k^2}{h^2+k^2} - 0\}$

$= \lim\limits_{k\to 0} \frac{h^2k}{h^2+k^2}$

$= \frac{0}{h^2+0}$

i.e. $f_y(h, 0) = 0$

$\therefore f_y(0, 0) = 0$

iii) Now $f_{xy}(0, 0) = (f_y)_x(0, 0) = \lim\limits_{h\to 0} \frac{f_y(h,\ 0)-f_y(0,\ 0)}{h}$

$= \lim\limits_{h\to 0} \frac{0-0}{h}$

$= 0$………(i)

& $f_{yx}(0, 0) = (f_x)_y(0, 0) = \lim\limits_{k\to 0} \frac{f_x(0,\ k)-f_x(0,\ 0)}{k}$

$= \lim\limits_{k\to 0} \frac{0-0}{k}$

i.e. $f_{yx}(0, 0) = 0$ ……….(ii)

By equation (i) and (ii), $f_{xy}(0, 0) = f_{yx}(0, 0)$ is proved.

==

Ex. Let $f(x, y) = x^2 \tan^{-1}(\frac{y}{x}) - y^2 \tan^{-1}(\frac{x}{y}),\ if\ (x, y) \neq (0, 0)$

$= 0,\ if\ (x, y) \neq (0, 0)$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Sol. Let $f(x, y) = x^2 \tan^{-1}(\frac{y}{x}) - y^2 \tan^{-1}(\frac{x}{y}),\ if\ (x, y) \neq (0, 0)$

$= 0,\ if\ (x, y) \neq (0, 0)$

$\therefore$ i) $f(0, 0) = 0$, $f(h, 0) = 0$, $f(0, k) = 0$ and $f(h, k) = h^2 \tan^{-1}(\frac{k}{h}) - k^2 \tan^{-1}(\frac{h}{k})$

ii) $f_x(0, k) = \lim\limits_{h\to 0} \frac{f(0+h,\ k)-f(0,\ k)}{h}$

$= \lim\limits_{h\to 0} \frac{1}{h}\{ h^2 \tan^{-1}(\frac{k}{h}) - k^2 \tan^{-1}(\frac{h}{k}) - 0\}$

$= \lim\limits_{h\to 0} h \tan^{-1}(\frac{k}{h}) - \lim\limits_{h\to 0} k \frac{\tan^{-1}(\frac{h}{k})}{(\frac{h}{k})}$

$= 0 - k\,(1)$

i.e. $f_x(0, k) = -k$

$\therefore f_x(0, 0) = 0$

and $f_y(h, 0) = \lim\limits_{k\to 0} \frac{f(h,\ 0+k)-f(h,\ 0)}{k}$

$= \lim\limits_{k\to 0} \frac{1}{k}\{ h^2 \tan^{-1}(\frac{k}{h}) - k^2 \tan^{-1}(\frac{h}{k}) - 0\}$

$= \lim\limits_{k\to 0} h \frac{\tan^{-1}(\frac{k}{h})}{(\frac{k}{h})} - \lim\limits_{k\to 0} k \tan^{-1}(\frac{h}{k})$

$= h\,(1) - 0$

i.e. $f_y(h, 0) = h$

$\therefore f_y(0, 0) = 0$

iii) Now $f_{xy}(0, 0) = (f_y)_x(0, 0) = \lim_{h\to 0} \frac{f_y(h,\ 0)-f_y(0,\ 0)}{h}$

$= \lim_{h\to 0} \frac{h-0}{h} = \lim_{h\to 0} 1 \quad \because h \neq 0$

$= 1\ldots\ldots\ldots(i)$

& $f_{yx}(0, 0) = (f_x)_y(0, 0) = \lim_{k\to 0} \frac{f_x(0,\ k)-f_x(0,\ 0)}{k}$

$= \lim_{k\to 0} \frac{-k-0}{k} = \lim_{h\to 0} (-1) \quad \because k \neq 0$

i.e. $f_{yx}(0, 0) = -1 \ldots\ldots\ldots(ii)$

By equation (i) and (ii), $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ is proved.

==

Ex. Examine the equality of f_{xy} and f_{yx} where $f(x, y) = x^3y + e^{xy^2}$

Sol. Let $f(x, y) = x^3y + e^{xy^2}$

First differentiating partially w. r. to x and y, we get

$f_x = 3x^2y + y^2e^{xy^2} \ldots\ldots\ldots(1)$

$f_y = x^3 + e^{xy^2}(2xy) \ldots\ldots\ldots(2)$

Differentiating equation (2) partially w. r. to x, we get

$f_{xy} = 3x^2 + 2ye^{xy^2} + 2xye^{xy^2}(y^2)$

$= 3x^2 + 2ye^{xy^2}(1+xy^2)\ldots\ldots(3)$

Differentiating equation (1) partially w. r. to y, we get

$f_{yx} = 3x^2 + 2ye^{xy^2} + y^2\, e^{xy^2}(2xy)$

$= 3x^2 + 2ye^{xy^2}(1+xy^2) \ldots\ldots.(4)$

From equation (3) & (4), $f_{xy} = f_{yx}$.

==

Ex. If $u = x^2 \tan^{-1}(\frac{y}{x}) - y^2 \tan^{-1}(\frac{x}{y})$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$

Sol. Let $u = x^2 \tan^{-1}(\frac{y}{x}) - y^2 \tan^{-1}(\frac{x}{y})$

First differentiating partially w. r. to y, we get

$\frac{\partial u}{\partial y} = x^2\frac{1}{1+(\frac{y}{x})^2}(\frac{1}{x}) - 2y \tan^{-1}(\frac{x}{y}) - y^2\frac{1}{1+(\frac{x}{y})^2}(\frac{-x}{y^2})$

$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}(\frac{x}{y}) + \frac{xy^2}{x^2+y^2}$

$= \frac{x^3+xy^2}{x^2+y^2} - 2y \tan^{-1}(\frac{x}{y})$

$= x - 2y \tan^{-1}(\frac{x}{y})$

Now differentiating it partially w. r. to x, we get

$\frac{\partial^2 u}{\partial x \partial y} = 1 - 2y\frac{1}{1+(\frac{x}{y})^2}(\frac{1}{y})$

$= 1 - \frac{2y^2}{x^2+y^2}$

$= \frac{x^2-y^2}{x^2+y^2}$

Hence proved.

==

Ex. Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, if f = log ($\frac{x^2+y^2}{xy}$)

Sol. Let f = log ($\frac{x^2+y^2}{xy}$) = log $(x^2+y^2) - logx - logy$

First differentiating partially w. r. to x, we get

$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$

$= \frac{2x^2-x^2-y^2}{x(x^2+y^2)}$

$\frac{\partial f}{\partial x} = \frac{x^2-y^2}{x(x^2+y^2)}$ ……(i)

Similarly $\frac{\partial f}{\partial y} = \frac{y^2-x^2}{y(x^2+y^2)}$ ……(ii)

Now by differentiating equation (ii) partially w. r. to x, we get

$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{y}\{\frac{(x^2+y^2)(-2x)-(y^2-x^2)(2x)}{(x^2+y^2)^2}\}$

$= \frac{1}{y}\{\frac{-2x^3-2xy^2-2xy^2+2x^3}{(x^2+y^2)^2}\}$

$= \frac{1}{y}\{\frac{-4xy^2}{(x^2+y^2)^2}\}$

$= \frac{-4xy}{(x^2+y^2)^2}$

Again by differentiating equation (i) partially w. r. to y, we get

$\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{x}\{\frac{(x^2+y^2)(-2y)-(x^2-y^2)(2y)}{(x^2+y^2)^2}\}$

$= \frac{1}{x}\{\frac{-2x^2y-2y^3-2x^2y+2y^3}{(x^2+y^2)^2}\}$

$= \frac{1}{x}\{\frac{-4x^2y}{(x^2+y^2)^2}\}$

$= \frac{-4xy}{(x^2+y^2)^2}$

Hence $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ is proved.

==

Ex. If u = $(x^2+y^2+z^2)^{-1/2}$, $x^2+y^2+z^2 \neq 0$, show that $u_{xx} + u_{yy} + u_{zz} = 0$

Sol. Let u = $(x^2+y^2+z^2)^{-1/2}$, $x^2+y^2+z^2 \neq 0$

First differentiating partially w. r. to x, we get

$u_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$

$u_x = -x(x^2+y^2+z^2)^{-3/2}$

Again differentiating it partially w. r. to x, we get

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x)$$

$$= -(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$= (x^2+y^2+z^2)^{-5/2}\{-(x^2+y^2+z^2) + 3x^2\}$$

$$u_{xx} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly $u_{yy} = (x^2+y^2+z^2)^{-5/2}(2y^2-x^2-z^2)$ & $u_{zz} = (x^2+y^2+z^2)^{-5/2}(2z^2-x^2-y^2)$

Adding we get,

$$u_{xx}+ u_{yy}+ u_{zz} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2)$$

$$= (x^2+y^2+z^2)^{-5/2}(0)$$

$$= 0$$

Hence proved.

==

Differentiable Function:

The function f (x, y) is said to be differentiable at point (a, b) if the change δf= f (a+h, b+k) – f (a, b) is expressed in the form δf= Ah + Bk + hΦ(h, k) + kΨ(h, k), where A and B are constants independent of h, k and $\Phi, \Psi \to 0$ *as* (h, k) → (0, 0).

Differentials:

Let u = f (x, y) be a differentiable function of two variables x and y, then the differential of u is denoted by du and is defined as du = $\frac{\partial u}{\partial x}$ dx + $\frac{\partial u}{\partial y}$ dy.

Approximate Value by Using Differentials:

Approximate value of a function at point (a+h, b+k) by using differentials is given by f (a+h, b+k) ≑ f (a, b) + hf_x(a, b) + kf_y(a, b)

==

Necessary Condition For Differentiability:

If a real valued function f (x, y) is differentiable at point (a, b) then

i) f is continuous at point (a, b), ii) f_x(a, b) and f_y(a, b) exists.

Proof: Let a real valued function f (x, y) is differentiable at point (a, b) then for any point (a+h, b+k) in a neighbourhood of point (a, b), we have,

$$\delta f = f(a+h, b+k) - f(a,b) = Ah + Bk + h\Phi(h, k) + k\Psi(h, k) \quad \text{.......(1)}$$

where A and B are constants independent of h, k and $\Phi, \Psi \to 0$ *as* (h, k) → (0, 0).

i) By taking limit (h, k) → (0, 0) on both sides of equation (1), we get,

$$\lim_{(h,k)\to(0,0)}\{f(a+h,b+k) - f(a,b)\} = \lim_{(h,k)\to(0,0)}\{Ah + Bk + h\Phi(h,k) + k\Psi(h,k)\}$$

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k) - \lim_{(h,k)\to(0,0)} f(a,\ b) = 0$$

i.e. $\lim_{(h,k)\to(0,0)} f(a+h,b+k) - f(a,\ b) = 0$

$\therefore \lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b)$

∴ f (x, y) is continuous at (a, b) is proved.

ii) Putting k = 0 in equation (1), we get,

$$f(a+h,b) - f(a,b) = Ah + h\Phi(h, 0)$$

$$\frac{f(a+h,b) - f(a,b)}{h} = A + \Phi(h, 0)$$

Taking limit as $h \to 0$, we get,

$$\lim_{h\to 0}\left\{\frac{f(a+h,\ b) - f(a,\ b)}{h}\right\} = \lim_{h\to 0}\{A + \Phi(h, 0)\}$$

∴ $f_x(a, b) = A$

Similarly we obtain

$f_y(a, b) = B$

Thus $f_x(a, b)$ and $f_y(a, b)$ exists is proved.

==

Sufficient Condition For Differentiability:

A real valued function f (x, y) is differentiable at point (a, b) if i) f_x is continuous at point (a, b) and ii) f_y exist at (a, b).

Proof: Let f (x, y) be a function defined in a domain $D \subseteq R^2$.

For any point (a+h, b+k) in a neighbourhood of point (a, b) we have

$$\delta f = f(a+h, b+k) - f(a,b)$$

$$= f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b) \text{ ....(1)}$$

As $f_x(a, b)$ is exist in a neighbourhood of (a, b).

∴ By Lagrange's Mean Value Theorem,

$f(a+h, b+k) - f(a, b+k) = hf_x(a+\theta h, b+k)$ ....(2) where $0 < \theta < 1$.

Again $f_x(a, b)$ is continuous at (a, b)

∴ $\lim\limits_{(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a, b)$

∴ $f_x(a+\theta h, b+k) = f_x(a, b) + \Phi(h, k)$ for some function $\Phi(h, k) \to 0$ *as* $(h, k) \to (0, 0)$

∴ Equation (2) is written as

$f(a+h, b+k) - f(a, b+k) = hf_x(a, b) + h\Phi(h, k)$......(3)

By condition (ii), $f_y(a, b)$ is exists.

$f_y(a, b) = \lim\limits_{k\to 0}\left\{\frac{f(a,\ b+k) - f(a,\ b)}{k}\right\}$ exists.

∴ $\frac{f(a,\ b+k) - f(a,\ b)}{k} = f_y(a, b) + \Psi(0, k)$

for some function $\Psi(0, k) \to 0$ *as* $(h, k) \to (0, 0)$

∴ $f(a,\ b+k) - f(a,\ b) = kf_y(a, b) + k\,\Psi(0, k)$.........(4)

Using equation (3) and (4) equation (1) becomes

$$\delta f = hf_x(a, b) + h\,\Phi(h, k) + kf_y(a, b) + k\Psi(0, k)$$

$$\delta f = hf_x(a, b) + kf_y(a, b) + h\Phi(h, k) + k\Psi(0, k)$$

where $\Phi, \Psi \to 0$ *as* (h, k) → (0, 0).
Hence f (x, y) is differentiable at (a, b) is proved.

==

Ex. Show that the function

$$f(x, y) = \frac{xy}{\sqrt{x^2+y^2}},\ if\ x^2 + y^2 \neq 0$$
$$= 0,\ if\ x = y = 0$$

possesses the first order partial derivatives but is not differentiable at the origin.

Proof. Let $f(x, y) = \frac{xy}{\sqrt{x^2+y^2}},\ if\ x^2 + y^2 \neq 0$

$= 0,\ if\ x = y = 0$

i.e. f (0, 0) = 0……….(i)

Consider

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\&\ f_y(0, 0) = \lim_{k \to 0} \{\frac{f(0, k) - f(0, 0)}{k}\} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

i. e. first partial derivatives are exists.
Suppose f (x, y) is differentiable at the origin.

$$\delta f = f(h, k) - f(0, 0) = hf_x(0, 0) + kf_y(0, 0) + h\Phi(h, k) + k\Psi(h, k)\ \text{…….(1)}$$

where $\Phi, \Psi \to 0$ *as* (h, k) → (0, 0).
From equation (1), we have,

$$\frac{hk}{\sqrt{h^2+k^2}} - 0 = 0 + 0 + h\Phi(h, k) + k\Psi(h, k)$$

i.e. $\frac{hk}{\sqrt{h^2+k^2}} = h\Phi(h, k) + k\Psi(h, k)$

Putting k = h, we get,

$$\frac{h^2}{\sqrt{h^2+h^2}} = h\Phi(h, h) + h\Psi(h, h)$$

$$\frac{1}{\sqrt{2}} = \Phi(h, h) + \Psi(h, h)$$

As $h \to 0 \Longrightarrow \Phi, \Psi \to 0$ we get,

$\frac{1}{\sqrt{2}} = 0$ which is absurd.

∴ f (x, y) is not differentiable at the origin.

==

Ex. Show that the function

$$f(x, y) = \frac{xy}{x^2+y^2},\ if\ x^2 + y^2 \neq 0$$
$$= 0,\ if\ x = y = 0$$

possesses the first order partial derivatives but is not differentiable at the origin.

Proof. Let f $(x, y) = \frac{xy}{x^2+y^2}$, $if\ x^2 + y^2 \neq 0$

$= 0$, $if\ x = y = 0$

i.e. f (0, 0) = 0……….(i)

Consider

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h,\ 0) - f(0,\ 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\& \ f_y(0, 0) = \lim_{k\to 0}\left\{\frac{f(0,\ k) - f(0,\ 0)}{k}\right\} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

i. e. first partial derivatives are exists.

Suppose f (x, y) is differentiable at the origin.

$$\delta f = f(h,\ k) - f(0,\ 0) = hf_x(0, 0) + kf_y(0, 0) + h\Phi(h, k) + k\Psi(h, k) \ …….(1)$$

where $\Phi, \Psi \to 0\ as\ (h, k) \to (0, 0)$.

From equation (1), we have,

$$\frac{hk}{h^2+k^2} - 0 = 0 + 0 + h\Phi(h, k) + k\Psi(h, k)$$

i.e. $\frac{hk}{h^2+k^2} = h\Phi(h, k) + k\Psi(h, k)$

Putting k = h, we get,

$$\frac{h^2}{h^2+h^2} = h\Phi(h, h) + h\Psi(h, h)$$

$$\frac{1}{2} = h\Phi(h, h) + h\Psi(h, h)$$

As $h \to 0 \Longrightarrow \Phi, \Psi \to 0\ we\ get,$

$\frac{1}{2} = 0$ which is absurd.

∴ f (x, y) is not differentiable at the origin.

==

Ex. Discuss the continuity and differentiability at the origin of the function

$f(x, y) = \frac{xy}{x^2+y^2}$, $if\ (x, y) \neq (0, 0)$

$= 0$, $if\ (x, y) = (0, 0)$

Sol. Let f $(x, y) = \frac{xy}{x^2+y^2}$, $if\ (x, y) \neq (0, 0)$

$= 0$, $if\ (x, y) = (0, 0)$

i. e. f (0, 0) = 0 is defined.

Let $L = \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

For the path y = 0, we have

$$L = \lim_{x\to 0} \frac{0}{x^2+0} = 0$$

For the path y = x, we have

$$L = \lim_{x\to 0} \frac{x^2}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{x^2}{2x^2}$$

$$= \lim_{x\to 0} \frac{1}{2}$$

$$= \frac{1}{2}$$

For two different paths we get two different limits.

$\therefore \lim_{(x,y)\to(0,0)} f(x, y)$ does not exists.

$\therefore$ the given function f (x, y) is not continuous at (0, 0).

As if function is not continuous then is not differentiable.

Hence the given function f (x, y) is neither continuous nor differentiable at (0, 0).

==

Ex. Discuss the differentiability of a function at (0, 0).

Where $f(x, y) = \frac{x^4+y^4}{x^2+y^2}$, when $x^2 + y^2 \neq 0$ and f (0, 0) = 0

Proof. Let $f(x, y) = \frac{x^4+y^4}{x^2+y^2}$, when $x^2 + y^2 \neq 0$ and f (0, 0) = 0

$$\therefore f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{1}{h}\{\frac{h^4+0}{h^2+0} - 0\} = \lim_{h\to 0} h = 0$$

$$\& \ f_y(0, 0) = \lim_{k\to 0}\{\frac{f(0, k) - f(0, 0)}{k}\} = \lim_{k\to 0} \frac{1}{k}\{\frac{0+k^4}{0+k^2} - 0\} = \lim_{k\to 0} k = 0$$

i. e. first partial derivatives are exists.

Now consider

$$\delta f = f(h, k) - f(0, 0) = \frac{h^4+k^4}{h^2+k^2} - 0 = h(0) + k(0) + h\{\frac{h^3}{h^2+k^2}\} + k\{\frac{k^3}{h^2+k^2}\}$$

$$= hf_x(0, 0) + kf_y(0, 0) + h\Phi(h, k) + k\Psi(h, k)$$

Where $\Phi = \frac{h^3}{h^2+k^2}$, $\Psi = \frac{k^3}{h^2+k^2} \to 0$ as $(h, k) \to (0, 0)$.

$\therefore$ f (x, y) is differentiable at (0, 0).

==

Ex. Using differentials find approximate value of $\sqrt{(1.02)^2 + (1.97)^3}$

Sol. Let $f(x, y) = \sqrt{x^2 + y^3} = (x^2 + y^3)^{1/2}$

$$\therefore f_x(x, y) = \frac{1}{2}(x^2 + y^3)^{-1/2}(2x) = \frac{x}{(x^2+y^3)^{1/2}}$$

$$f_y(x, y) = \frac{1}{2}(x^2 + y^3)^{-1/2}(3y^2) = \frac{3y^2}{2(x^2+y^3)^{1/2}}$$

Using differentials approximate value is given by

$$f(a+h, b+k) \doteqdot f(a, b) + hf_x(a, b) + kf_y(a, b)$$

By taking a = 1, b = 2, h = 0.02 and k = – 0.03 we get,

$f(1.02, 1.97) \doteqdot f(1,2) + (0.02)f_x(1,2) + (-0.03)f_y(1,2)$

$$\sqrt{(1.02)^2 + (1.97)^3} \doteqdot \sqrt{(1)^2 + (2)^3} + (0.02)\left\{\frac{1}{(1^2+2^3)^{1/2}}\right\} - (0.03)\left\{\frac{3(2)^2}{2(1^2+2^3)^{1/2}}\right\}$$

$$\doteqdot 3 + \frac{0.02}{3} - \frac{0.36}{6}$$

$$\doteqdot 3 + 0.0067 - 0.06$$

$\therefore \sqrt{(1.02)^2 + (1.97)^3} \doteqdot 2.9467$

==

Ex. Using differentials find approximate value of $(3.9)^2(2.05) + (2.05)^3$.

Sol. Let $f(x, y) = x^2y + y^3$ $\quad\therefore f_x(x, y) = 2xy$ & $f_y(x, y) = x^2 + 3y^2$

Using differentials approximate value is given by

$f(a+h, b+k) \doteqdot f(a, b) + hf_x(a, b) + kf_y(a, b)$

By taking $a = 4$, $b = 2$, $h = -0.1$ and $k = 0.05$ we get,

$f(3.9, 2.05) \doteqdot f(4, 2) + (-0.1)f_x(4, 2) + (0.05)f_y(4, 2)$

$(3.9)^2(2.05) + (2.05)^3 \doteqdot \{(4)^2(2) + 2^3\} - (0.1)(2\times 4 \times 2) + (0.05)\{4^2+3\times2^2\}$

$\doteqdot 40 - 1.60 + 1.40$

$(3.9)^2(2.05) + (2.05)^3 \doteqdot 39.80$

==

Ex. Find the approximate value of $(5.12)^2(6.85) - 3(6.85)$.

Sol. Let $f(x, y) = x^2y - 3y$ $\quad\therefore f_x(x, y) = 2xy$ & $f_y(x, y) = x^2 - 3$

Using differentials approximate value is given by

$f(a+h, b+k) \doteqdot f(a, b) + hf_x(a, b) + kf_y(a, b)$

By taking $a = 5$, $b = 7$, $h = 0.12$ and $k = -0.15$ we get,

$f(5.12, 6.85) \doteqdot f(5, 7) + (0.12)f_x(5, 7) + (-0.15)f_y(5, 7)$

$(5.12)^2(6.85) - 3(6.85) \doteqdot \{(5)^2(7) - 3\times 7\} + (0.12)(2\times 5 \times 7) - (0.15)\{5^2-3\}$

$\doteqdot 154 + 8.40 - 3.30$

$(5.12)^2(6.85) - 3(6.85) \doteqdot 159.10$

==

Schwarz's Theorem:

If f_y exists in a neighbourood of a point (a, b) of a domain of a function f and f_{yx} is continuous at (a, b) then $f_{xy}(a, b)$ exists, and $f_{yx}(a, b) = f_{xy}(a, b)$.

Proof: By the given conditions f_x, f_y and f_{yx} all exists in a neighbourood of a point (a, b). Let (a+h, b+k) be any point lies in this neighbourood.

Let $\Phi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$

Write $G(x) = f(x, b+k) - f(x, b)$

$\therefore \Phi(h, k) = G(a+h) - G(a)$

$\because$ f_x exists in a neighbourood of (a, b) $\Longrightarrow$ G(x) is differentiable in (a, a+h) with

$G'(x) = f_x(x, b+k) - f_x(x, b)$.

$\therefore$ By Lagranges M.V.T. we get,

$\Phi(h, k) = hG'(a+\theta h)$, where $0 < \theta < 1$

$= h\{ f_x(a+\theta h, b+k) - f_x(a+\theta h, b)\}$

Again f_{yx} exists in a neighbourood of (a, b) $\Longrightarrow$ f_x is differentiable in (b, b+k).

$\therefore$ By Lagranges M.V.T. we get,

$\Phi(h, k) = hkf_{yx}(a+\theta h, b+\theta' k)$ where $0 < \theta' < 1$

$\therefore \frac{\Phi(h,k)}{hk} = f_{yx}(a+\theta h, b+\theta' k)$

$\therefore \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk} = f_{yx}(a+\theta h, b+\theta' k)$

$\therefore \frac{1}{h}\{\frac{f(a+h,b+k) - f(a+h,b)}{k} - \frac{f(a,b+k) - f(a,b)}{k}\} = f_{yx}(a+\theta h, b+\theta' k)$

By taking limit as $k \to 0$ on both sides, we get,

$\frac{f_y(a+h, b) - f_y(a, b)}{h} = f_{yx}(a+\theta h, b)$ $\because$ f_y and f_{yx} are exists in a nhd of (a, b)

Again taking limit as $h \to 0$ on both sides, we get,

$f_{xy}(a, b) = f_{yx}(a, b)$

Hence proved.

==

Young's Theorem:

If f_x and f_y both are differentiable at a point (a, b) of a domain of a function f, then $f_{yx}(a, b) = f_{xy}(a, b)$.

Proof: By the given conditions f_x and f_y both are differentiable at a point (a, b) of a domain of a function f.

$\therefore$ f_{xx}, f_{xy}, f_{yx} and f_{yy} are exists at point (a, b) and its neighbourood.

Let (a+h, b+h) be any point lies in this neighbourood.

Let $\Phi(h, h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a, b)$

Write G(x) = f (x, b+h) – f (x, b)

$\therefore \Phi$(h, h) = G(a+h) – G(a)

$\because$ f_x exists in a neighbourood of (a, b) $\Longrightarrow$ G(x) is differentiable in (a, a+h) with

G'(x) = f_x(x, b+h) – f_x(x, b)

$\therefore$ By Lagranges M.V.T. we get,

Φ(h, h) = hG'(a+θh), where 0 < θ < 1

= h{ f_x(a+θh, b+h) – f_x(a+θh, b)}

= h{ [f_x(a+θh, b+h) – f_x(a, b)] – [f_x(a+θh, b) – f_x(a, b)]} ….(1)

As f_x is differentiable at point (a, b) $\Longrightarrow$

f_x(a+θh, b+h) – f_x(a, b) = θh f_{xx}(a, b) + hf_{yx}(a, b) + θhΦ_1(h, h)+hΨ_1(h, h)

and f_x(a+θh, b) – f_x(a, b) = θh f_{xx}(a, b) + θhΦ_2(h, 0)

Putting these values in equation (1), we get,

Φ(h, h) = h{ hf_{yx}(a, b) + θhΦ_1(h, h) + hΨ_1(h, h) – θhΦ_2(h, 0)}

$\therefore \frac{\Phi(h,h)}{h^2}$ = f_{yx}(a, b) + $\theta\Phi_1$(h, h)+Ψ_1(h, h) – $\theta\Phi_2$(h, 0)

By taking limit as h→0 on both sides, we get,

$\lim_{h\to0}\frac{\Phi(h,h)}{h^2}$ = f_{yx}(a, b) $\because$ $\Phi_1,\Psi_1,\Phi_2\to0$ as h→0

Similarly, if we consider H(y) = f (a+h, y) – f (a, y) and proceed as above, we can obtain $\lim_{h\to0}\frac{\Phi(h,h)}{h^2}$ = f_{xy}(a, b)

$\therefore$ f_{xy}(a, b) = f_{yx}(a, b) Hence proved.

==

Note: i) If both f_{xy} and f_{yx} are continuous at (a, b), then f_{xy}(a, b) = f_{yx}(a, b).

ii) The conditions in Schwarz's & Young's Theorem are sufficient but they are not necessary.

==

Ex. Show that for the function

$$f(x, y) = \frac{x^2y^2}{x^2+y^2}, \; if\ (x, y) \neq (0, 0)$$
$$= 0, \; if\ (x, y) = (0, 0)$$

fxy (0, 0) = fyx (0, 0), even though the conditions of Schwarz's theorem and Young's theorem are not satisfied.

Proof: We have Let f $(x, y) = \frac{x^2y^2}{x^2+y^2}$, (x, y) ≠ (0, 0) and f (0, 0) = 0

∴ i) f (0, 0) = 0, f (h, 0) = 0, f (0, k) = 0 and f (h, k) = $\frac{h^2k^2}{h^2+k^2}$

ii) $f_x(0, k) = \lim\limits_{h\to 0} \frac{f(0+h,\ k)-f(0,\ k)}{h} = \lim\limits_{h\to 0} \frac{1}{h}\{\frac{h^2k^2}{h^2+k^2} - 0\}$

$= \lim\limits_{h\to 0} \frac{hk^2}{h^2+k^2} = \lim\limits_{h\to 0} \frac{0}{0+k^2} = 0$

i.e. $f_x(0, k) = 0$ ∴ $f_x(0, 0) = 0$

and $f_y(h, 0) = \lim\limits_{k\to 0} \frac{f(h,\ 0+k)-f(h,\ 0)}{k} = \lim\limits_{k\to 0} \frac{1}{k}\{\frac{h^2k^2}{h^2+k^2} - 0\}$

$= \lim\limits_{k\to 0} \frac{h^2k}{h^2+k^2} = \lim\limits_{k\to 0} \frac{0}{h^2+0} = 0$

i.e. $f_y(h, 0) = 0$ ∴ $f_y(0, 0) = 0$

iii) Now $f_{xy}(0, 0) = (f_y)_x(0, 0) = \lim\limits_{h\to 0} \frac{f_y(h,\ 0)-f_y(0,\ 0)}{h}$

$= \lim\limits_{h\to 0} \frac{0-0}{h} = 0$………(i)

& $f_{yx}(0, 0) = (f_x)_y(0, 0) = \lim\limits_{k\to 0} \frac{f_x(0,\ k)-f_x(0,\ 0)}{k}$

$= \lim\limits_{k\to 0} \frac{0-0}{k}$

i.e. $f_{yx}(0, 0) = 0$ ……….(ii)

By equation (i) and (ii), $f_{xy}(0, 0) = f_{yx}(0, 0)$ is proved.

Now $f_x(x, y) = y^2\{\frac{(x^2+y^2)(2x)-(x^2)(2x)}{(x^2+y^2)^2}\} = \frac{2xy^4}{(x^2+y^2)^2}$

∴ $f_{yx}(x, y) = 2x\{\frac{(x^2+y^2)^2(4y^3)-(y^4)2(x^2+y^2)(2y)}{(x^2+y^2)^4} = 2x\{\frac{(x^2+y^2)(4y^3)-(4y^5)}{(x^2+y^2)^3}\}$

$= 2x\{\frac{(4x^2y^3)}{(x^2+y^2)^3}\} = \frac{8x^3y^3}{(x^2+y^2)^3}$

$L = \lim\limits_{(x,y)\to(0,0)} f_{yx}(x, y) = \lim\limits_{(x,y)\to(0,0)} \frac{8x^3y^3}{(x^2+y^2)^3}$

For the path y = 0, we have

$L = \lim\limits_{x\to 0} \frac{0}{x^6} = 0$

For the path y = x, we have

$L = \lim\limits_{x\to 0} \frac{8x^6}{8x^6} = \lim\limits_{x\to 0} 1 = 1$

For two different paths we get two different limits.

∴ $\lim\limits_{(x,y)\to(0,0)} f_{yx}(x, y)$ does not exists.

$\therefore$ $f_{yx}(x, y)$ is not continuous at (0, 0) i.e. condition of Schwarz's theorem is not satisfied.

Suppose $f_x(x, y)$ is differentiable at (0, 0).

$\delta f = f_x(0+h, 0+k) - f_x(0, 0) = hf_{xx}(0, 0) + kf_{yx}(0, 0) + h\Phi(h, k) + k\Psi(h, k)$ …….(1)

where $\Phi, \Psi \to 0$ *as* $(h, k) \to (0, 0)$.

From equation (1), we have,

$\frac{2hk^4}{(h^2+k^2)^2} - 0 = 0 + 0 + h\Phi(h, k) + k\Psi(h, k)$

$\because$ $f_x(0, 0)$ gives $f_{xx}(0, 0) = 0$ & $f_{yx}(0, 0) = 0$

i.e. $\frac{2hk^4}{(h^2+k^2)^2} = h\Phi(h, k) + k\Psi(h, k)$

Putting k = h, we get,

$\frac{2h^5}{(h^2+h^2)^2} = h\Phi(h, h) + h\Psi(h, h)$

$\frac{1}{2} = \Phi(h, h) + \Psi(h, h)$

As $h \to 0 \Longrightarrow \Phi, \Psi \to 0$ *we get,*

$\frac{1}{2} = 0$ which is absurd.

$\therefore$ $f_x(x, y)$ is not differentiable at (0, 0). i.e. condition of Young's theorem is not satisfied. But $f_{xy}(0, 0) = f_{yx}(0, 0)$.

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UNIT- 1: FUNCTIONS OF TWO AND THREE VARIABLES [MCQ'S]

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1) A set $\delta N(a, b) = \{(x, y)/ \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$ is called … … … … of a point (a, b) in xy-plane.

a) δ neighbourhood b) deleted δ neighbourhood c) None of these

2) A set $\delta N'(a, b) = \{(x, y)/ 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$ is called ………… of a point (a, b) in xy-plane.

a) δ neighbourhood b) deleted δ neighbourhood c) None of these

3) $\lim_{(x,y)\to(0,0)} \frac{x-a}{y-b}$ along the path y = 0 is ……

a) 0 b) $\frac{a}{b}$ c) $-\frac{a}{b}$ d) None of these

4) $\lim_{(x,y)\to(0,0)} \frac{x-1}{y-1}$ along the path y = 2x is ……

a) 1 b) $\frac{1}{2}$ c) $-\frac{1}{2}$ d) None of these

5) Along the path y = x, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ =…..

a) 1　b) 0　c) -1　d) None of these

6) Along the path x = 0, $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y)}{x+y}$=…..

a) 1　b) -1　c) 0　d) None of these

7) Along the path y = 0, $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y)}{x+y}$=…..

a) 1　b) -1　c) 0　d) None of these

8) Along the path x = 0, $\lim\limits_{(x,y)\to(0,0)} \frac{\tan(x^2+y)}{x+y}$=…..

a) 1　b) -1　c) 0　d) None of these

9) Along the path y = 0, $\lim\limits_{(x,y)\to(0,0)} \frac{\tan(x^2+y)}{x+y}$=…..

a) 1　b) -1　c) 0　d) None of these

10) Along the path y^3 = x, $\lim\limits_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ =…..

a) 1　b) -1　c) $\frac{1}{2}$　d) None of these

11) Along the path x^2 = y, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ =…..

a) 1　b) -1　c) $\frac{1}{2}$　d) None of these

8) $\lim\limits_{y\to0} \lim\limits_{x\to0} x \sin\frac{1}{y}$ =…..

a) 1　b) 0　c) $\frac{1}{2}$　d) None of these

9) $\lim\limits_{y\to0}[\lim\limits_{x\to0} \frac{x^2y^2}{x^4+y^4-x^2y^2}]$ =…..

a) 1　b) 0　c) $\frac{1}{2}$　d) None of these

10) A function f(x, y) is said to be continuous at point (a, b) if f (a, b) is defined, $\lim\limits_{(x,\ y)\to(a,\ b)}$ f (x, y) is exist and $\lim\limits_{(x,\ y)\to(a,\ b)}$ f (x, y) …. f (a, b).

a) =　b) <　c) >　d) ≠

11) If $\lim\limits_{h\to0} \frac{f(x+h,\ y)-f(x,\ y)}{h}$ is exist, then it is denoted by……

a) $f_y(x, y)$　b) $f_x(x, y)$　c) $f_{xx}(x, y)$　d) $f_{yy}(x, y)$

12) If $\lim\limits_{k\to0} \frac{f(x,\ y+k)-f(x,\ y)}{k}$ is exist, then it is denoted by……

a) $f_y(x, y)$　b) $f_x(x, y)$　c) $f_{xx}(x, y)$　d) $f_{yy}(x, y)$

13) If u = $x^3z + xy^2 - 2yz$ then $\frac{\partial u}{\partial x}$ at point (1, 2, 3) is………..

a) 13　b) -2　c) -3　d) None of these

14) If u = $x^3z + xy^2 - 2yz$ then $\frac{\partial u}{\partial y}$ at point (1, 2, 3) is ……….

a) 13 b) -2 c) -3 d) None of these

15) If u = $x^3z + xy^2 - 2yz$ then $\frac{\partial u}{\partial z}$ at point (1, 2, 3) is ……….

a) 13 b) -2 c) -3 d) None of these

16) If u = $e^x \sin xy$ then $\frac{\partial u}{\partial x}$ at point (0, 0) is ………..

a) 0 b) 1 c) 2 d) None of these

17) If u = $e^x \sin xy$ then $\frac{\partial u}{\partial y}$ at point (0, 0) is …………

a) 0 b) 1 c) 2 d) None of these

18) If u = $\tan^{-1}\frac{y}{x}$ then $\frac{\partial u}{\partial x}$ = ……….

a) $\frac{x}{x^2+y^2}$ b) $\frac{y}{x^2+y^2}$ c) $\frac{-y}{x^2+y^2}$ d) None of these

19) If u = $\tan^{-1}\frac{y}{x}$ then $\frac{\partial u}{\partial y}$ = ……….

a) $\frac{x}{x^2+y^2}$ b) $\frac{y}{x^2+y^2}$ c) $\frac{-y}{x^2+y^2}$ d) None of these

20) If u = $\log(x^2+y^2+z^2)$ then $\frac{\partial u}{\partial x}$ = ……….

a) $\frac{2z}{x^2+y^2+z^2}$ b) $\frac{2y}{x^2+y^2+z^2}$ c) $\frac{2x}{x^2+y^2+z^2}$ d) None of these

21) If u = $x^2y + y^2z + z^2x$ then $\frac{\partial u}{\partial y}$ at (1, 0, -1) ……….

a) 0 b) 1 c) -1 d) None of these

22) If u = $\log(\tan x + \tan y + \tan z)$ then $\frac{\partial u}{\partial z}$ = ……….

a) $\frac{\sec^2 z}{\tan x + \tan y + \tan z}$ b) $\frac{\sec^2 y}{\tan x + \tan y + \tan z}$ c) $\frac{\sec^2 x}{\tan x + \tan y + \tan z}$ d) None of these

23) $\lim\limits_{h\to 0} \frac{f_x(a+h,\ b)-f_x(a,\ b)}{h}$ = …………

a) $f_{xx}(a, b)$ b) $f_{xy}(a, b)$ c) $f_{yx}(a, b)$ d) $f_{yy}(a, b)$

24) $\lim\limits_{h\to 0} \frac{f_y(a+h,\ b)-f_y(a,\ b)}{h}$ = …………

a) $f_{xx}(a, b)$ b) $f_{xy}(a, b)$ c) $f_{yx}(a, b)$ d) $f_{yy}(a, b)$

25) $\lim\limits_{k\to 0} \frac{f_x(a,\ b+k)-f_x(a,\ b)}{k}$ = …………

a) $f_{xx}(a, b)$ b) $f_{xy}(a, b)$ c) $f_{yx}(a, b)$ d) $f_{yy}(a, b)$

26) $\lim\limits_{k\to 0} \frac{f_y(a,\ b+k)-f_y(a,\ b)}{k}$ = …………

a) $f_{xx}(a, b)$ b) $f_{xy}(a, b)$ c) $f_{yx}(a, b)$ d) $f_{yy}(a, b)$

27) If u = f (x, y) is a differentiable function of two variables x and y, then

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ is called ………of u.

a) differential b) derivative c) partial derivative d) None of these

28) By differentials approximate value f(a+h, b+k) is given by

a) $f(a, b)+hf_x(a, b)+kf_x(a, b)$ b) $f(a, b)+hf_x(a, b)+kf_y(a, b)$

c) $f(a, b)+hf_y(a, b)+kf_x(a, b)$ d) $f(a, b)+hf_y(a, b)+kf_y(a, b)$

29) Every continuous function f (x, y) is …….

a) always differentiable b) always not differentiable

b) may or may not be differentiable d) None of these

30) If f_y exists in a neighbourood of a point (a, b) of a domain of a function f and f_{yx} is continuous at (a, b) then $f_{xy}(a, b)$ exists, and ……….

a) $f_{yx}(a, b) = f_{xy}(a, b)$ b) $f_{xx}(a, b) = f_{xy}(a, b)$

c) $f_{yx}(a, b) = f_{yy}(a, b)$ d) $f_{xx}(a, b) = f_{yy}(a, b)$

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UNIT-2: JACOBIAN, COMPOSITE FUNCTIONS AND MEAN VALUE THEOREM

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Jacobians:

If u and v are functions of two independent variables x and y, then

$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called jacobian of u and v w.r.to x and y.

Jacobians:

If u, v and w are functions of three independent variables x, y and z, then

$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ is called jacobian of u, v and w w.r.to x, y and z.

Note: i) $J\left(\frac{u,v}{x,y}\right) J\left(\frac{x,y}{u,v}\right) = 1$ i.e. $J\left(\frac{x,y}{u,v}\right) = \frac{1}{J\left(\frac{u,v}{x,y}\right)}$

ii) $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(r,\theta)}$

Functionally Dependent Functions: Functions u, v and w of three independent variables x, y and z are functionally dependent or functionally related if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

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Ex. If u = x^2, v = y^2, find $\frac{\partial(u,v)}{\partial(x,y)}$,

Sol. Let u = x^2, v = y^2

$\therefore \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 0$

$\& \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2y$

$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$

==

Ex. If u = x(1-y), v = xy, then show that $\frac{\partial(u,v)}{\partial(x,y)}$ = u + v.

Proof. Let u = x(1-y), v = xy

$\therefore \frac{\partial u}{\partial x} = 1-y, \frac{\partial u}{\partial y} = -x$

$\& \frac{\partial v}{\partial x} = y, \frac{\partial v}{\partial y} = x$

$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$

$= x - xy$ + xy

= x(1-y) + xy

$\therefore \frac{\partial(u,v)}{\partial(x,y)}$ = u + v.

Hence proved.

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Ex. If x = rcosθ, y = rsinθ, then evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ & $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Sol. Let x = rcosθ, y = rsinθ

$\therefore \frac{\partial x}{\partial r}$ = cosθ, $\frac{\partial x}{\partial \theta}$ = -rsinθ

& $\frac{\partial y}{\partial r}$ = sinθ, $\frac{\partial y}{\partial \theta}$ = rcosθ

$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = rcos^2\theta + rsin^2\theta = r$

Now $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}}$ gives $\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$.

==

Ex. Find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$,

where u = x^2-y^2, v = 2xy and x = rcosθ, y = rsinθ

Sol. Let u = x^2-y^2, v = 2xy

$\therefore \frac{\partial u}{\partial x}$ = 2x, $\frac{\partial u}{\partial y}$ = -2y

& $\frac{\partial v}{\partial x}$ = 2y, $\frac{\partial v}{\partial y}$ = 2x

$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2 = 4(r^2cos^2\theta + r^2sin^2\theta) = 4r^2$

Again x = rcosθ, y = rsinθ

$\therefore \frac{\partial x}{\partial r}$ = cosθ, $\frac{\partial x}{\partial \theta}$ = -rsinθ

& $\frac{\partial y}{\partial r}$ = sinθ, $\frac{\partial y}{\partial \theta}$ = rcosθ

$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = rcos^2\theta + rsin^2\theta = r$

Now $\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(r,\theta)}$ gives

$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^2.\text{r} = 4r^3$

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Ex. If x = rcosθ, y = rsinθ, z = z then evaluate $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Sol. Let x = rcosθ, y = rsinθ, z = z

$\therefore \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial x}{\partial z} = 0$

$\frac{\partial y}{\partial r} = \sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta, \frac{\partial y}{\partial z} = 0$

& $\frac{\partial z}{\partial r} = 0, \frac{\partial z}{\partial \theta} = 0, \frac{\partial z}{\partial z} = 1$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta = \text{r}$

===

Ex. Verify whether the following functions are functionally dependent, and if so, find the relation between them. u = $\frac{x+y}{1-xy}$ and v = $\tan^{-1}x + \tan^{-1}y$.

Proof. Let u = $\frac{x+y}{1-xy}$ and v = $\tan^{-1}x + \tan^{-1}y$

$\therefore \frac{\partial u}{\partial x} = \frac{1-xy-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}, \frac{\partial u}{\partial y} = \frac{1-xy-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$

& $\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \frac{\partial v}{\partial y} = \frac{1}{1+y^2},$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$\therefore \frac{\partial(u,v)}{\partial(x,y)} = 0$

Hence u = $\frac{x+y}{1-xy}$ and v = $\tan^{-1}x + \tan^{-1}y$ are functionally dependent is proved.

Now v = $\tan^{-1}x + \tan^{-1}y$ = = $\tan^{-1}(\frac{x+y}{1-xy}) = \tan^{-1}u$

$\therefore$ u = tanv be the relation between them.

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Ex. Show that u = xy + yz + zx, v = $x^2 + y^2 + z^2$ and w = x + y + z are functionally related.

Proof. Let u = xy + yz + zx, v = $x^2 + y^2 + z^2$ and w = x + y + z

$\therefore \frac{\partial u}{\partial x}$ = y + z, $\frac{\partial u}{\partial y}$ = x + z, $\frac{\partial u}{\partial z}$ = y + x

$\frac{\partial v}{\partial x}$ = 2x, $\frac{\partial v}{\partial y}$ = 2y, $\frac{\partial v}{\partial z}$ = 2z

& $\frac{\partial w}{\partial x}$ = 1, $\frac{\partial w}{\partial y}$ = 1, $\frac{\partial w}{\partial z}$ = 1

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \mathrm{y+z} & \mathrm{x+y} & \mathrm{y+x} \\ 2\mathrm{x} & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} \mathrm{y+z} & \mathrm{x+z} & \mathrm{y+x} \\ \mathrm{x} & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} \mathrm{y+z+x} & \mathrm{x+z+y} & \mathrm{y+x+z} \\ \mathrm{x} & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad \text{by } R_1 + R_2$$

$$= 2(\mathrm{x+y+z}) \begin{vmatrix} 1 & 1 & 1 \\ \mathrm{x} & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(\mathrm{x+y+z})(0) \qquad \because R_1 = R_3$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Hence u = xy + yz + zx, v = $x^2 + y^2 + z^2$ and w = x + y + z are functionally related is proved.

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Ex. If u = cosx, v = $sinxcosy$ and w = sinxsinycosz then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 sin^3 x sin^2 y sinz$$

Proof. Let u = cosx, v = $sinxcosy$ and w = sinxsinycosz

$\therefore \frac{\partial u}{\partial x}$ = -sinx, $\frac{\partial u}{\partial y}$ = 0, $\frac{\partial u}{\partial z}$ = 0

$\frac{\partial v}{\partial x}$ = cosxcosy, $\frac{\partial v}{\partial y}$ = -sinxsiny, $\frac{\partial v}{\partial z}$ = 0

& $\frac{\partial w}{\partial x}$ = cosxsinycosz, $\frac{\partial w}{\partial y}$ = sinxcosycosz, $\frac{\partial w}{\partial z}$ = -sinxsinysinz

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\text{sinx} & 0 & 0 \\ \text{cosxcosy} & -sinxsiny & 0 \\ \text{cosxsinycosz} & \text{sinxcosycosz} & -\text{sinxsinysinz} \end{vmatrix}$$

= (-sinx)(-sinxsiny)(-sinxsinysinz)

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 sin^3 x sin^2 y sinz$$

Hence proved.

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Composite Function:

If u is a function of two variables x, y and x, y are functions of a real variable t, then u is said to be composite function of variable t.

Composite Function:

If w is a function of two variables u, v and u, v are functions of two variables x, y then w is said to be composite function of variables x, y.

Chain Rule-I: If u = f (x, y) is a differential function of x, y and x = $\emptyset(t)$, y = $\Psi(t)$ are differential functions of t, then composite function u = f [$\emptyset(t)$, $\Psi(t)$] is differential function of t and $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

Proof: Let δx, δy and δu are the increments in x, y and u respectively, corresponding to the increment δt in t,

Let u = f (x, y) is a differential function of x, y.

$\therefore \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \alpha\delta x + \beta\delta y = (\frac{\partial u}{\partial x} + \alpha)\delta x + (\frac{\partial u}{\partial y} + \beta)\delta y$ …….. (1)

Where $\alpha, \beta \to 0$ as $(\delta x, \delta y) \to (0,0)$.

Dividing equation (1) by δt and taking limit as $\delta t \to 0$, we get,

$$\lim_{\delta t\to 0}\frac{\delta u}{\delta t} = \lim_{\delta t\to 0}[\left(\frac{\partial u}{\partial x} + \alpha\right)\frac{\delta x}{\delta t} + (\frac{\partial u}{\partial y} + \beta)\frac{\delta y}{\delta t}]$$

As x = $\emptyset(t)$, y = $\Psi(t)$ are differential functions of t,

$\therefore \lim_{\delta t\to 0}\frac{\delta x}{\delta t} = \frac{dx}{dt}, \lim_{\delta t\to 0}\frac{\delta y}{\delta t} = \frac{dy}{dt}$ and every differentiable function is continuous,

$\therefore \delta t \to 0 \Longrightarrow \delta x, \delta y \to 0$ and hence $\alpha, \beta \to 0$.

$$\therefore \lim_{\delta t\to 0}\frac{\delta u}{\delta t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$\therefore$ The composite function u = f [$\emptyset(t)$, $\Psi(t)$] is differential function of t and $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ is proved.

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Chain Rule-II: If w = f (u, v) is a differential function of u, v and u = $\emptyset(x, y)$, v = $\Psi(x, y)$ are differential functions of x and y, then composite function w = f [$\emptyset(x, y), \Psi(x, y)$] is differential function of x and y and its partial derivatives are given by

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \text{ and } \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Proof: Let $\delta u, \delta v$ and δw are the increments in u, v and w respectively, corresponding to the increments δx in x and δy in y,

Let w = f (u, v) is a differential function of two variables u, v.

$$\therefore \delta w = \frac{\partial w}{\partial u}\delta u + \frac{\partial w}{\partial v}\delta v + \alpha_1\delta u + \beta_1\delta v$$

$$\therefore \delta w = (\frac{\partial w}{\partial u} + \alpha_1)\delta u + (\frac{\partial w}{\partial v} + \beta_1)\delta v \text{ ....(1)}$$

Where $\alpha_1, \beta_1 \rightarrow 0$ as $(\delta u, \delta v) \rightarrow (0, 0)$.

Again u = $\emptyset(x, y)$, v = $\Psi(x, y)$ are differential functions of x and y

$$\therefore \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \alpha_2\delta x + \beta_2\delta y$$

$$\& \quad \delta v = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \alpha_3\delta x + \beta_3\delta y$$

$$\text{i.e. } \delta u = (\frac{\partial u}{\partial x} + \alpha_2)\delta x + (\frac{\partial u}{\partial y} + \beta_2)\delta y \text{ ....(2)}$$

$$\& \quad \delta v = (\frac{\partial v}{\partial x} + \alpha_3)\delta x + (\frac{\partial v}{\partial y} + \beta_3)\delta y \text{ ....(3)}$$

Where $\alpha_2, \beta_2, \alpha_3, \beta_3 \rightarrow 0$ as $(\delta x, \delta x) \rightarrow (0, 0)$.

By using equation (2) and (3) equation (1) becomes

$$\delta w = (\frac{\partial w}{\partial u} + \alpha_1)[(\frac{\partial u}{\partial x} + \alpha_2)\delta x + (\frac{\partial u}{\partial y} + \beta_2)\delta y] + (\frac{\partial w}{\partial v} + \beta_1)[(\frac{\partial v}{\partial x} + \alpha_3)\delta x + (\frac{\partial v}{\partial y} + \beta_3)\delta y]$$

$$\delta w = (\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x})\delta x + (\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y})\delta y + \alpha\delta x + \beta\delta y$$

We observe that each term in α and β contain at least one of $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$.

$\therefore \delta x, \delta y \rightarrow 0 \Longrightarrow \alpha, \beta \rightarrow 0$.

∴ By definition of differentiability, the composite function w = f [$\emptyset(x, y), \Psi(x, y)$] is differential function of x and y with

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \text{ and } \frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}.$$ Hence proved.

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Remark: If u = f (x, y, z) is a differential function of x, y, z and x, y, z are differential functions of t, then composite function u is differential function of t and

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

==

Ex. Find $\frac{dz}{dt}$ when z = $xy^2 + x^2y$, x = at^2, y = 2at

Sol. Let z = $xy^2 + x^2y$, x = at^2, y = 2at.

$$\therefore \frac{\partial z}{\partial x} = y^2 + 2xy, \frac{\partial z}{\partial y} = 2xy + x^2, \frac{dx}{dt} = 2at \ \& \ \frac{dy}{dt} = 2a$$

As z is function of x, y and x, y are functions of t.

∴ z is composite function of t.

∴ By using Chain Rule-I, we get,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$

$= (4a^2t^2 + 4a^2t^3)(2at) + (4a^2t^3 + a^2t^4)(2a)$

$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$

$= 16a^3t^3 + 10a^3t^4$

$= 2a^3t^3(8 + 5t)$

==

Ex. If $z = f(x, y) = x^2+y^2$ where $x = t^2 +1$, $y = 2t$, find $\frac{dz}{dt}$ at $t = 1$. (Oct.2019)

Sol. Let $z = f(x, y) = x^2 + y^2$, $x = t^2 +1$, $y = 2t$.

∴ $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = 2y$, $\frac{dx}{dt} = 2t$ & $\frac{dy}{dt} = 2$

As z is function of x, y and x, y are functions of t.

∴ z is composite function of t.

∴ By using Chain Rule-I, we get,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

$= (2x)(2t)+(2y)(2)$

$= 4(xt + y)$

$= 4[(t^2 +1)t+2t]$

$\frac{dz}{dt} = 4(t^3 + 3t)$

At $t = 1$,

$\left[\frac{dz}{dt}\right]_{t=1} = 16.$

==

Ex. Find $\frac{du}{dt}$ if $u = x^3+ y^3$, $x = a\cos t$, $y = b\sin t$

Sol. Let $u = x^3+ y^3$, $x = a\cos t$, $y = b\sin t$.

∴ $\frac{\partial u}{\partial x} = 3x^2$, $\frac{\partial u}{\partial y} = 3y^2$, $\frac{dx}{dt} = -a\sin t$ & $\frac{dy}{dt} = b\cos t$.

As u is function of x, y and x, y are functions of t.

∴ u is composite function of t.

∴ By using Chain Rule-I, we get,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$= (3x^2)(-a\sin t) + (3y^2)(b\cos t)$

$= (3a^2\cos^2 t)(-a\sin t) + (3b^2\sin^2 t)(b\cos t)$

$\frac{du}{dt} = 3\sin t\cos t(b^3\sin t - a^3\cos t)$

==

Ex. If z = f (x, y) where x = r$\cos\theta$, y = r$\sin\theta$, prove that
$\frac{\partial z}{\partial r} = \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}$ & $\frac{\partial z}{\partial \theta} = -r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}$

Proof. Let z = f (x, y) where x = r$\cos\theta$, y = r$\sin\theta$.

$\therefore \frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial r} = \sin\theta$ & $\frac{\partial y}{\partial \theta} = r\cos\theta$.

As z is function of x, y and x, y are functions of r and θ.

$\therefore$ z is composite function of r and θ.

$\therefore$ By using Chain Rule-II,

$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$ & $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$, we get,

$\frac{\partial z}{\partial r} = \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}$ & $\frac{\partial z}{\partial \theta} = -r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}$

Hence proved.

==

Ex. If z = f (x, y)= $tan^{-1}(\frac{x}{y})$ where x = u + v, y = u - v, then show that $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u-v}{u^2+v^2}$

Proof. Let z = f (x, y)= $tan^{-1}(\frac{x}{y})$ where x = u + v, y = u - v.

$\therefore \frac{\partial z}{\partial x} = \frac{1}{1+(\frac{x}{y})^2}\frac{1}{y} = \frac{y}{y^2+x^2} = \frac{u-v}{(u-v)^2+(u+v)^2} = \frac{u-v}{2u^2+2v^2}$

$\frac{\partial z}{\partial y} = \frac{1}{1+(\frac{x}{y})^2}\frac{-x}{y^2} = \frac{-x}{y^2+x^2} = \frac{-(u+v)}{(u-v)^2+(u+v)^2} = \frac{-u-v}{2u^2+2v^2}$

$\frac{\partial x}{\partial u} = 1$, $\frac{\partial x}{\partial v} = 1$, $\frac{\partial y}{\partial u} = 1$ & $\frac{\partial y}{\partial v} = -1$

As z is function of x, y and x, y are functions of u and v.

$\therefore$ z is composite function of u and v.

$\therefore$ By using Chain Rule-II,

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ & $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$, we get,

$\frac{\partial z}{\partial u} = \frac{u-v}{2u^2+2v^2}(1) + \frac{-u-v}{2u^2+2v^2}(1) = \frac{u-v-u-v}{2u^2+2v^2} = \frac{-v}{u^2+v^2}$

& $\frac{\partial z}{\partial v} = \frac{u-v}{2u^2+2v^2}(1) + \frac{-u-v}{2u^2+2v^2}(-1) = \frac{u-v+u+v}{2u^2+2v^2} = \frac{u}{u^2+v^2}$

Adding, we get,

$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{-v}{u^2+v^2} + \frac{u}{u^2+v^2} = \frac{u-v}{u^2+v^2}$

Hence proved.

==

Ex. If z is function of x and y and x = $e^u + e^{-v}$, y = $e^{-u} - e^v$, then show that
$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$

Proof. Let z is function of x and y and x = $e^u + e^{-v}$, y = $e^{-u} - e^v$.

$\therefore \frac{\partial x}{\partial u} = e^u$, $\frac{\partial x}{\partial v} = -e^{-v}$, $\frac{\partial y}{\partial u} = -e^{-u}$ & $\frac{\partial y}{\partial v} = -e^v$

As z is function of x, y and x, y are functions of u and v.

∴ z is composite function of u and v.

∴ By using Chain Rule-II,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \& \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}, \text{ we get,}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(e^{u}) + \frac{\partial z}{\partial y}(-e^{-u}) = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$$

$$\& \ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}) = -e^{-v}\frac{\partial z}{\partial x} - e^{v}\frac{\partial z}{\partial y}$$

Consider

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$= (e^{u} + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^{v})\frac{\partial z}{\partial y}$$

$$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Hence proved.

==

Ex. If z = f (u, v) where u = 2x - 3y, v = x + 2y, then show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}$

Proof. Let z = f (u, v) where u = 2x - 3y, v = x + 2y

$$\therefore \frac{\partial u}{\partial x} = 2, \ \frac{\partial u}{\partial y} = -3, \ \frac{\partial v}{\partial x} = 1 \ \& \ \frac{\partial v}{\partial y} = 2$$

As z is function of u, v and u, v are functions of x and y.

∴ z is composite function of x and y.

∴ By using Chain Rule-II,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}, \text{ we get,}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(2) + \frac{\partial z}{\partial v}(1) = 2\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\& \ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-3) + \frac{\partial z}{\partial v}(2) = -3\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}$$

Adding, we get,

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = -\frac{\partial z}{\partial u} + 3\frac{\partial z}{\partial v} = 3\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}$$

Hence proved.

==

Ex. If $u = f(y - z, \ z - x, x - y)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ (Oct.2019)

Proof. Let $u = f(y - z, \ z - x, x - y) = f(p, q, r)$

where $p = y - z$, $q = z - x$ & $r = x - y$

$$\therefore \frac{\partial p}{\partial x} = 0, \ \frac{\partial p}{\partial y} = 1, \ \frac{\partial p}{\partial z} = -1, \ \frac{\partial q}{\partial x} = -1, \ \frac{\partial q}{\partial y} = 0, \ \frac{\partial q}{\partial z} = 1$$

$$\& \ \frac{\partial r}{\partial x} = 1, \ \frac{\partial r}{\partial y} = -1, \ \frac{\partial r}{\partial z} = 0$$

As u is function of p, q, r and p, q, r are functions of x, y and z.

∴ u is composite function of x, y and z.

∴ By using Chain Rule-II,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

& $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$, we get,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1) = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}$$

& $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0) = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p}$

Adding, we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} = 0$$

Hence proved.

==

Ex. If $u = f(y^2-z^2, z^2-x^2, x^2-y^2)$ then show that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

Proof. Let $u = f(y^2-z^2, z^2-x^2, x^2-y^2) = f(p, q, r)$

where $p = y^2-z^2$, $q = z^2-x^2$ & $r = x^2-y^2$

∴ $\frac{\partial p}{\partial x} = 0$, $\frac{\partial p}{\partial y} = 2y$, $\frac{\partial p}{\partial z} = -2z$, $\frac{\partial q}{\partial x} = -2x$, $\frac{\partial q}{\partial y} = 0$, $\frac{\partial q}{\partial z} = 2z$

& $\frac{\partial r}{\partial x} = 2x$, $\frac{\partial r}{\partial y} = -2y$, $\frac{\partial r}{\partial z} = 0$

As u is function of p, q, r and p, q, r are functions of x, y and z.

∴ u is composite function of x, y and z.

∴ By using Chain Rule-II,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

& $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$, we get,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-2x) + \frac{\partial u}{\partial r}(2x) = 2x\frac{\partial u}{\partial r} - 2x\frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}(2y) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-2y) = 2y\frac{\partial u}{\partial p} - 2y\frac{\partial u}{\partial r}$$

& $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(-2z) + \frac{\partial u}{\partial q}(2z) + \frac{\partial u}{\partial r}(0) = 2z\frac{\partial u}{\partial q} - 2z\frac{\partial u}{\partial p}$

∴ $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial q} + 2\frac{\partial u}{\partial p} - 2\frac{\partial u}{\partial r} + 2\frac{\partial u}{\partial q} - 2\frac{\partial u}{\partial p}$

∴ $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$. Hence proved.

==

Ex. If u = f ($e^{y-z}, e^{z-x}, e^{x-y}$), then show that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Proof. Let u = f ($e^{y-z}, e^{z-x}, e^{x-y}$) = f (p, q, r)

where p = e^{y-z}, q = e^{z-x} & r = e^{x-y}

$\therefore \frac{\partial p}{\partial x}=0,\ \frac{\partial p}{\partial y}=e^{y-z},\ \frac{\partial p}{\partial z}=-e^{y-z}$

$\frac{\partial q}{\partial x}=-e^{z-x},\ \frac{\partial q}{\partial y}=0,\ \frac{\partial q}{\partial z}=e^{z-x}$

& $\frac{\partial r}{\partial x}=e^{x-y},\ \frac{\partial r}{\partial y}=-e^{x-y},\ \frac{\partial r}{\partial z}=0$

As u is function of p, q, r and p, q, r are functions of x, y and z.

∴ u is composite function of x, y and z.

∴ By using Chain Rule-II,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial y}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

& $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial z}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial z}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$, we get,

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}(0)+\frac{\partial u}{\partial q}(-e^{z-x})+\frac{\partial u}{\partial r}(e^{x-y})=e^{x-y}\frac{\partial u}{\partial r}-e^{z-x}\frac{\partial u}{\partial q}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial p}(e^{y-z})+\frac{\partial u}{\partial q}(0)+\frac{\partial u}{\partial r}(-e^{x-y})=e^{y-z}\frac{\partial u}{\partial p}-e^{x-y}\frac{\partial u}{\partial r}$$

& $\frac{\partial u}{\partial z}=\frac{\partial u}{\partial p}(-e^{y-z})+\frac{\partial u}{\partial q}(e^{z-x})+\frac{\partial u}{\partial r}(0)=e^{z-x}\frac{\partial u}{\partial q}-e^{y-z}\frac{\partial u}{\partial p}$

Adding, we get,

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=e^{x-y}\frac{\partial u}{\partial r}-e^{z-x}\frac{\partial u}{\partial q}+e^{y-z}\frac{\partial u}{\partial p}-e^{x-y}\frac{\partial u}{\partial r}+e^{z-x}\frac{\partial u}{\partial q}-e^{y-z}\frac{\partial u}{\partial p}=0$$

$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ Hence proved.

==

Homogeneous Function: A function u = f (x, y) is said to be homogeneous function of degree n, if it can be expressed as u = f (x, y) = $x^n\phi(\frac{y}{x})$.

Homogeneous Function: A function f (x, y, z) is said to be homogeneous function of degree n, if f (xt, yt, zt) = t^nf (x, y, z).

==

Euler's Theorem: If f (x, y) is homogeneous function of degree n in two variables x and y having first order partial derivatives then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$.

Proof: Let f (x, y) is homogeneous function of degree n in two variables x and y

∴ f (x, y) = $x^n\phi(\frac{y}{x})$ ……….(1)

Differentiating (1) partially w.r.to x, we get,

$$\frac{\partial f}{\partial x}=nx^{n-1}\phi(\frac{y}{x})+x^n\phi'(\frac{y}{x})(\frac{-y}{x^2})$$

i.e. $\frac{\partial f}{\partial x} = nx^{n-1}\emptyset(\frac{y}{x}) - yx^{n-2}\emptyset'(\frac{y}{x})$

Multiplying both sides by x, we get,

$x\frac{\partial f}{\partial x} = nx^{n}\emptyset(\frac{y}{x}) - yx^{n-1}\emptyset'(\frac{y}{x})$ ………(2)

Differentiating (1) partially w.r.to y, we get,

$\frac{\partial f}{\partial y} = x^{n}\emptyset'(\frac{y}{x})(\frac{1}{x})$

i.e. $\frac{\partial f}{\partial y} = x^{n-1}\emptyset'(\frac{y}{x})$

Multiplying both sides by y, we get,

$y\frac{\partial f}{\partial y} = yx^{n-1}\emptyset'(\frac{y}{x})$ ………(3)

Adding (2) and (3), we get,

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^{n}\emptyset(\frac{y}{x}) - yx^{n-1}\emptyset'(\frac{y}{x}) + yx^{n-1}\emptyset'(\frac{y}{x}) = nx^{n}\emptyset(\frac{y}{x})$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$. Hence proved.

==

Corollary: If $u = G^{-1}\{x^{n}f(\frac{y}{x})\}$ and $G'(u) \neq 0$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{G(u)}{G'(u)}$.

Proof: Let $u = G^{-1}\{x^{n}f(\frac{y}{x})\}$

$\therefore G(u) = x^{n}f(\frac{y}{x}) = z$

is homogeneous function of degree n in two variables x and y

$\therefore$ By Euler's theorem

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

As $z = G(u) \therefore \frac{\partial z}{\partial x} = G'(u)\frac{\partial u}{\partial x}$ & $\frac{\partial z}{\partial y} = G'(u)\frac{\partial u}{\partial y}$

$\therefore xG'(u)\frac{\partial u}{\partial x} + yG'(u)\frac{\partial u}{\partial y} = nG(u)$.

$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{G(u)}{G'(u)} \qquad \because G'(u) \neq 0$

Hence proved.

==

Corollary: If u = f (x, y) is homogeneous function of degree n in two variables x and y having continuous first and second order partial derivatives, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Proof: Let u = f (x, y) is homogeneous function of degree n in two variables x and y

$\therefore$ By Euler's theorem

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ ……….(1)

Differentiating (1) partially w.r.to x, we get,

$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$

$\therefore x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$

Multiplying both sides by x, we get,

$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (n-1)x\frac{\partial u}{\partial x}$ ………(2)

Differentiating (1) partially w.r.to y, we get,

$x\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y}$

$\therefore x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$ $\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Multiplying both sides by y, we get,

$xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)y\frac{\partial u}{\partial y}$ ………(3)

Adding (2) and (3), we get,

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y})$

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$ Hence proved.

==

Ex. Verify Euler's Theorem for the function $f(x, y) = x^3 + y^3 - 3x^2y$

Proof: Let $f(x, y) = x^3 + y^3 - 3x^2y$ …………(1)

$= x^3[1 + (\frac{y}{x})^3 - 3(\frac{y}{x})]$

i.e. $f(x, y) = x^3\phi(\frac{y}{x})$

$\therefore$ f (x, y) is homogeneous function of degree 3 in two variables x and y

$\therefore$ By Euler's theorem

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$ ……….(2)

Differentiating (1) partially w.r.to x, we get,

$\frac{\partial f}{\partial x} = 3x^2 - 6xy$

Multiplying both sides by x, we get,

$x\frac{\partial f}{\partial x} = 3x^3 - 6x^2y$………(3)

Differentiating (1) partially w.r.to y, we get,

$\frac{\partial f}{\partial y} = 3y^2 - 3x^2$

Multiplying both sides by y, we get,

$y\frac{\partial f}{\partial y} = 3y^3 - 3x^2y$ ………(4)

Adding (3) and (4), we get,

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3x^3 - 6x^2y + 3y^3 - 3x^2y = 3(x^3 + y^3 - 3x^2y)$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f.$ Hence Euler's theorem is verified.

==

Ex. Verify Euler's Theorem for the function f (x, y) = $\tan^{-1}(\frac{x}{y})$ (Oct.2019)

Proof: Let f (x, y) = $\tan^{-1}(\frac{x}{y})$ …………(1)

= $x^0 \tan^{-1}(\frac{x}{y})$

i.e. f (x, y) = $x^0 \emptyset(\frac{y}{x})$

∴ f (x, y) is homogeneous function of degree 0 in two variables x and y

∴ By Euler's theorem

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0f = 0$ ……….(2)

Differentiating (1) partially w.r.to x, we get,

$\frac{\partial f}{\partial x} = \frac{1}{1+(\frac{x}{y})^2}\frac{1}{y} = \frac{y}{y^2+x^2}$

Multiplying both sides by x, we get,

$x\frac{\partial f}{\partial x} = \frac{xy}{x^2+y^2}$………(3)

Differentiating (1) partially w.r.to y, we get,

$\frac{\partial f}{\partial y} = \frac{1}{1+(\frac{x}{y})^2}\frac{-x}{y^2} = \frac{-x}{y^2+x^2}$

Multiplying both sides by y, we get,

$y\frac{\partial f}{\partial y} = \frac{-xy}{x^2+y^2}$ ………(4)

Adding (3) and (4), we get,

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{xy}{x^2+y^2} - \frac{xy}{x^2+y^2} = 0$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0$. Hence Euler's theorem is verified.

==

Ex. If u = $\sin^{-1}(\frac{x^2+y^2}{x+y})$ then find the value of $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$

Sol: Let u = $\sin^{-1}(\frac{x^2+y^2}{x+y})$

∴ sinu = $\frac{x^2+y^2}{x+y}$ = z

∴ z = sinu is homogeneous function of degree 1 in two variables x and y

∴ By Euler's theorem

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1z = z$

As z = sinu ∴ $\frac{\partial z}{\partial x}$ = cosu $\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}$ = cosu $\frac{\partial u}{\partial y}$

∴ $x$cosu $\frac{\partial u}{\partial x}$ + ycosu $\frac{\partial u}{\partial y}$ = sinu

∴ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = tanu

==

Ex.: If u = log ($x^3+y^3-x^2y-xy^2$), prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$ = -3. (Oct.2019)

Proof: Let u = log ($x^3 + y^3 - x^2y - xy^2$)

$\therefore\ e^u = x^3 + y^3 - x^2y - xy^2 = z$

$\therefore$ z = e^u is homogeneous function of degree 3 in two variables x and y

$\therefore$ By Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 3z$$

As z = e^u $\therefore$ $\frac{\partial z}{\partial x} = e^u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = e^u\frac{\partial u}{\partial y}$

$$\therefore\ xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3 \text{ ....(1)}$$

Differentiating (1) partially w.r.to x, we get,

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = 0$$

$$\therefore\ x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = -\frac{\partial u}{\partial x}$$

Multiplying both sides by x, we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y} = -x\frac{\partial u}{\partial x} \text{ ....(2)}$$

Differentiating (1) partially w.r.to y, we get,

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore\ x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial y} \qquad \because \frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$$

Multiplying both sides by y, we get,

$$xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -y\frac{\partial u}{\partial y} \text{ ....(3)}$$

Adding (2) and (3), we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y})$$

i.e. $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -3$ by (1)

Hence proved.

==

Ex.: If u = $\tan^{-1}(\frac{\sqrt{x^2+y^2}}{x-y})$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$.

Sol: Let u = $\tan^{-1}(\frac{\sqrt{x^2+y^2}}{x-y})$= $\tan^{-1}(\frac{\sqrt{1+(\frac{y}{x})^2}}{1-\frac{y}{x}})$

$\therefore\ u = x^0\emptyset(\frac{y}{x})$

$\therefore$ u is homogeneous function of degree 0 in two variables x and y

∴ By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \text{ ………(1)}$$

Differentiating (1) partially w.r.to x, we get,

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = 0$$

$$\therefore x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = -\frac{\partial u}{\partial x}$$

Multiplying both sides by x, we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = -x\frac{\partial u}{\partial x} \text{ ………(2)}$$

Differentiating (1) partially w.r.to y, we get,

$$x\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial y} \quad \because \frac{\partial^2 u}{\partial y\partial x} = \frac{\partial^2 u}{\partial x\partial y}$$

Multiplying both sides by y, we get,

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -y\frac{\partial u}{\partial y} \text{ ………(3)}$$

Adding (2) and (3), we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$ by (1)

===

Ex.: If u = $\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ = sin2u. hence deduce that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$

Proof: Let u = $\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\therefore \tan u = \frac{x^3+y^3}{x-y} = z$$

∴ z = tanu is homogeneous function of degree 2 in two variables x and y

∴ By Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

As z = tanu ∴ $\frac{\partial z}{\partial x} = \sec^2 u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$

$$\therefore x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \frac{2\sin u}{\cos u}\times\cos^2 u = 2\sin u.\cos u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u \text{ ………(1)}$$

Differentiating (1) partially w.r.to x, we get,

$\frac{\partial u}{\partial x}$ + x $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y}$ = 2cos2u$\frac{\partial u}{\partial x}$

∴ x $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (2cos2u - 1)\ \frac{\partial u}{\partial x}$

Multiplying both sides by x, we get,

$x^2 \frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (2cos2u - 1)\ x\frac{\partial u}{\partial x}$ ………(2)

Differentiating (1) partially w.r.to y, we get,

x $\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2}$ = 2cos2u$\frac{\partial u}{\partial y}$

∴ x $\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (2cos2u - 1)\ \frac{\partial u}{\partial y}$ $\quad \because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Multiplying both sides by y, we get,

xy $\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2cos2u - 1)y\frac{\partial u}{\partial y}$ ………(3)

Adding (2) and (3), we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2cos2u - 1)\ x\frac{\partial u}{\partial x} + (2cos2u - 1)y\frac{\partial u}{\partial y}$$

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (2cos2u - 1)(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y})$

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [2(1 - 2sin^2u) - 1]sin2u$ by (1)

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4sin^2u)sin2u$

Hence proved.

===

Ex.: If u = $\sin^{-1}[\frac{x^2+2xy}{\sqrt{x-y}}]^{1/5}$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Sol: Let u = $\sin^{-1}[\frac{x^2+2xy}{\sqrt{x-y}}]^{1/5}$

∴ $sinu = [\frac{x^2+2xy}{\sqrt{x-y}}]^{1/5}$ = z

∴ z = sinu is homogeneous function of degree $\frac{3}{10}$ in two variables x and y

∴ By Euler's theorem

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{3}{10}$z

As z = sinu ∴ $\frac{\partial z}{\partial x}$ = cosu $\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}$ = cosu $\frac{\partial u}{\partial y}$

∴ $x$cosu $\frac{\partial u}{\partial x}$ + ycosu $\frac{\partial u}{\partial y} = \frac{3}{10}$sinu

∴ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{10}$tanu ………(1)

Differentiating (1) partially w.r.to x, we get,

$\frac{\partial u}{\partial x}$ + x $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{3}{10}sec^2u\frac{\partial u}{\partial x}$

$\therefore$ x $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (\frac{3}{10}sec^2 u - 1)\ \frac{\partial u}{\partial x}$

Multiplying both sides by x, we get,

$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = (\frac{3}{10}sec^2 u - 1)\ x\frac{\partial u}{\partial x}$ ....(2)

Differentiating (1) partially w.r.to y, we get,

x $\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{3}{10}sec^2 u\frac{\partial u}{\partial y}$

$\therefore$ x $\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (\frac{3}{10}sec^2 u - 1)\ \frac{\partial u}{\partial y}$ $\quad \because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Multiplying both sides by y, we get,

xy $\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\frac{3}{10}sec^2 u - 1)y\frac{\partial u}{\partial y}$ ....(3)

Adding (2) and (3), we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\frac{3}{10}sec^2 u - 1)\ x\frac{\partial u}{\partial x} + (\frac{3}{10}sec^2 u - 1)y\frac{\partial u}{\partial y}$$

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\frac{3}{10}sec^2 u - 1)\ (x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y})$

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left[\frac{3}{10}(1 + tan^2 u\right) - 1](\frac{3}{10}$tanu) by (1)

i.e. $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{3}{100}(3tan^2 u - 7)$tanu

==

Mean Value Theorem: Let f (x, y) is continuous in a closed region R and differential in the interior of R. Let P (a, b) and Q (a+h, b+k) be any two points of R such that all points, (a+θh, b+θk), where o < $\theta < 1$, of the straight line segment joining P and Q belongs to the interior of R.

Then f (a+h, b+k) = f (a, b) + hf_x(a+θh, b+θk) + kf_y(a+θh, b+θk).

Proof: We take x = a+ht, y = b+kt

$\therefore \frac{dx}{dt}$ = h & $\frac{dy}{dt}$ = k

Let F(t) = f (x, y) = f (a+ht, b+kt)

$\therefore$ F'(t) = $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

i.e. F'(t) = hf_x(a+ht, b+kt) + kf_y(a+ht, b+kt)

As F(t) is continuous in [0, 1] and differentiable in (0, 1).

$\therefore$ By Lagrange's Mean Value Theorem, we get,

F (1) – F (0) = F'(θ) for some 0 < $\theta < 1$

$\therefore$ f (a+h, b+k) – f (a, b) = hf_x(a+hθ, b+kθ) + kf_y(a+hθ, b+kθ)

i.e. f (a+h, b+k) = f (a, b) + hf_x(a+θh, b+θk) + kf_y(a+θh, b+θk)

where 0 < $\theta < 1$ Hence proved.

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Ex.: If $f(x, y) = x^3 - xy^2$, show that θ used in the mean value theorem applied to the points (2, 1) and (4, 1) satisfies the quadratic equation $3\theta^2 + 6\theta - 4 = 0$.

Proof: Let $f(x, y) = x^3 - xy^2$

$\therefore f_x(x, y) = 3x^2 - y^2$ & $f_y(x, y) = -2xy$

By Mean Value Theorem,

$f(a+h, b+k) = f(a, b) + hf_x(a+h\theta, b+k\theta) + kf_y(a+h\theta, b+k\theta)$ ……(1)

where $0 < \theta < 1$

Given points are (2, 1) and (4, 1)

i.e. a = 2, b = 1, a+h = 4 & b+k = 1.

$\therefore$ h = 2 & k = 0.

$\therefore$ From (1), we get,

$f(4, 1) = f(2, 1) + 2f_x(2+2\theta, 1)$

i.e. $[4^3 - 4x1^2] = [2^3 - 2x1^2] + 2[3(2+2\theta)^2 - 1^2]$

i.e. $[64 - 4] = [8 - 2] + 2[3(4+8\theta + 4\theta^2) - 1]$

i.e. $60 = 6 + 2[12+24\theta + 12\theta^2 - 1]$

i.e. $60 = 6 + 24 + 48\theta + 24\theta^2 - 2$

i.e. $24\theta^2 + 48\theta + 28 - 60 = 0$

i.e. $24\theta^2 + 48\theta - 32 = 0$

i.e. $3\theta^2 + 6\theta - 4 = 0$

Hence proved.

==

Ex.: If $f(x, y) = x^2y + 2xy^2$, show that the value of θ used in the expression of the mean value theorem applied to the line segment joining the points (1, 2) and (3, 3) satisfies the equation $12\theta^2 + 30\theta - 19 = 0$.

Proof: Let $f(x, y) = x^2y + 2xy^2$

$\therefore f_x(x, y) = 2xy + 2y^2$ & $f_y(x, y) = x^2 + 4xy$

By Mean Value Theorem,

$f(a+h, b+k) = f(a, b) + hf_x(a+h\theta, b+k\theta) + kf_y(a+h\theta, b+k\theta)$ ……(1)

where $0 < \theta < 1$

Given points are (1, 2) and (3, 3)

i.e. a = 1, b = 2, a+h = 3 & b+k = 3.

$\therefore$ h = 2 & k = 1.

$\therefore$ From (1), we get,

$f(3, 3) = f(1, 2) + 2f_x(1+2\theta, 2+\theta) + f_y(1+2\theta, 2+\theta)$

i.e. $(3^2 x 3 + 2x3x3^2) = (1^2 x2 + 2x1x2^2) + 2[2(1+2\theta)(2+\theta) + 2(2+\theta)^2]$

$+ [(1+2\theta)^2 + 4(1+2\theta)(2+\theta)]$

i.e. $(27+54) = (2 + 8) + 2[2(2+\theta + 4\theta+2\theta^2) + 2(4+4\theta+\theta^2)]$

$+ [1+4\theta+4\theta^2 + 4(2+\theta + 4\theta+2\theta^2)]$

i.e. $81 = 10 + 2[4+2\theta + 8\theta+4\theta^2+8+8\theta+2\theta^2]$

$+ [1+4\theta+4\theta^2 + 8+4\theta + 16\theta+8\theta^2]$

i.e. $71 = 2(12+18\theta+6\theta^2) +(9+24\theta+12\theta^2)$

i.e. $24+36\theta+12\theta^2+9+24\theta+12\theta^2-71 = 0$

i.e. $24\theta^2+60\theta -38 = 0$

i.e. $12\theta^2+30\theta - 19 = 0$

Hence proved.

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UNIT-2: JACOBIAN, COMPOSITE FUNCTIONS AND MEAN VALUE THEOREMS [MCQ'S]

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1) If u and v are functions of two independent variables x and y, then jacobian of u and v w. r. to x and y i.e. $J(\frac{u,v}{x,y}) = \frac{\partial(u,v)}{\partial(x,y)} = \ldots$

a) $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$ b) $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ c) $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$ d) None of these

2) $J(\frac{u,v}{x,y})\, J(\frac{x,y}{u,v}) = \ldots$

a) 0 b) -1 c) 1 d) None of these

3) $\frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,\theta)} = \ldots$

a) $\frac{\partial(u,v)}{\partial(r,\theta)}$ b) $\frac{\partial(r,\theta)}{\partial(u,v)}$ c) 1 d) None of these

4) Functions u, v and w of three independent variables x, y and z are functionally related (or dependent) if and only if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \ldots$

a) 0 b) -1 c) 1 d) None of these

5) If $u = x^2$ and $v = y^2$, then $\frac{\partial(u,v)}{\partial(x,y)} = \ldots\ldots$

a) 4xy b) 2x c) 2y d) None of these

6) If $u = x(1-y)$ and $v = xy$, then $\frac{\partial(u,v)}{\partial(x,y)} = \ldots\ldots$

a) xy b) x c) y d) None of these

7) If $u = f(x, y)$, $x = \emptyset(t)$, $y = \Psi(t)$, then u is a composite function of …

a) x b) t c) y d) None of these

8) If $z = f(x, y)$, $x = \emptyset(u, v)$, $y = \Psi(u, v)$, then z is a composite function of

a) u and v b) x and y c) u and x d) None of these

9) If z = f(u, v) , u = $\emptyset$(x, y), v = Ψ(x, y), then z is a composite function of

a) u and v b) x and y c) u and x d) None of these

10) If z = f(x, y) , x = $r\cos\theta$, y = $r\sin\theta$, then z is a composite function of

a) u and v b) x and y c) r and θ d) None of these

11) If z = log(x^2+y^2), x = u + v, y = u - v, then z is a composite function of

a) u and v b) x and y c) u and x d) None of these

12) If u = f (x, y) is a differential function of x, y and x = $\emptyset$(t), y = Ψ(t) are differential functions of t, then composite function u = f [$\emptyset$(t), Ψ(t)] is differential function of t and $\frac{du}{dt}$ = …

a) $\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$ b) $\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$ c) $\frac{du}{dx}\frac{dx}{dt}+\frac{du}{dy}\frac{dy}{dt}$ d) None of these

13) If z = f (x, y) = x^2+y^2 where x = t^2+1, y = 2t, then $\frac{dz}{dt}$ at t = 1 is …

a) 0 b) 2 c) 16 d) None of these

14) If u = f ($y-z,\ z-x, x-y$), then $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ = …

a) 0 b) 1 c) -1 d) None of these

15) If u = x^3 + y^2 + z^3 , then u is…….

a) Homogenous function b) non- homogenous function

c) both homogenous & non- homogenous function d) None of these

16) u = x^2 + xy + y^2 is homogenous function of degree…….

a) 3 b) 2 c) 1 d) None of these

17) u = x^3 + xy^2 is homogenous function of degree…….

a) 3 b) 2 c) 1 d) None of these

18) u = $\frac{x^4+y^4}{x+y}$ is homogenous function of degree…….

a) 3 b) 2 c) 1 d) None of these

19) If u = $\sin^{-1}\frac{x^4+y^4}{x+y}$, then sinu is homogenous function of degree…….

a) 3 b) 2 c) 1 d) None of these

20) If u = $\tan^{-1}\frac{x^3+y^3}{x+y}$, then tanu is homogenous function of degree…….

a) 3 b) 2 c) 1 d) None of these

21) Let u = $\frac{x^2+y^2}{x+y}$ is a homogenous function. What is the degree of u?

a) 3 b) 2 c) 1 d) None of these

22) Let u = $\frac{x^3+y^3}{x+y}$ is a homogenous function. What is the degree of u?

a) 3 b) 2 c) 1 d) None of these

23) u = $\tan^{-1}\frac{y}{x}$ is homogenous function of degree…….

a) 0 b) 1 c) 2 d) None of these

24) u = $\tan^{-1}\frac{y}{x} + \sin^{-1}\frac{x}{y}$ is homogenous function of degree…….

a) 0 b) 1 c) 2 d) None of these

25) f(x, y) = $\Phi(\frac{y}{x}) + \Psi\left(\frac{x}{y}\right)$is homogenous function of degree…….

a) n b) 1 c) 0 d) None of these

26) By Euler's Theorem, if f (x, y) is homogeneous function of degree n in two variables x and y having first order partial derivatives then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \ldots$

a) nf b) f c) 0 d) None of these

27) If z is homogenous function of degree 2 then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = \ldots$

a) 2z b) 2 c) z d) None of these

28) If z is homogenous function of degree 3 then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = \ldots$

a) z b) 3z c) 5 d) None of these

29) If u is homogenous function of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \ldots$

a) nu b) n c) u d) None of these

30) If u is homogenous function of degree 0 then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \ldots$

a) 0 b) 1 c) 2 d) None of these

31) If u is homogenous function of degree 7 then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \ldots$

a) 7u b) 7 c) u d) None of these

32) A function f(x, y) is said to be homogenous function of degree n if it expressed as

a) f(x, y) = $\Phi(\frac{x}{y})$ b) f(x, y) = $\Phi(\frac{y}{x})$ c) f(x, y) = $x^n\Phi(\frac{y}{x})$ d) None of these

33) A function f(x, y) is said to be homogenous function of degree n if f(tx, ty) = …

a) t f(x, y) b) t^nf(x, y) c) t^2 f(x, y) d) None of these

34) A function f(x, y, z) is said to be homogenous function of degree n if f(tx, ty, tz) = …

a) t^nf(x, y, z) b) tf(x, y, z) c) f(x, y, z) d) None of these

35) u = $\tan^{-1}\frac{y}{x}$ is homogenous function of degree…….

a) 0 b) 1 c) 2 d) None of these

36) If u = $G^{-1}\{x^n f(\frac{y}{x})\}$ and $G'(u) \neq 0$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \ldots$

a) nu b) n$\frac{G(u)}{G'(u)}$ c) nG(u) d) None of these

37) If u = f(x, y) is homogenous function of degree n, then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \ldots$

a) n(n-1)u b) (n-1)u c) nu d) None of these

38) If u = f(x, y) is homogenous function of degree 0, then $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \ldots$

a) 3 b) 2 c) 0 d) None of these

39) If u = f(x, y) is homogenous function of degree 1, then $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \ldots$

a) 0 b) u c) 2u d) None of these

40) If u = f(x, y) is homogenous function of degree 2, then $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \ldots$

a) 3u b) 2u c) u d) None of these

UNIT-3: TAYLOR'S THEOREM AND EXTREME VALUES

❖ Taylor's theorem:

If $f(x,y)$ possesses continuous n^{th} order partial derivatives in the neighborhood of point $(a\,,b)$ and point $(a+h\,,b+k)$ lies in the neighbourhood of point $(a\,,b)$ then there exists $\theta,\ 0<\theta<1$ such that

$$\mathrm{f}(\mathrm{a}+\mathrm{h}\,,\mathrm{b}+\mathrm{k})=\mathrm{f}(\mathrm{a}\,,\mathrm{b})+\left(\mathrm{h}\frac{\partial}{\partial \mathrm{x}}+\mathrm{k}\frac{\partial}{\partial \mathrm{y}}\right)\mathrm{f}(\mathrm{a},\mathrm{b})+\frac{1}{2!}\left(\mathrm{h}\frac{\partial}{\partial \mathrm{x}}+\mathrm{k}\frac{\partial}{\partial \mathrm{y}}\right)^2\mathrm{f}(\mathrm{a},\mathrm{b})+\cdots+\frac{1}{(\mathrm{n}-1)!}\left(\mathrm{h}\frac{\partial}{\partial \mathrm{x}}+\mathrm{k}\frac{\partial}{\partial \mathrm{y}}\right)^{\mathrm{n}-1}\mathrm{f}(\mathrm{a},\mathrm{b})+\frac{1}{\mathrm{n}!}\left(\mathrm{h}\frac{\partial}{\partial \mathrm{x}}+\mathrm{k}\frac{\partial}{\partial \mathrm{y}}\right)^{\mathrm{n}}\mathrm{f}(\mathrm{a}+\theta\mathrm{h},\mathrm{b}+\theta\mathrm{k}).$$

Proof: Let us write $x=a+ht,\ y=b+kt$

$$\therefore \frac{dx}{dt}=h\ \&\ \frac{dy}{dt}=k$$

$$\therefore\ f(x\,,y)=f\,(a+ht,b+kt)=\emptyset(t)$$

As $f(x\,,y)$ possesses continuous n^{th} order partial derivatives in the neighborhood of point $(a\,,b)$.

$\therefore \emptyset(t)$ is continuous $[0,t]$ and derivable in $(0\,,t)$.

$\therefore$ By Maclaurin's series expansion of $\emptyset(t)$ in $[0,t]$

$$\therefore\ \emptyset(t)=\emptyset(0)+t\emptyset'(0)+\frac{t^2}{2!}\emptyset''(0)+\ldots\ldots+\frac{t^{n-1}}{(n-1)!}\emptyset^{n-1}(0)+\frac{t^n}{n!}\emptyset^n(\theta t)$$

Putting $t=1$, we get

$$\emptyset(1)=\emptyset(0)+\emptyset'(0)+\frac{1}{2!}\emptyset''(0)+\ldots+\frac{1}{(n-1)!}\emptyset^{n-1}(0)+\frac{1}{n!}\emptyset^n(\theta)\ldots\ldots\ldots(1)$$

As $\emptyset(t)=f(x\,,y)=f(a+ht\,,b+kt)$

$$\therefore\ \emptyset'(t)=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f$$

Again, differentiating w.r.t. t, we get

$$\emptyset''(t)=\frac{d}{dt}\left[h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial x}\left[h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right]\frac{dx}{dt}+\frac{\partial}{\partial y}\left[h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right]\frac{dy}{dt}$$

$$=\left(h\frac{\partial^2 f}{\partial x^2}+k\frac{\partial^2 f}{\partial x\partial y}\right)h+(h\frac{\partial^2 f}{\partial y\partial x}+k\frac{\partial^2 f}{\partial y^2})k$$

$$=h^2\frac{\partial^2 f}{\partial x^2}+hk\frac{\partial^2 f}{\partial x\partial y}+hk\frac{\partial^2 f}{\partial y\partial x}+k^2\frac{\partial^2 f}{\partial y^2}$$

$$=h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\qquad\because\ \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

$\therefore\ \emptyset''(t)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2 f$ and so on.

In general, $\quad \emptyset^r(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^r f(a+ht,b+kt)$

$\therefore$ We have, $\emptyset(1)=f(a+h,b+k)\ \&\ \emptyset(0)=f(a\,,b)$

$\&\ \emptyset^r(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^r f(a\,,b)\qquad$ for $\ 1\le r\le n-1$

$$\therefore\ \emptyset^n(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k)$$

Putting these values in equation (1), we get

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\ \dots\ +\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+\ \frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n} f(a\ +\ \theta h,b+\ \theta k).$$

Hence proved.

==

❖ Maclaurin's theorem for a function of two variables:-

If $f(x,y)$ possesses continuous n^{th} order partial derivatives in the neighborhood of point $(0,0)$ and point (x,y) lies in the neighbourhood of point $(0,0)$ then there exists θ, $0<\theta<1$ such that

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\ \dots$$

$$+\frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n-1} f(0,0)+\ \frac{1}{n!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{n} f(\theta x,\theta y).$$

➢ **REMARK:-**

1] $f(x,y)=f(a,b)+\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right]+\frac{1}{3!}\left[(x-a)^3f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2f_{xyy}(a,b)+(y-b)^3f_{yyy}(a,b)\right]+\cdots$ is called Taylor's series expansion of $f(x,y)$ in powers of $(x-a)$& $(y-b)$or about point (a,b).

2]$f(x,y)=f(0,0)+\left[xf_x(0,0)+yf_y(0,0)\right]+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]+\frac{1}{3!}\left[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)\right]+\cdots$ is called Maclaurin's series expansion of $f(x,y)$ in powers of x & y or about point (0, 0).

==

Ex. Expand the function $f(x,y)=x^2+xy-y^2$ by Taylor's theorem in powers of $(x-1)$ & $(y+2)$.

Solution: By Taylor's theorem expansion of $f(x,y)$ in powers of $(x-1)$ & $(y+2)$ $i.e.$ about the point $(1,-2)$ is

$$f(x,y)=f(1,-2)+\left[(x-1)f_x(1,-2)+(y+2)f_y(1,-2)\right]+\frac{1}{2!}\left[(x-1)^2f_{xx}(1,-2)+2(x-1)(y+2)f_{xy}(1,-2)+(y+2)^2f_{yy}(1,-2)\right]+\cdots\ (1)$$

Here, $f(x,y)=x^2+xy-y^2$ $\therefore f(1,-2)=1-2-4=-5$

$f_x(x,y)=2x+y$ $\therefore f_x(1,-2)=2-2=0$

$f_y(x,y)=x-2y$ $\therefore f_y(1,-2)=1+4=5$

$f_{xx}(x,y)=2$ $\therefore f_{xx}(1,-2)=2$

$f_{xy}(x,y)=1$ $\therefore f_{xy}(1,-2)=1$

$f_{yy}(x,y)=-2$ $\therefore f_{yy}(1,-2)=-2$

And all higher order partial derivatives are 0. Putting these values in equation (1), we get

$$x^2+xy-y^2 = -5+[0+5(y+2)]$$
$$+\frac{1}{2}[2(x-1)^2+2(x-1)(y+2)-2(y+2)^2]+0$$
$$\therefore x^2+xy-y^2 = -5+5(y+2)+(x-1)^2+(x-1)(y+2)-(y+2)^2$$

be the required expansion.

==

Ex. Expand the function $f(x,y)=x^3+y^3+xy^2$ by Taylor's theorem in powers of $(x-1)$ & $(y-2)$.

Solution: By Taylor's theorem expansion of $f(x,y)$ in powers of $(x-1)$ & $(y-2)$

$i.e.$ about the point $(1,2)$ is

$$f(x,y)=f(1,2)+\left[(x-1)f_x(1,2)+(y-2)f_y(1,2)\right]+\frac{1}{2!}\left[(x-1)^2f_{xx}(1,2)+2(x-1)(y-2)f_{xy}(1,2)+(y-2)^2f_{yy}(1,2)\right]+\frac{1}{3!}\left[(x-1)^3f_{xxx}(1,2)+3(x-1)^2(y-2)f_{xxy}(1,2)+3(x-1)(y-2)^2f_{xyy}(1,2)+(y-2)^3f_{yyy}(1,2)\right]+\cdots \quad (1)$$

Here, $f(x,y)=x^3+y^3+xy^2$ $\therefore f(1,2)=1+8+4=13$

$f_x(x,y)=3x^2+y^2$ $\therefore f_x(1,2)=3+4=7$

$f_y(x,y)=3y^2+2xy$ $\therefore f_y(1,2)=12+4=16$

$f_{xx}(x,y)=6x$ $\therefore f_{xx}(1,2)=6$

$f_{xy}(x,y)=2y$ $\therefore f_{xy}(1,2)=4$

$f_{yy}(x,y)=6y+2x$ $\therefore f_{yy}(1,2)=12+2=14$

$f_{xxx}(x,y)=6$ $\therefore f_{xxx}(1,2)=6$

$f_{xxy}(x,y)=0$ $\therefore f_{xxy}(1,2)=0$

$f_{xyy}(x,y)=2$ $\therefore f_{xyy}(1,2)=2$

$f_{yyy}(x,y)=6$ $\therefore f_{yyy}(1,2)=6$

and all higher order partial derivatives are 0. Putting these values in equation (1), We get

$$x^3+y^3+xy^2=13+[7(x-1)+16(y-2)]$$
$$+\frac{1}{2}[6(x-1)^2+8(x-1)(y-2)+14\,(y+2)^2]$$
$$+\frac{1}{6}[6(x-1)^3+0+6(x-1)(y-2)^2+6(y-2)^3]+0$$
$$\therefore\ x^3+y^3+xy^2=13+[7(x-1)+16(y-2)]$$
$$+3(x-1)^2+4(x-1)(y-2)+7(y-2)^2$$
$$+(x-1)^3+(x-1)(y-2)^2+(y-2)^3$$

be the required expansion.

==

Ex. Express x^2y as polynomial in $(x-1)$ & $(y+2)$by using Taylor's theorem.

Solution: By Taylor's theorem expansion of $f(x,y)$ in powers of $(x-1)$&$(y+2)$

$i.e.$ about the point $(1,-2)$ is

$$f(x,y)=f(1,-2)+\left[(x-1)f_x(1,-2)+(y+2)f_y(1,-2)\right]$$

$$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2)+2(x-1)(y+2)f_{xy}(1,-2)+(y+2)^2 f_{yy}(1,-2)\right]$$
$$+\tfrac{1}{3!}\left[(x-1)^3 f_{xxx}(1,-2)+\cdots \quad \ldots\ldots\ldots\ldots\ldots (1)\right.$$

Here, $f(x,y)=x^2y \therefore f(1,-2)=-2$

$f_x(x,y)=2xy \qquad \therefore f_x(1,-2)=-4$

$f_y(x,y)=x^2 \qquad \therefore f_y(1,-2)=1$

$f_{xx}(x,y)=2y \qquad \therefore f_{xx}(1,-2)=-4$

$f_{xy}(x,y)=2x \qquad \therefore f_{xy}(1,-2)=2$

$f_{yy}(x,y)=0 \qquad \therefore f_{yy}(1,-2)=0$

$f_{xxx}(x,y)=0 \qquad \therefore f_{xxx}(1,-2)=0$

$f_{xxy}(x,y)=2 \qquad \therefore f_{xxy}(1,-2)=2$

$f_{xyy}(x,y)=0 \qquad \therefore f_{xyy}(1,-2)=0$

$f_{yyy}(x,y)=0 \qquad \therefore f_{yyy}(1,-2)=0$

and all higher order partial derivatives are 0.Putting these values in equation (1), we get

$$x^2y=-2+[(x-1)(-4)+(y+2)(1)]$$
$$+\frac{1}{2}[(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)]$$
$$+\frac{1}{6}[0+3(x-1)^2(y+2)(2)+0+0]$$
$$\therefore\ x^2y=-2-4(x-1)+(y+2)-2(x-1)^2+2(x-1)(y+2)$$
$$+(x-1)^2(y+2)$$

be the required expansion.

==

Ex. Show that expansion of $\sin(xy)$ in powers of $(x-1)$ & $\left(y-\frac{\pi}{2}\right)$ upto and including second term is $1-\frac{\pi^2}{8}(x-1)^2-\frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$.

Proof: Expansion of $f(x,y)$ inpowers of $(x-1)$& $\left(y-\frac{\pi}{2}\right)$ upto and including second degree term is $f(x,y)=f\left(1,\frac{\pi}{2}\right)+(x-1)f_x\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)f_y\left(1,\frac{\pi}{2}\right)$

$+\frac{1}{2!}\left[(x-1)^2 f_{xx}\left(1,\frac{\pi}{2}\right)+2(x-1)\left(y-\frac{\pi}{2}\right)f_{xy}\left(1,\frac{\pi}{2}\right)+\left(y-\frac{\pi}{2}\right)^2 f_{yy}\left(1,\frac{\pi}{2}\right)\right]$ ----- (1)

Here, $f(x,y)=\sin(xy) \qquad \therefore f\left(1,\frac{\pi}{2}\right)=1$

$f_x(x,y)=y\cos(xy) \qquad \therefore f_x\left(1,\frac{\pi}{2}\right)=0$

$f_y(x,y)=x\cos(xy) \qquad \therefore f_y\left(1,\frac{\pi}{2}\right)=0$

$f_{xx}(x,y)=-y^2\sin(xy) \qquad \therefore f_{xx}\left(1,\frac{\pi}{2}\right)=-\frac{\pi^2}{4}$

$f_{xy}(x,y) = \cos(xy) - xy\sin(xy) \quad \therefore\ f_{xy}\left(1,\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$f_{yy}(x,y) = -x^2\sin(xy) \quad \therefore\ f_{yy}\left(1,\frac{\pi}{2}\right) = -1$

Putting these values in equation (1), we get

$\sin(xy)$

$$= 1 + [0+0] + \frac{1}{2}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)\left(y-\frac{\pi}{2}\right)\left(-\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2(-1)\right]$$

$$\therefore \sin(xy) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

Hence Proved.

==

Ex. Prove that $\sin(x+y) = (x+y) - \frac{(x+y)^3}{3!} + \cdots$

Proof: Maclaurin's series expansion of $f(x,y)$ inpowers of x & y is

$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2f_{yy}(0,0)] + \frac{1}{3!}[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)] + \cdots$ -------- (1)

Here, $f(x,y) = \sin(x+y) \quad \therefore f(0,0) = 0$

$f_x(x,y) = \cos(x+y) \quad \therefore f_x(0,0) = 1$

$f_y(x,y) = \cos(x+y) \quad \therefore f_y(0,0) = 1$

$f_{xx}(x,y) = -\sin(x+y) \quad \therefore f_{xx}(0,0) = 0$

$f_{xy}(x,y) = -\sin(x+y) \quad \therefore f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -\sin(x+y) \quad \therefore f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = -\cos(x+y) \quad \therefore f_{xxx}(0,0) = -1$

$f_{xxy}(x,y) = -\cos(x+y) \quad \therefore f_{xxy}(0,0) = -1$

$f_{xyy}(x,y) = -\cos(x+y) \quad \therefore f_{xyy}(0,0) = -1$

$f_{yyy}(x,y) = -\cos(x+y) \quad \therefore f_{yyy}(0,0) = -1$

And so on. Putting these values in equation (1), we get

$$sin(x+y) = 0 + [x(1) + y(1)] + \frac{1}{2!}[0+0+0] + \frac{1}{3!}[x^3(-1) + 3x^2y(-1) + 3xy^2(-1) + y^3(-1)] + \cdots$$

$$\therefore \sin(x+y) = (x+y) - \frac{(x+y)^3}{3!} + \cdots$$

Hence proved.

==

Ex. Show that for $0 < \theta < 1, \sin x \sin y = xy - \frac{1}{6}[(x^3 + 3xy^2)(\cos\theta x \sin\theta y) + (y^3 + 3x^2y)(\sin\theta x \cos\theta y)]$.

Proof: Maclaurin's series expansion of $f(x,y)$ inpowers of x & y with remainder after third term. $f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)]$

$$+\frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)]$$

$$+\frac{1}{3!}[x^3 f_{xxx}(\theta x,\theta y) + 3x^2 y f_{xxy}(\theta x,\theta y) + 3xy^2 f_{xyy}(\theta x,\theta y) + y^3 f_{yyy}(\theta x,\theta y)]$$
---(1)

Here, $f(x,y) = \sin x \sin y \qquad \therefore f(0,0) = 0$

$f_x(x,y) = \cos x \sin y \qquad \therefore f_x(0,0) = 0$

$f_y(x,y) = \sin x \cos y \qquad \therefore f_y(0,0) = 0$

$f_{xx}(x,y) = -\sin x \sin y \qquad \therefore f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \cos x \cos y \qquad \therefore f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -\sin x \sin y \qquad \therefore f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = -\cos x \sin y \qquad \therefore f_{xxx}(\theta x,\theta y) = -\cos\theta x \sin\theta y$

$f_{xxy}(x,y) = -\sin x \cos y \qquad \therefore f_{xxy}(\theta x,\theta y) = -\sin\theta x \sin\theta y$

$f_{xyy}(x,y) = -\cos x \sin y \qquad \therefore f_{xyy}(\theta x,\theta y) = -\cos\theta x \sin\theta y$

$f_{yyy}(x,y) = -\sin x \cos y \qquad \therefore f_{yyy}(\theta x,\theta y) = -\sin\theta x \cos\theta y$

And so on. Putting these values in equation (1), we get,

$$\sin x \cos y = 0 + 0 + 0 + \frac{1}{2}[0 + 2xy + 0] + \frac{1}{6}[-x^3 \cos\theta x \sin\theta y - 3x^2 y \sin\theta x \cos\theta y - 3xy^2 \cos\theta x \sin\theta y - y^3 \sin\theta x \cos\theta y]$$

$$\therefore \sin x \cos y = xy - \frac{1}{6}[(x^3 + 3x^2 y)\cos\theta x \sin\theta y + (y^3 + 3x^2 y)\sin\theta x \cos\theta y]$$

Hence proved.

==

Ex. Expand $e^{2x}\cos y$ as Taylor's series about $(0,0)$ upto first three terms.

Solution: Taylor series expansion for $f(x,y)$ about $(0,0)$ up to first three terms is

$$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)]$$

$$+\frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] \text{ --------- (1)}$$

Here, $f(x,y) = e^{2x}\cos y \qquad \therefore f(0,0) = 1$

$f_x(x,y) = 2e^{2x}\cos y \qquad \therefore f_x(0,0) = 2$

$f_y(x,y) = -e^{2x}\sin y \qquad \therefore f_y(0,0) = 0$

$f_{xx}(x,y) = 4e^{2x}\cos y \qquad \therefore f_{xx}(0,0) = 4$

$f_{xy}(x,y) = -2e^{2x}\sin y \qquad \therefore f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -e^{2x}\cos y \qquad \therefore f_{yy}(0,0) = -1$

Putting these values in equation (1), we get

$$e^{2x}\cos y = 1 + [2x + 0] + \frac{1}{2}[4x^2 + 0 - y^2]$$

$$i.e.\ \ e^{2x}\cos y = 1 + 2x + 2x^2 - \frac{1}{2}y^2$$

==

Ex. Expand e^{x+y} as Taylor's series about $(0,0)$.

Solution. Taylor series expansion for $f(x,y)$ about $(0,0)$ is

$$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)]$$
$$+\frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots \quad (1)$$

Here, $f(x,y) = e^{x+y}$ $\therefore f(0,0) = 1$
$f_x(x,y) = e^{x+y}$ $\therefore f_x(0,0) = 1$
$f_y(x,y) = e^{x+y}$ $\therefore f_y(0,0) = 1$
$f_{xx}(x,y) = e^{x+y}$ $\therefore f_{xx}(0,0) = 1$
$f_{xy}(x,y) = e^{x+y}$ $\therefore f_{xy}(0,0) = 1$
$f_{yy}(x,y) = e^{x+y}$ $\therefore f_{yy}(0,0) = 1$
$f_{xxx}(x,y) = e^{x+y}$ $\therefore f_{xxx}(0,0) = 1$
$f_{xxy}(x,y) = e^{x+y}$ $\therefore f_{xxy}(0,0) = 1$
$f_{xyy}(x,y) = e^{x+y}$ $\therefore f_{xyy}(0,0) = 1$
$f_{yyy}(x,y) = e^{x+y}$ $\therefore f_{yyy}(0,0) = 1$

And so on. Putting these values in equation (1), we get

$$e^{x+y} = 1 + [x+y] + \frac{1}{2}[x^2 + 2xy + y^2] + \frac{1}{3!}[x^3 + 3x^2 y + 3y^2 x + y^3] + \cdots$$

$$i.e.\ \ e^{x+y} = 1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \cdots$$

be the required expansion.

==

❖ Absolute maximum:

A function $f(x,y)$ is said to have absolute maximum at point (a,b) of the region R if $f(x,y) \le f(a,b) \quad \forall\ (x,y) \in R$.

❖ Absolute minimum:

A function $f(x,y)$ is said to have absolute minimum at point (a,b) of the region R if $f(x,y) \ge f(a,b) \quad \forall\ (x,y) \in R$.

❖ Relative maximum:

A function $f(x,y)$ is said to have relative maximum at point (a,b) if $f(x,y) \le f(a,b) \quad \forall\ (x,y) \in N\delta(a,b)$.

❖ Relative minimum:

A function $f(x,y)$ is said to have relative minimum at point (a,b) if $f(a,b) \le f(x,y) \quad \forall\ (x,y) \in N\delta(a,b)$.

➢ **REMARK:**

1] An Absolute maximum or an Absolute minimum is called an Absolute extremum.

2] Relative maximum or Relative minimum is called as Relative extremum.

❖ Critical point or Stationary point:

A point (a, b) is said to be critical point or stationary point of a function $f(x,y)$, if $f_x(a,b) = 0$ & $f_y(a,b) = 0$ or they does not exists.

❖ Saddle point or Minimax point:

A critical point (a, b) is said to be a saddle point or minimax point if $f(x,y)$ have no extremum at point (a,b).

==

- NECESSARY CONDITION FOR EXTREMUM:

If a function $f(x,y)$ have an extremum at point (a,b) then

1) $f_x(a,b) = 0$ or it does not exists.

2) $f_y(a,b) = 0$ or it does not exists.

Proof: Let, function $f(x,y)$ have an extremum at point (a,b).

By taking $y = b$, we have a function $f(x,b)$ of one variable x.

$\therefore\ f_x(a,b) = 0$ or it does not exists.

Similarly, by taking $x = a$, we get $f_y(a,b) = 0$ or it does not exists.

==

- Sufficient Condition For Extremum:

If $f(x,y)$ possesses n^{th} order partial derivative in a neighbourhood of point (a,b) of the region R with $f_x(a,b) = 0, f_y(a,b) = 0, r = f_{xx}(a,b), s = f_{xy}(a,b)$, $t = f_{yy}(a,b)$ & $\Delta = rt - s^2$, then the function $f(x,y)$ is

i) Minimum at point (a,b) if $\Delta > 0$ & $r > 0$.

ii) Maximum at point (a,b) if $\Delta > 0$ & $r < 0$.

iii) No extremum at point (a,b) if $\Delta < 0$ $i.e.$ (a,b) is saddle point if $\Delta < 0$.

iv) The next investigation is needed if $\Delta = 0$.

Proof: By Taylor's theorem,

$$f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + R_3.$$

$$\therefore f(a+h,b+k) - f(a,b) = \frac{1}{2!}(h^2r + 2hks + k^2t) + R_3$$

$$\therefore f(a+h,b+k) - f(a,b) = \frac{1}{2!r}(h^2r^2 + 2hksr + k^2rt) + R_3.$$

$$\therefore f(a+h,b+k) - f(a,b) = \frac{1}{2!\,r}(h^2r^2 + 2hksr + k^2s^2 + k^2rt - k^2s^2) + R_3.$$

$$= \frac{1}{2!r}[(hr+ks)^2 + k^2(rt - s^2)] + R_3$$

For small values of h, k, the sign of RHS is independent of R_3.

i) If $\Delta = rt - s^2 > 0$ & $r > 0$ then $f(a+h,b+k) - f(a,b) \geq 0$

$i.e.$ $f(a+h,b+k) \geq f(a,b)$. $\therefore f$ is minimum at point (a,b).

ii) If $\Delta = rt - s^2 > 0$ & $r < 0$ then $f(a+h,b+k) - f(a,b) \leq 0$

$i.e.$ $f(a+h,b+k) \leq f(a,b)$. $\therefore f$ is maximum at point (a,b).

iii) If $\Delta = rt - s^2 < 0$ then the function $f(x,y)$ have no extremum at point (a,b).

iv) If $\Delta = 0$ then we can't say the function $f(x,y)$ have an extremum.

$\therefore$ We need further investigation.

==

❖ Working rule to find the extremum by using second order partial derivatives:-

1. Find critical points by solving $f_x(x,y) = 0$ & $f_y(x,y) = 0$.
2. At these critical points, find $r = f_{xx}$, $s = f_{xy}$ & $t = f_{yy}$ and $\Delta = rt - s^2$.
3. f is minimum if $\Delta > 0$ & $r > 0$.
4. f is maximum if $\Delta > 0$ & $r < 0$.
5. f has no extremum if $\Delta < 0$.

==

Ex: Discuss the maxima and minima of the function $u = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$.

Solution: Let, $u = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$

$\therefore u_x = 2x - \frac{2}{x^2}$ and $u_y = 2y - \frac{2}{y^2}$

$\therefore u_{xx} = 2 + \frac{4}{x^3}$, $u_{xy} = 0$ and $u_{yy} = 2 + \frac{4}{y^3}$

By solving $u_x = 0$ and $u_y = 0$, We get

$\Rightarrow$ $2x - \frac{2}{x^2} = 0$ and $2y - \frac{2}{y^2} = 0$

$i.e.$ $x - \frac{1}{x^2} = 0$ and $y - \frac{1}{y^2} = 0$

$i.e.$ $x^3 - 1 = 0$ and $y^3 - 1 = 0$

$\therefore$ $x = 1$ and $y = 1$

$\therefore$ Critical point is $(1,1)$. At the critical point, we get

$r = u_{xx}(1,1) = 2 + 4 = 6$

$s = u_{xy}(1,1) = 0$

$t = u_{yy}(1,1) = 6$

$\therefore$ $\Delta = rt - s^2 = 36 - 0 = 36$

Here $\Delta = 36 > 0$ and $r = 6 > 0$

$\therefore$ The function u is minimum at point $(1,1)$ and its minimum value is $u_{min.} = u(1,1) = 1 + 1 + 2 + 2 = 6$.

==

Ex: Find the points (x,y) where the function $u = xy(a - x - y)$ is maximum or minimum. What is the maximum value of function?

Solution: Let $u = xy(a - x - y)$ $i.e.$ $u = axy - x^2y - xy^2$

$\therefore$ $u_x = ay - 2xy - y^2$ and $u_y = ax - x^2 - 2xy$

$u_{xx} = -2y$, $u_{xy} = a - 2x - 2y$ and $u_{yy} = -2x$

Now $u_x = 0$ and $u_y = 0$ gives

$ay - 2xy - y^2 = 0$ and $ax - x^2 - 2xy = 0$

$i.e. -y(y + 2x - a) = 0$ and $-x(x + 2y - a) = 0$

$i.e.$ $y = 0$ or $y + 2x = a$ ------- (1)

$x = 0$ or $x + 2y = a$ ------- (2)

For $y = 0$, from (2), we get $x = a$

For $x = 0$, from (1), we get $y = a$

To solve equation (1) and (2), Consider equations $2 \times (1) - (2)$,

$2y + 4x = 2a$

$-2y - x = -a$

$3x = a \therefore x = \dfrac{a}{3}$

Putting $x = \frac{a}{3}$ in equation (1), we get, $y + \frac{2a}{3} = a \therefore y = a - \frac{2a}{3} = \frac{a}{3}$.

$\therefore$ The critical points are $(0,0), (a,0), (0,a)$ & $\left(\frac{a}{3}, \frac{a}{3}\right)$.

Critical point	$r = u_{xx} = -2y$	$s = u_{xy} = a - 2x - 2y$	$t = u_{yy} = -2x$	$\Delta = rt - s^2$	Remark
$(0,0)$	0	a	0	$-a^2 < 0$	Saddle point
$(a,0)$	0	$-a$	$-2a$	$-a^2 < 0$	Saddle point
$(0,a)$	$-2a$	$-a$	0	$-a^2 < 0$	Saddle point
$(\frac{a}{3}, \frac{a}{3})$	$-\frac{2a}{3}$	$-\frac{a}{3}$	$-\frac{2a}{3}$	$\frac{a^2}{3} > 0$	Point of maximum

$$u_{max.} = u\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a^2}{9}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}$$

==

Ex: Find the least value of the function $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$.

Solution: Let, $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$

$\therefore f_x(x,y) = y - \frac{50}{x^2}$ and $f_y(x,y) = x - \frac{20}{y^2}$

$f_{xx}(x,y) = \frac{100}{x^3}$, $f_{xy}(x,y) = 1$ and $f_{yy}(x,y) = \frac{40}{y^3}$

Now, $f_x(x,y) = 0$ and $f_y(x,y) = 0$

$i.e.\ y - \frac{50}{x^2} = 0$ and $x - \frac{20}{y^2} = 0$

$i.e.\ x^2y - 50 = 0$ and $xy^2 - 20 = 0$

$i.e.\ x^2y = 50$ ----- (1) and $xy^2 = 20$ ----- (2)

$i.e.\ \dfrac{x^2y}{xy^2} = \dfrac{50}{20} \Rightarrow \dfrac{x}{y} = \dfrac{5}{2}\ i.e.\ y = \dfrac{2}{5}x$

Putting $y = \frac{2}{5}x$ in equation (1), we get

$\dfrac{2}{5}x^3 = 50 \Rightarrow x^3 = 125 \therefore x = 5 \Rightarrow y = \dfrac{2}{5}(5) = 2$

$\therefore$ Critical point is $(5,2)$.

$r = f_{xx}(5,2) = \dfrac{100}{125} = \dfrac{4}{5}$

$s = f_{xy}(5,2) = 1$

$t = f_{yy}(5,2) = \dfrac{40}{8} = 5$

$\therefore\ \ \Delta = rt - s^2 = 4 - 1 = 3$

Here $\Delta = 3\ > 0$ and $r = \frac{4}{5} > 0.$

$\therefore$ Given function is minimum at point $(5, 2)$ and its minimum value is

$i.e.$ Least value is $f_{min.} = f(5,2) = 10 + \frac{50}{5} + \frac{20}{2} = 30.$

==

Ex: Investigate the maximum and minimum values of the function

$f(x,y) = 3x^2y - 3x^2 - 3y^2 + y^3 + 2.$

Solution: Let, $f(x,y) = 3x^2y - 3x^2 - 3y^2 + y^3 + 2$

$\therefore f_x(x,y) = 6xy - 6x$ and $f_y(x,y) = 3x^2 - 6y + 3y^2$

$f_{xx}(x,y) = 6y - 6,\ \ f_{xy}(x,y) = 6x$ and $f_{yy}(x,y) = -6 + 6y$

Now, $f_x(x,y) = 0$ and $f_y(x,y) = 0$ gives

$6xy - 6x = 0$ and $3x^2 - 6y + 3y^2 = 0$

$i.e.\ 6x(y-1) = 0$ and $x^2 - 2y + y^2 = 0$

$i.e.\ \ x = 0$ or $y = 1$ and $x^2 - 2y + y^2 = 0$ ------- (1)

For $x = 0$, from equation (1), we get $-2y + y^2 = 0$

$i.e.\ y(y-2) = 0\ i.e.\ y = 0$ or $y = 2$

For $y = 1$, from equation (1), we get $x^2 - 2 + 1 = 0$

$i.e.\ \ x^2 - 1 = 0\ i.e.\ \ x = \pm 1$

$\therefore$ The critical points are $(0,0), (0,2), (1,1)$ & $(-1,1)$.

Critical point	$\boldsymbol{r = f_{xx} = 6y - 6}$	$\boldsymbol{s = f_{xy} = 6x}$	$\boldsymbol{t = f_{yy} = -6 + 6y}$	$\boldsymbol{\Delta = rt - s^2}$	Remark
$\boldsymbol{(0,0)}$	$-6 < 0$	0	-6	$36 > 0$	Point of maximum
$\boldsymbol{(0,2)}$	$6 > 0$	0	6	$36 > 0$	Point of minimum
$\boldsymbol{(1,1)}$	0	6	0	$-36 < 0$	Saddle point
$\boldsymbol{(-1,1)}$	0	-6	0	$-36 < 0$	Saddle point

$f_{max.} = f(0,0) = 2$

$f_{min.} = f(0,2) = 0 - 0 - 12 + 8 + 2 = -2$

==

Ex: Find the stationary points and determine the nature of the function

$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$

Solution: Let, $f(x,y) = x^3 + y^3 - 3x - 12y + 20.$

$\therefore\ \ f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = 3y^2 - 12$

$f_{xx}(x,y) = 6x,\ \ f_{xy}(x,y) = 0$ and $f_{yy}(x,y) = 6y$

Now, $f_x(x,y) = 0$ and $f_y(x,y) = 0$ gives

$3x^2 - 3 = 0$ and $3y^2 - 12 = 0$

$i.e. x^2 - 1 = 0$ and $y^2 - 4 = 0$

$i.e. \quad x = \pm 1$ or $y = \pm 2$

$\therefore$ The critical points are $(1, 2), (1, -2), (-1, 2)$ & $(-1, -2)$.

Nature of the function at these critical points is as follows:-

Critical point	$r = f_{xx} = 6x$	$s = f_{xy} = 0$	$t = f_{yy} = 6y$	$\Delta = rt - s^2$	Remark
$(\mathbf{1}, \mathbf{2})$	$6 > 0$	0	12	$72 > 0$	Point of minimum
$(\mathbf{1}, -\mathbf{2})$	6	0	-12	$-72 < 0$	Saddle Point
$(-\mathbf{1}, \mathbf{2})$	-6	0	12	$-72 < 0$	Saddle point
$(-\mathbf{1}, -\mathbf{2})$	$-6 < 0$	0	-12	$72 > 0$	Point of maximum

$f_{max} = f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38$

$f_{min.} = f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$

==

6. Discuss the extreme value of the function $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

Solution: Let $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$.

$\therefore f(x, y) = 4x - 4x^3$

$f_y(x, y) = -4y + 4y^3$

$f_{xx}(x, y) = 4 - 12x^2$,

$(x, y) = 0$

and $f_{yy}(x, y) = -4 + 12y^2$

Now $f_x(x, y) = 0$ and $f_y(x, y) = 0$ gives

$4x - 4x^3 = 0$ and $-4y + 4y^3 = 0$

$i.e.$ $x(x^2 - 1) = 0$ and $y(y^2 - 1) = 0$

$i.e.$ $x = 0, \pm 1$ or $y = 0, \pm 1$

$\therefore$ The critical points are $(0, 0), (0, \pm 1), (\pm 1, 0)$ & $(\pm 1, \pm 1)$.

Nature of the function at these critical points is as follows:

Critical point	$r = f_{xx} = 4 - 12x^2$	$s = f_{xy} = \mathbf{0}$	$t = f_{yy} = -4 + 12y^2$	$\Delta = rt - s^2$	Remark
$(0, 0)$	4	0	-4	$-16 < 0$	Saddle point
$(0, \pm 1)$	$4 > 0$	0	8	$32 > 0$	Point of minimum
$(\pm 1, 0)$	$-8 < 0$	0	-4	$32 > 0$	Point of maximum
$(\pm 1, \pm 1)$	$-8 < 0$	0	8	$-64 < 0$	Saddle point

$f_{max.} = f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

$f_{min.} = f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$

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UNIT-3: TAYLOR'S THEOREM AND EXTREME VALUES [MCQ'S]

===

1) $f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2 (y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)] + \cdots$ is called ……series expansion of $f(x,y)$ in powers of $(x-a)$& $(y-b)$or about point (a,b).
A) Taylor's B) Maclaurin's C) Laurent's D) None of these

2) $f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \cdots$ is called ……. series expansion of $f(x,y)$ in powers of x & y or about point (0, 0).
A) Taylor's B) Maclaurin's C) Laurent's D) None of these

3) Expression of x – y + 3 in powers of (x-1) and (y-1) is……………….
A) 3 + (x-1) - (y-1) B) (x-1) - (y-1) C) 3 + (x-1) D) None of these

4) Expression of x + y + 3 in powers of (x-1) and (y-1) is……………….
A) 5 + (x-1) + (y-1) B) (x-1) - (y-1) C) 3 + (x-1) D) None of these

5) Maclaurin's theorem for a function of two variables obtained from Taylor's theorem by putting…….
A) 3 B) a = x, b = y, h = 0, k = 0
C) a = 0, b = x, h = y, k = 0 D) a = 0, b = 0, h =0, k = 0

6) $1 + (x+y) + \frac{1}{2!}(x+y)^2 + \frac{1}{3!}(x+y)^3 + \cdots$ is an expansion of ….
A) sin(x+y) B) cos(x+y) C) e^{x+y} D) tan(x+y)

7) $(x+y) - \frac{1}{3!}(x+y)^3 + \frac{1}{5!}(x+y)^5 - \cdots$ is an expansion of ….
A) sin(x+y) B) cos(x+y) C) e^{x+y} D) tan(x+y)

8) $1 - \frac{1}{2!}(x+y)^2 + \frac{1}{4!}(x+y)^4 - \cdots$ is an expansion of ….
A) sin(x+y) B) cos(x+y) C) e^{x+y} D) tan(x+y)

9) If f(x, y) = $x^2 + y^2$ then f has extreme value at…….
A) (1,1) B) (0,0) C) (1,2) D) (2, 1)

10) If f(x, y) = $x^2 + y^2 + 3$ then f has extreme value at…….
A) (0,0) B) (1,0) C) (0,1) D) (1, 1)

11) If f(x, y) = $3x^2 + 3y^2 - 2$ then f has extreme value at…….
A) (0,0) B) (1,0) C) (0,1) D) (1, 1)

12) If f(x, y) = $x^2 - 2y^2 + 1$ then f has extreme value at…….
A) (1,1) B) (0,0) C) (1,0) D) (0, 1)

13) If f(x, y) = $2x^2 - y^2 + 3$ then f has extreme value at…….
A) (1,1) B) (0,0) C) (1,0) D) (0, 1)

14) If f(x, y) = $x^2 - y^2 + 4$ then f has extreme value at…….
A) (1,1) B) (0,0) C) (1,0) D) (0, 1)

15) If $u = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}$ then f has extreme value at…….

A) (1,1) B) (0,0) C) (1,2) D) (0, 2)

16) If $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$ then f has extreme value at…….

A) (0,0) B) (0,2) C) (5,2) D) (5, 0)

17) If $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ then f has extreme value at…….

A) (1,1) B) (0,0) C) (1,2) D) (2, 1)

18) If $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$ then f has extreme value at…….

A) (0,0) B) (0,1) C) (1,0) D) all the above

19) If $u = xy(a - x - y)$ then u has extreme value at…….

A) (a,a) B) (0,0) C) (a,0) D) (0, 1)

20) Stationary point of the function f(x, y) are obtained by…..

A) $f_x = 0$ B) $f_x = 0$ & $f_y = 0$ C) $f_y = 0$ D) none of these

21) Stationary point of the function u(x, y) are obtained by…..

A) $u_x = 0$ B) $u_x = 0$ & $u_y = 0$ C) $u_y = 0$ D) none of these

22) A function $f(x,y)$ is said to have absolute maximum at point (a,b) of the region R if $f(x,y) \dots\dots f(a,b) \quad \forall \ (x,y) \in R$.

A) $\leq$ B) $\geq$ C) $\neq$ D) $=$

23) A function $f(x,y)$ is said to have absolute minimum at point (a,b) of the region R if $f(x,y) \dots\dots f(a,b) \quad \forall \ (x,y) \in R$.

A) $\leq$ B) $\geq$ C) $\neq$ D) $=$

24) A function $f(x,y)$ is said to have relative maximum at point (a,b) If ……… $\forall \ (x,y) \in N\delta(a,b)$.

A) $f(a,b) \leq f(x,y)$ B) $f(a,b) \geq f(x,y)$

C) $f(a,b) \neq f(x,y)$ D) $f(a,b) = f(x,y)$

25) A function $f(x,y)$ is said to have relative minimum at point (a,b) If ……… $\forall \ (x,y) \in N\delta(a,b)$.

A) $f(a,b) \leq f(x,y)$ B) $f(a,b) \geq f(x,y)$

C) $f(a,b) \neq f(x,y)$ D) $f(a,b) = f(x,y)$

26) Let $r = f_{xx}(a,b), s = f_{xy}(a,b), t = f_{yy}(a,b)$ & $\Delta = rt - s^2$, then the function $f(x,y)$ have maximum at point (a,b) if ……..

A) $\Delta > 0$ a & $r < 0$ B) $\Delta > 0$ & $r < 0$

C) $\Delta < 0$ and $r > 0$ D) none of these

27) Let $r = f_{xx}(a,b), s = f_{xy}(a,b), t = f_{yy}(a,b)$ & $\Delta = rt - s^2$, then the function $f(x,y)$ have minimum at point (a,b) if ……..

A) $\Delta > 0$ & $r < 0$ B) $\Delta > 0$ & $r > 0$

C) $\Delta < 0$ and $r > 0$ D) none of these

28) Let $r = f_{xx}(a,b), s = f_{xy}(a,b), t = f_{yy}(a,b)$ & $\Delta = rt - s^2$, then the function $f(x,y)$ have saddle at point (a,b) if ……..

A) $\Delta > 0$ and $r > 0$ B) $\Delta > 0$ and $r < 0$ C) $\Delta < 0$ and $r > 0$ D) none of these

==

UNIT-4: DOUBLE AND TRIPLE INTEGRALS

==

Double Integration:

If $f(x, y)$ is a function of two variables x and y defined in a region R and R is divided into n subregions $\delta R_1, \delta R_2, \ldots, \delta R_n$ then for any point (x_r, y_r) in subregion δR_r double integration over R is denoted by $\iint_R f(x, y)dA$ and defined as

$$\iint_R f(x, y)dA = \lim_{\substack{n\to\infty \\ \delta R_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\, \delta R_r$$

==

Remark:

1) If region R is bounded by $x = a, x = b, y = c$ & $y = d$ then

$$\iint_R f(x, y)dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\, dy\, dx$$

2) If region R is bounded by $y = f_1(x), y = f_2(x), x = a$ & $x = b$ then

$$\iint_R f(x, y)dA = \int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x, y)\, dy\, dx$$

3) If region R is bounded by $x = g_1(y), x = g_2(y), y = c$ & $y = d$ then

$$\iint_R f(x, y)dA = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} f(x, y)\, dx\, dy$$

==

Area of region by Double integration:

The area of the region R by double integration is given by

Area of region $R = \iint_R dx\, dy$

==

Ex: Evaluate $\int_0^a \int_0^b (x^2 + y^2)\, dx\, dy$.

Solution: Let $I = \int_0^a \int_0^b (x^2 + y^2)\, dy\, dx$

$$= \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^b dx$$

$$= \int_0^a \left[bx^2 + \frac{b^3}{3} - 0\right] dx$$

$$= \left[\frac{bx^3}{3} + \frac{b^3}{3} x\right]_0^a$$

$$= [\frac{a^3 b}{3} + \frac{ab^3}{3} - 0]$$

$$\therefore \quad \boxed{I = \frac{1}{3}\, ab\,(a^2 + b^2)}$$

===

Ex: Evaluate $\int_0^a \int_{x/a}^{x} \frac{x}{x^2 + y^2}\, dx\, dy$

Solution: Let, $I = \int_0^a \int_{x/a}^{x} \frac{x}{x^2 + y^2}\, dy\, dx$

$$= \int_0^a x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)\right]_{x/a}^{x} dx$$

$$= \int_0^a \left[\tan^{-1} 1 - \tan^{-1}\left(\frac{1}{a}\right)\right] dx$$

$$= \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{a}\right)\right] [x]_0^a$$

$$= \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{a}\right)\right] (a - 0)$$

$$\boxed{\therefore \quad I = a\left(\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right)}$$

===

Ex: Evaluate $\int_1^2 \int_0^1 (x^2 + y^2)\, dx\, dy$

Solution: Let, $I = \int_1^2 \int_0^1 (x^2 + y^2)\, dy\, dx$

$$= \int_1^2 \left[x^2 y + \frac{y^3}{3}\right]_0^1 dx$$

$$= \int_1^2 \left[x^2 + \frac{1}{3} - 0\right] dx$$

$$= \left[\frac{x^3}{3} + \frac{1}{3} x\right]_1^2$$

$$\therefore \ \mathrm{I} = \frac{8}{3} + \frac{2}{3} - \frac{1}{3} - \frac{1}{3} = \frac{8}{3}$$

===

Ex: Evaluate $\int_0^4 \int_0^{\sqrt{y}} xy \ \ dx\, dy$

Solution: Let, $I = \int_0^4 \int_0^{\sqrt{y}} xy \ \ dx\, dy$

$$= \int_0^4 y \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} \ dy$$

$$= \int_0^4 y \left[\frac{y}{2} - 0\right] \ dy$$

$$= \frac{1}{2} \int_0^4 y^2 \ \ dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3}\right]_0^4$$

$$= \frac{1}{2} \left[\frac{64}{3} - 0\right]$$

$$\boxed{\therefore \quad I = \frac{32}{3}}$$

==

Ex: Evaluate $\int_1^2 \int_0^x \frac{1}{x^2 + y^2} \ dx\, dy$

Solution: Let $I = \int_1^2 \int_0^x \frac{1}{x^2 + y^2} \ \ dx\, dy$

$$= \int_1^2 \int_0^x \ \left[\frac{1}{x^2 + y^2} dy\right] dx$$

$$= \int_1^2 \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right)\right]_0^x \ dx$$

$$= \int_1^2 \frac{1}{x}\left[\frac{\pi}{4} - 0\right] \ dx$$

$$= \frac{\pi}{4} \ [\log x\,]_1^2$$

$$= \frac{\pi}{4} \ [\log 2 - 0]$$

$$\boxed{\therefore \quad I = \frac{\pi}{4} \ \log 2}$$

==

Ex: Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} \; dx\, dy$

Solution: Let I $= \int_0^1 \int_0^{x^2} [e^{y/x} \; dy] \; dx$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} \; dx$$

$$= \int_0^1 x[e^x - 1] \; dx$$

$$= \int_0^1 xe^x \, dx - \int_0^1 x \; dx$$

$$= [xe^x - \int (1)e^x \; dx]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= [xe^x - e^x]_0^1 - [\tfrac{1}{2} - 0]$$

$$= [0 - 0 + 1] - \frac{1}{2}$$

$$\boxed{\therefore \quad I = \frac{1}{2}}$$

==

Ex: Show that $\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} \; dy\right] dx \; \neq \; \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} \; dx\right] \; dy$

Proof: Consider,

L.H.S. $= \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3} \; dy\right] \; dx$

$$= \int_0^1 \left[\int_0^1 \frac{2x-(x+y)}{(x+y)^3} \; dy\right] \; dx$$

$$= \int_0^1 \left[\int_0^1 \{\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\} \, dy\right] \; dx$$

$$= \int_0^1 \left[\frac{2x(x+y)^{-2}}{-2} - \frac{(x+y)^{-1}}{-1}\right]_0^1 \; dx$$

$$= \int_0^1 \left[\frac{1}{x+y} - \frac{x}{(x+y)^2}\right]_0^1 \; dx$$

$$= \int_0^1 \left[\frac{x+y-x}{(x+y)^2}\right]_0^1 \; dx$$

$$= \int_0^1 \left[\frac{y}{(x+y)^2}\right]_0^1 \; dx$$

$$= \int_0^1 \left[\frac{1}{(x+1)^2} - 0\right] \; dx$$

$$= \left[\frac{(x+1)^{-1}}{-1}\right]_0^1$$

$$= \left[\frac{-1}{x+1}\right]_0^1$$

$$= \left[\frac{-1}{2} + 1\right]$$

$\therefore$ L.H.S. $= \frac{1}{2}$ -------- [1]

Consider,

R.H.S. $= \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dx\right] dy$

$$= \int_0^1 \left[\int_0^1 \frac{(x+y)-2y}{(x+y)^3}\, dx\right] dy$$

$$= \int_0^1 \left[\int_0^1 \{(x+y)^{-2} - 2y(x+y)^{-3}\}\, dx\right] dy$$

$$= \int_0^1 \left[\frac{(x+y)^{-1}}{-1} - \frac{2y(x+y)^{-2}}{-2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y}{(x+y)^2} - \frac{1}{x+y}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y-x-y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{-x}{(x+y)^2}\right]_0^1 dy$$

$$= -\int_0^1 \left[\frac{1}{(1+y)^2} - 0\right] dy$$

$$= -\left[\frac{(1+y)^{-1}}{-1}\right]_0^1$$

$$= \left[\frac{1}{1+y}\right]_0^1$$

$$= \frac{1}{2} - 1$$

$\therefore$ R.H.S. $= -\frac{1}{2}$ -------- [2]

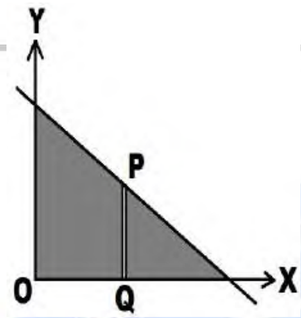
By equations [1] and [2]

$\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dy\right] dx \neq \int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dx\right] dy$ Hence proved.

===

Ex: Evaluate $\iint_R xy \;\; dx\, dy$ over the region in the positive quadrant for which $x + y \;\leq 1$.

Solution: Let, R be the region in the positive quadrant for which $x + y \leq 1$ which is shown in figure.



By taking strip PQ parallel to the y-axis and moving it from

$x = 0$ to $x = 1$, we get limits $0 \leq x \leq 1$ & $0 \leq y \leq 1 - x$

$$\therefore \iint_R xy \;\; dx\, dy = \int_0^1 \int_0^{1-x} x\, [y\, dy]\, dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 x[(1-x)^2 - 0]\; dx$$

$$= \frac{1}{2} \int_0^1 x(1 - 2x + x^2)\; dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} - 0\right]$$

$$= \frac{1}{2}\left[\frac{6-8+3}{12}\right]$$

$$\therefore \iint_R xy \;\; dx\, dy = \frac{1}{24}$$

==

Ex: Evaluate $\iint_R xy(x + y)\; dx\, dy$ where R is the area between $y = x^2$ & $y = x$.

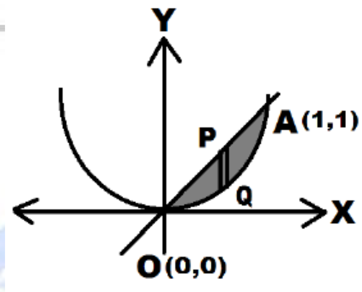
Solution: First we find the point of intersection of

$y = x^2$ & $y = x$ by solving together as follows

$x = x^2 \;\; i.e. x^2 - x = 0 \;\; i.e.\; x(x-1) = 0 \;\; i.e. x = 0$ & $x = 1$

For $x = 0 \Rightarrow y = 0$ & $x = 1 \Rightarrow y = 1$

The point of intersections are $O(0,0)$ & $A(1,1)$. The region between the curves $y = x^2$ & $y = x$ is shown in figure.



By taking strip PQ parallel to y-axis and moving it from $x = 0$ to $x = 1$. We get limit as $0 \le x \le 1$ & $x^2 \le y \le x$

$$\therefore \iint_R xy\,(x+y)\;dx\,dy = \int_0^1 \int_{x^2}^{x} x(xy+y^2)\;dy\,dx$$

$$= \int_0^1 x\left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{x^2}^{x} dx$$

$$= \int_0^1 x\left[\frac{1}{2}x^3 + \frac{1}{3}x^3 - \frac{1}{2}x^5 - \frac{1}{3}x^6\right] dx$$

$$= \int_0^1 \left[\frac{1}{2}x^4 + \frac{1}{3}x^4 - \frac{1}{2}x^6 - \frac{1}{3}x^7\right] dx$$

$$= \left[\frac{1}{6}x^5 - \frac{1}{14}x^7 - \frac{1}{24}x^8\right]_0^1$$

$$= \left[\frac{1}{6} - \frac{1}{14} - \frac{1}{24}\right] - 0$$

$$= \left[\frac{28-12-7}{168}\right] = \frac{9}{168}$$

$$\therefore \iint_R xy\,(x+y)\;dx\,dy = \frac{3}{56}$$

==

Ex: Evaluate $\iint_R y\;dx\,dy$ over the area bounded by $y = x^2$ and $x + y = 2$.

Solution: First we find point of intersection of $y = x^2$ & $x + y = 2$

By solving together as follows

$-x + 2 = x^2$ $i.e.\, x^2 + x - 2 = 0$ $i.e.\, (x+2)(x-1) = 0$

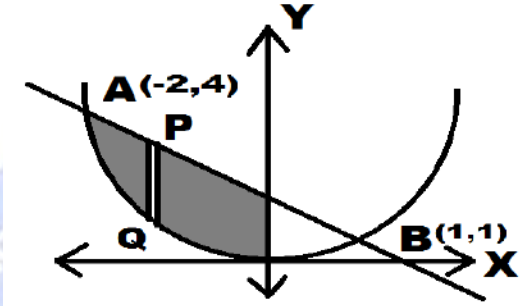
$i.e.$ $x = -2$ & $x = 1$

For $x = -2 \Rightarrow y = 4$ & $x = 1 \Rightarrow y = 1$

The point of intersections are $A(-2, 4)$ & $B(1, 1)$. The region between the curves $y = x^2$ & $x + y = 2$ as shown in figure.

By taking strip PQ parallel to y-axis and moving it from $x = -2$ to $x = 1$,

We get limits as $-2 \le x \le 1$ & $x^2 \le y \le 2 - x$



$$\therefore \iint_R y\ dx\, dy = \int_{-2}^{1}\int_{x^2}^{2-x} y\ dy\, dx$$

$$= \int_{-2}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx$$

$$= \frac{1}{2}\int_{-2}^{1} [(2-x)^2 - (x^2)^2]\ dx$$

$$= \frac{1}{2}\int_{-2}^{1} [4 - 4x + x^2 - x^4]\ dx$$

$$= \frac{1}{2}\left[4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(4 - 2 + \frac{1}{3} - \frac{1}{5}\right) - \left(-8 - 8 - \frac{8}{3} + \frac{32}{5}\right)\right]$$

$$= \frac{1}{2}\left[2 + \frac{1}{3} - \frac{1}{5} + 16 + \frac{8}{3} - \frac{32}{5}\right]$$

$$= \frac{1}{2}\left[18 + 3 - \frac{33}{5}\right]$$

$$= \frac{1}{2}\left[21 - \frac{33}{5}\right] = \frac{1}{2}\left(\frac{72}{5}\right)$$

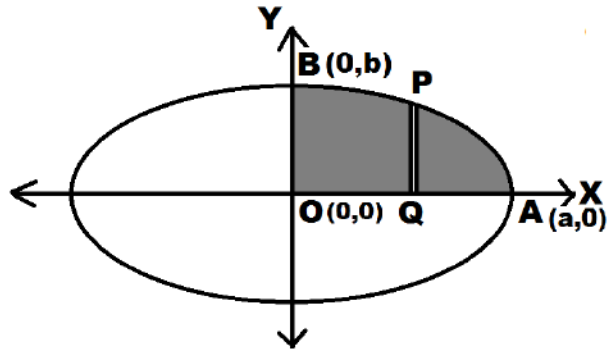
$$\therefore \iint_R y\ dx\, dy = \frac{36}{5}$$

==

Ex: Using double integral, find the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: The area of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

Area $=4$(Area of region $OABO$) $=4\iint_R dx\,dy$



By taking strip PQ parallel to y-axis and moving it from $x=0$ to $x=a$

$\therefore$ The limits of region $OABO$ are $0\le x\le a$ & $0\le y\le \frac{b}{a}\sqrt{a^2-x^2}$

$\therefore$ Area of ellipse $=4\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\,dx$

$$=4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx$$

$$=4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{4b}{a}\left[0+\frac{a^2}{2}\left(\frac{\pi}{2}\right)-0\right]$$

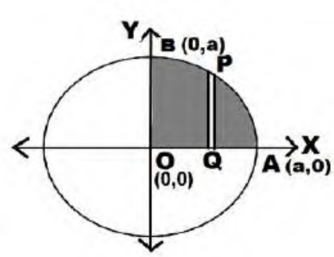
$=\pi ab$ squre units.

==

Ex: Using double integral, find the area of a circle of radius a.

Solution: The area of a circle of radius $x^2+y^2=a^2$ is given by

Area $=$ 4 (Area of region $OABO$) $=4\iint_R dx\,dy$



By taking strip PQ parallel to y-axis and moving it from $x=0$ to $x=a$.

$\therefore$ The limits of region $OABO$ are $0\le x\le a$ & $0\le y\le\sqrt{a^2-x^2}$

$\therefore$ Area of circle $=4\int_0^a\int_0^{\sqrt{a^2-x^2}} dy\,dx$

$$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}}\ dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\ dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[0+\frac{a^2}{2}\left(\frac{\pi}{2}\right)-0\right]$$

$= a^2\pi$ Square units.

==

Ex: By using double integral, Find the area of the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Solution: First we find point of intersection of the parabolas $y^2 = 4x$ and $x^2 = 4y$ by solving together as follows

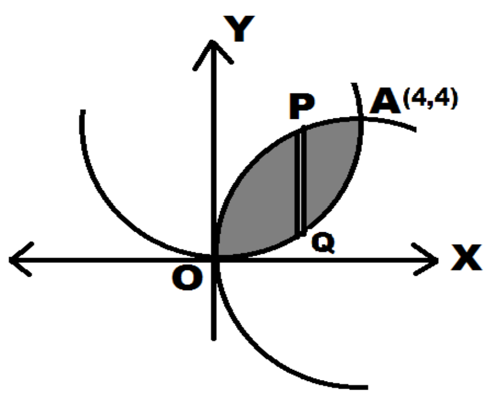
$$\left(\frac{x^2}{4}\right)^2 = 4x \quad i.e.\ \ x^4 = 64x \quad i.e.\ \ x(x^3-64) = 0$$

$\Rightarrow\ x = 0\ \ or\ \ x = 4$

For $x = 0 \Rightarrow y = 0$ & $x = 4 \Rightarrow y = 4.$

$\therefore$ The point of intersections are $O(0,0)$ & $A(4,4)$.

The required region is shown in figure.



By taking strip PQ parallel to y-axis and moving it from $x = 0$ and $x = 4$.

We get the limits of region as $0 \le x \le 4$ & $\frac{x^2}{4} \le y \le 2\sqrt{x}$.

$\therefore$ Area of region $= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\, dx$

$$= \int_0^4 [y]_{x^2/4}^{2(x)^{\frac{1}{2}}} \; dx$$

$$= \int_0^4 \left[2x^{\frac{1}{2}} - \frac{x^2}{4}\right] \; dx$$

$$= \left[2\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}(4)^{\frac{3}{2}} - \frac{4^3}{12}\right]$$

$$= \frac{32}{3} - \frac{16}{3}$$

$= \frac{16}{3}$ square units.

==

Ex: By using double integral, Find the area of the region bounded by the parabolas $y^2 = 2x$ and $x^2 = 2y$.

Solution: First we find point of intersection of the parabolas $y^2 = 2x$ and $x^2 = 2y$

by solving together as follows

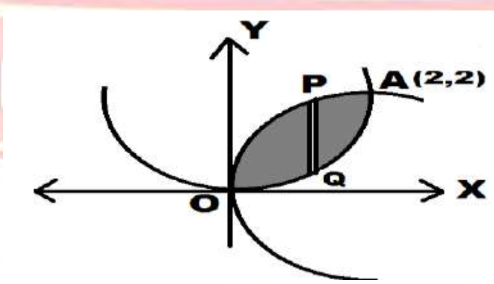
$$\left(\frac{x^2}{2}\right)^2 = 2x \quad i.e. \;\; x^4 = 8x \quad i.e. \;\; x(x^3 - 8) = 0$$

$\Rightarrow \;\; x = 0 \;\; or \;\; x = 2$

For $x = 0 \;\; \Rightarrow \;\; y = 0 \;\;$ & $\;\; x = 2 \;\; \Rightarrow \;\; y = 2.$

$\therefore$ The point of intersections are $O(0,0)$ & $A(2,2)$.

The required region is shown in figure.



By taking strip PQ parallel to y-axis and moving it from $x = 0$ and $x = 2$.

We get the limits of region as $0 \le x \le 2$ & $\frac{x^2}{2} \le y \le \sqrt{2x}$.

$\therefore$ Area of region $= \int_0^2 \int_{x^2/2}^{\sqrt{2x}} dy\, dx$

$$= \int_0^2 [y]_{x^2/_2}^{(2x)^{\frac{1}{2}}} \; dx$$

$$= \int_0^2 \left[(2x)^{\frac{1}{2}} - \frac{x^2}{2}\right] \; dx$$

$$= \left[2^{\frac{1}{2}} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{6}\right]_0^2$$

$$= \left[\frac{2^{\frac{3}{2}}}{3}(2)^{\frac{3}{2}} - \frac{2^3}{6}\right]$$

$$= \frac{8}{3} - \frac{4}{3}$$

$= \frac{4}{3}$ square units.

===

Ex: By using double integral, Find the area of the region bounded by the parabolas $y^2 = x$ and $x^2 = y$.

Solution: First we find point of intersection of the parabolas $y^2 = x$ and $x^2 = y$

by solving together as follows

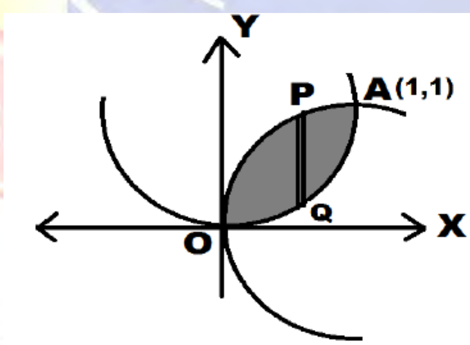
$(x^2)^2 = x \quad i.e. \;\; x^4 - x = 0 \quad i.e. \;\; x(x^3 - 1) = 0$

$\Rightarrow \;\; x = 0 \;\; or \;\; x = 1$

For $x = 0 \;\; \Rightarrow \;\; y = 0 \;\; \& \;\; x = 1 \;\; \Rightarrow \;\; y = 1.$

$\therefore$ The point of intersections are $O(0,0)$ & $A(1,1)$.

The required region is shown in figure.



By taking strip PQ parallel to y-axis and moving it from $x = 0$ and $x = 1$.

We get the limits of region as $0 \le x \le 1 \;\; \& \;\; x^2 \le y \le \sqrt{x}$.

$\therefore$ Area of region $= \int_0^1 \int_{x^2}^{\sqrt{x}} dy \, dx$

$$= \int_0^1 [y]_{x^2}^{(x)^{\frac{1}{2}}} \; dx$$

$$= \int_0^1 \left[(x)^{\frac{1}{2}} - x^2\right] \; dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1^3}{3}\right]$$

$$= \frac{2}{3} - \frac{1}{3}$$

$= \frac{1}{3}$ square units.

Ex: Calculate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^3 \sin\theta \,.\cos\theta \quad d\theta \;\; dr.$

Solution: Let, $I \; = \; \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^3 \sin\theta \,.\cos\theta \quad dr \;\; d\theta$

$$= \int_0^{\pi} \sin\theta.\cos\theta \; \left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} \; d\theta$$

$$= \frac{1}{4} \int_0^{\pi} \sin\theta.\cos\theta \;\; [a^4(1+\cos\theta)^4 - 0]d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi} (1+\cos\theta)^4 \cos\theta.\sin\theta \;\; d\theta$$

Put $1 + \cos\theta = t \quad \therefore \; -\sin\theta \;\; d\theta = dt \quad \therefore \; \sin\theta \;\; d\theta = -dt$

When $\theta = 0 \;\; \Rightarrow \;\; t = 2 \;\; \& \;\; \theta = \pi \;\; \Rightarrow \;\; t = 0$

$$= \frac{a^4}{4} \int_2^0 t^4 \,(t-1)(-dt)$$

$$= \frac{a^4}{4} \int_0^2 (t^5 - t^4) \;\; dt$$

$$= \frac{a^4}{4} \left[\frac{t^6}{6} - \frac{t^5}{5}\right]_0^2$$

$$= \frac{a^4}{4} \left[\frac{2^6}{6} - \frac{2^5}{5} - 0\right]$$

$$= \frac{a^4}{4} [\frac{32}{3} - \frac{32}{5}]$$

$$= 32 \, \frac{a^4}{4} [\frac{1}{3} - \frac{1}{5}]$$

$$= 8\,a^4 \left[\frac{2}{15}\right]$$

$$= \frac{16}{15} a^4$$

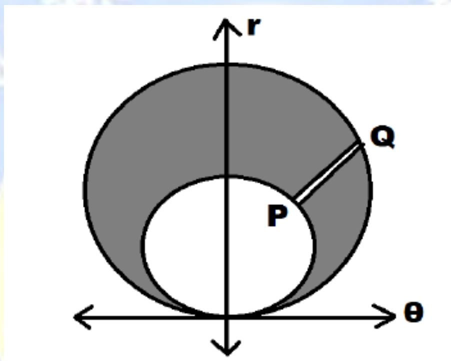
$$\boxed{\therefore \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^3 \sin\theta \,.\cos\theta \quad dr \;\; d\theta = \frac{16}{15}\, a^4}$$

===

Ex: Evaluate $\iint r^3\, dr\, d\theta$ over area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Solution: Let, region R is the area between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

By taking the strip PQ from $\theta = 0$ to $\theta = \pi$ then r lies between $2\sin\theta$ to $4\sin\theta$.



$$\therefore \iint_R r^3\, dr\;\, d\theta = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3\, dr.\, d\theta$$

$$= \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4} \int_0^{\pi} [256\, \sin^4\theta - 16\, \sin^4\theta] d\theta$$

$$= 60 \int_0^{\pi} \sin^4\theta \;\; d\theta$$

$$= 60 \times 2 \int_0^{\frac{\pi}{2}} \sin^4\theta \quad d\theta$$

$$= 120 \left[\frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}\right] \quad \text{by reduction formula}$$

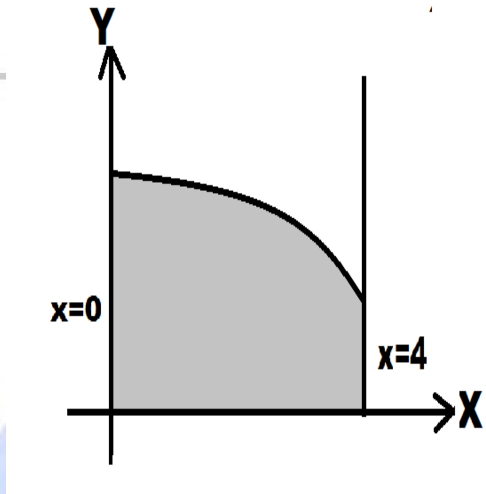
$$= \frac{45}{2} \pi.$$

===

Ex: Draw a sketch of the region of integration $\int_0^4 dx \int_0^{\sqrt{25-x^2}} f(x, y)\;\, dy$.

Solution: From given integration, the region bounded by $y = 0$ and $y = \sqrt{25 - x^2}$

$i.e. x^2 + y^2 = 25$ between the lines $x = 0$ & $x = 4$ as shown in figure.

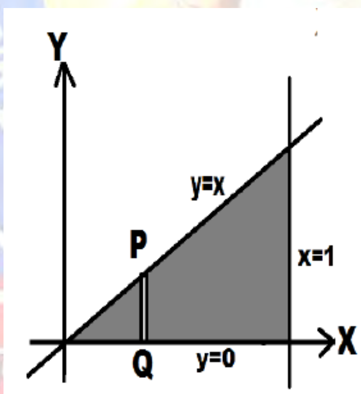


Be the sketch of the given region.

==

Ex: Evaluate $\iint_R e^{-x^2}\, dx\, dy,$ where R is the region bounded by the lines $y = 0, x = 1$ & $y = x$

Solution: Let region R is bounded by the lines $y = 0, x = 1$ & $y = x$ as shown in figure.



By taking strip PQ parallel to y-axis and moving it from $x = 0$ to $x = 1$, We get limits of region R as $0 \le x \le 1$ & $0 \le y \le x$.

$$\therefore \iint_R e^{-x^2}\, dx\, dy = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx$$

$$= \int_0^1 e^{-x^2}\, [y]_0^x\, dx$$

$$= \int_0^1 e^{-x^2}\, x\, dx$$

$$= \frac{-1}{2} \int_0^1 e^{-x^2} \ (-2x\, dx)$$

$$= \frac{-1}{2} \left[e^{-x^2}\right]_0^1$$

$$= \frac{-1}{2} \left[e^{-1} - 1\right]$$

$$= \frac{1}{2} \left(1 - \frac{1}{e}\right)$$

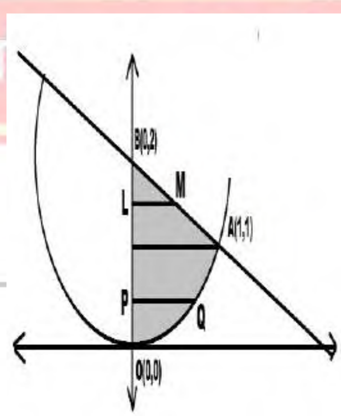
Change the order of Integration:

1) If given integration is $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y) \ dx\, dy,$ then the region is bounded by the curves $y = f_1(x)$ to $y = f_2(x)$ between the lines $x = a$ and $x = b$. We sketch this region first and then take strip PQ parallel to the x-axis and find the limits which give the change of order of integration.

2) If given integration is $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y) \ dx\, dy,$ then the region is bounded by the curves $x = g_1(y)$ to $x = g_2(y)$ between the lines $y = c$ and $y = d$. We sketch this region first and then take strip PQ parallel to the y-axis and find the limits which give the change of order of integration.

Ex: Change the order of integration $\int_0^1 \int_{x^2}^{2-x} f(x,y) \ dx\, dy.$

Solution: From given integration the region is bounded by $y = x^2$ and $y = 2 - x$ $i.e. x + y = 2$ between the lines $x = 0$ and $x = 1$ as shown in the figure.



To change the order of integration, we take strip PQ parallel to the x-axis. We observe that Q lies on curve $y = x^2$ up to A and on $x + y = 2$ from point A.

$\therefore$ We divide region into two sub-regions R_1 & R_2. For R_1 by taking strip PQ parallel to x-axis and moving it from $y = 0$ to $y = 1$. We get limit as $0 \le y \le 1$ and $0 \le x \le \sqrt{y}$.

For R_2 by taking strip LM parallel to the x-axis and moving it from $y = 1$ to $y = 2$.

We get limit as $1 \le y \le 2$ and $0 \le x \le 2 - y$

$\therefore$ Change of order of integration is

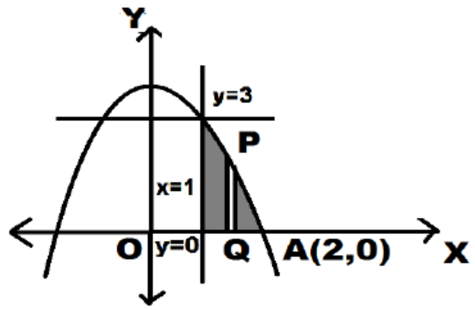
$$\therefore \int_0^1 \int_{x^2}^{2-x} f(x,y)\ dx\, dy = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\ dx\, dy + \int_1^2 \int_0^{2-y} f(x,y)\, dx\, dy$$

===

Ex: Change the order of integration $\int_0^3 \int_1^{\sqrt{4-y}} f(x,y)\ dx\, dy.$

Solution: From given integration the region is bounded by $x = 1$ and $x = \sqrt{4-y}$

$i.e.\ x^2 = 4 - y$ between the lines $y = 0$ and $y = 3$ as shown in the figure.



To change the order of integration, we have to integrate first w.r.t. y.

$\therefore$ We take strip PQ parallel to the y-axis and moving it from $x = 1$ to $x = 2$.

We get limits as $1 \le x \le 2$ and $0 \le y \le 4 - x^2$.

$\therefore$ Change of order of integration is

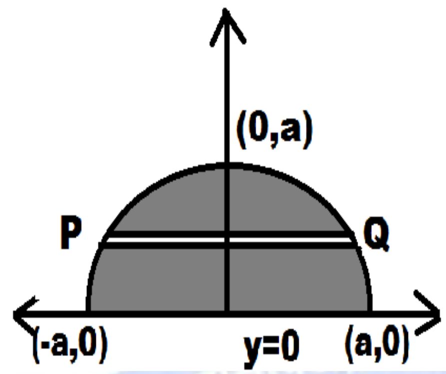
$$\therefore \int_0^3 \int_1^{\sqrt{4-y}} f(x,y)\ dx\, dy = \int_1^2 \int_0^{4-x^2} f(x,y)\ dy\, dx.$$

===

Ex: Change the order of integration $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} f(x,y)\ dx\, dy.$

Solution: From given integration the region is bounded by $y = 0$ and $y = \sqrt{a^2 - x^2}$

i.e. $x^2+y^2=a^2$ between the lines $x=-a$ and $x=a$ as shown in the figure.



To change the order of integration, we have to integrate first w.r.t. x.

∴We take strip PQ parallel to the x-axis and moving it from $y=0$ to $y=a$.

We get limit as $0\le y\le a$ and $-\sqrt{a^2-y^2}\le x\le\sqrt{a^2-y^2}$.

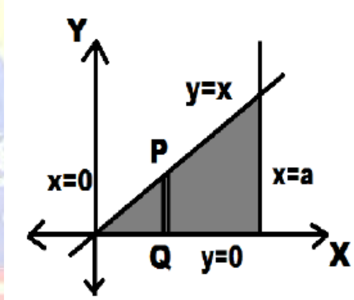
∴ Change of order of integration is

$$\therefore \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} f(x,y)\ dy\,dx=\int_{0}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\ dx\,dy.$$

==

Ex: Change the order of integration $\int_0^a\int_y^a \frac{x}{x^2+y^2}\ dx\,dy$ and hence evaluate it.

Solution: From given integration the region is bounded by $x=y$ and $x=a$ between the lines $y=0$ and $y=a$ as shown in the figure.



To change the order of integration, we have to integrate first w.r.t. y.

∴ We take strip PQ parallel to the y-axis and moving it from $x=0$ to $x=a$.

We get limits as $0\le x\le a$ and $0\le y\le x$.

∴ Change of order of integration is

$$\therefore \int_0^a\int_y^a \frac{x}{x^2+y^2}\ dx\,dy=\int_0^a\int_0^x \frac{x}{x^2+y^2}\ dy\,dx$$

$$=\int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x\ dx$$

$$=\int_0^a\left[\frac{\pi}{4}-0\right]\mathrm{dx}$$

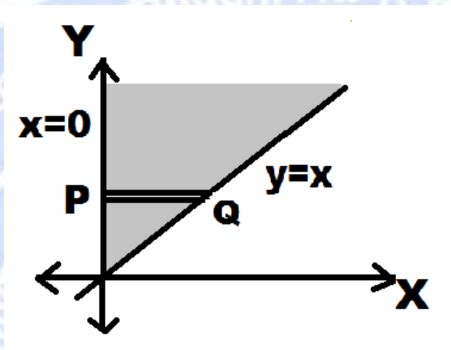
$$= \frac{\pi}{4}\,[x]_0^a$$

$$= \frac{\pi a}{4}$$

==

Ex: Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$ and hence evaluate it.

Solution: From given integration the region is bounded by $y = x$ and $y \to \infty$ between the lines $x = 0$ and $x \to \infty$ as shown in the figure.



To change the order of integration, we have to integrate first w.r.t. x.

∴We take strip PQ parallel to the x-axis and moving it from $y = 0$ to $y \to \infty$.

We get limits of region as $0 \le y < \infty$ and $0 \le x \le y$.

∴ Change of order of integration is

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, [x]_0^y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= [-e^{-y}]_0^\infty$$

$$= 0 + 1$$

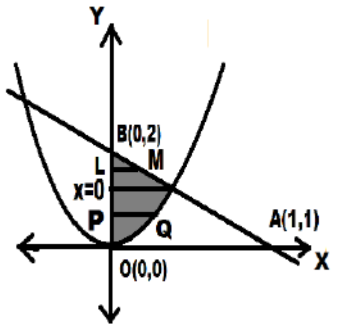
$$= 1$$

==

Ex: Change the order of integration $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate it.

Solution: From given integration the region is bounded by $y = x^2$ and $y = 2 - x$

$i.e. x + y = 2$ between the lines $x = 0$ and $x = 1$ as shown in the figure.



To change the order of integration, we take strip PQ parallel to the x-axis. We observe that Q lies on curve $y = x^2$ up to A and $x + y = 2$ from point A.

$\therefore$ We divide region into two sub-regions R_1 & R_2.

For R_1 by taking strip PQ parallel to x-axis and moving it from $y = 0$ to $y = 1$.

We get limit of region as $0 \le y \le 1$ and $0 \le x \le \sqrt{y}$.

For R_2 by taking strip LM parallel to the x-axis and moving it from $y = 1$ to $y = 2$.

We get limit as $1 \le y \le 2$ & $0 \le x \le 2 - y$.

$\therefore$ Change of order of integration is

$$\therefore \int_0^1 \int_{x^2}^{2-x} f(x,y)\ dx\ dy = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\ dx\ dy + \int_1^2 \int_0^{2-y} f(x,y)\ dx\ dy.$$

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\ dx\ dy = \int_0^1 \int_0^{\sqrt{y}} xy\ dx\ dy + \int_1^2 \int_0^{2-y} xy\ dx\ dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 y\left[\frac{y}{2} - 0\right] dy + \int_1^2 y\left[\frac{(2-y)^2}{2} - 0\right] dy$$

$$= \frac{1}{2}\int_0^1 y^2\ dy + \frac{1}{2}\int_1^2 y(4 - 4y + y^2)\ dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\, dy$$

$$= \frac{1}{2}\left[\frac{1}{3} - 0\right] + \frac{1}{2}\left[2y^2 - \frac{4}{3}y^3 + \frac{1}{4}y^4\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[12 - \frac{32}{3} - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{120-112-3}{12}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{5}{12}\right]$$
$$= \frac{1}{6} + \frac{5}{24}$$
$$= \frac{9}{24}$$
$$= \frac{3}{8}$$

==

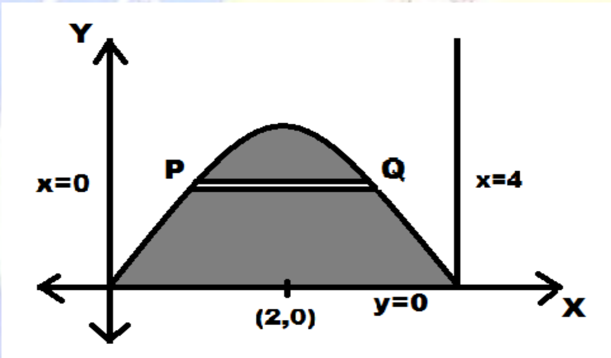
Ex: Change the order of integration $\int_0^4 \int_0^{\sqrt{4x-x^2}} f(x,y)\ dx\, dy$.

Solution: From given integration the region is bounded by

$y = 0$ and $y = \sqrt{4x - x^2}$ $i.e.$ $x^2 + y^2 - 4x = 0$

$i.e. (x-2)^2 + (y-0)^2 = 2^2$ $i.e.$ circle with centre at $(2, 0)$ and radius 2

between the lines $x = 0$ and $x = 4$ as shown in the figure.



To change the order of integration, we have to integrate first w.r.t. x and then w.r.t. y.

For that we take strip PQ parallel to the x-axis and moving it from $y = 0$ to $y = 2$.

We get limit as $0 \le y \le 2$ and $2 - \sqrt{4 - y^2} \le x \le 2 + \sqrt{4 - y^2}$.

∴ Change of order of integration is

$$\therefore \int_0^4 \int_0^{\sqrt{4x-x^2}} f(x,y)\ dy\, dx = \int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} f(x,y)\ dx\, dy.$$

==

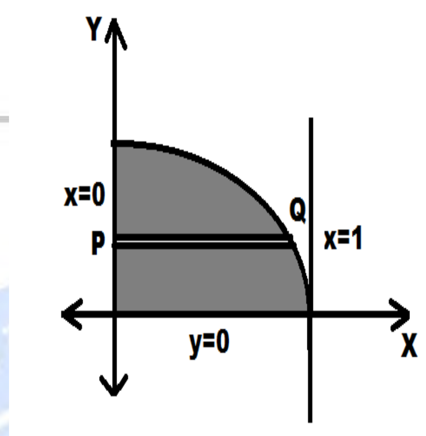
Ex: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{(1+e^y)\sqrt{1-x^2-y^2}}\ dx\, dy$.

Solution: We observe that, integration first w.r.t. y is not possible.

∴ We change the order of integration first then evaluate it.

∴ From given integration, the region is bounded by $y = 0$ & $y = \sqrt{1 - x^2}$

$i.e. x^2 + y^2 = 1$ circle with centre at origin and radius 1 between the lines $x = 0$ and $x = 1$ as shown in figure.



To change the order of integration we have to integrate first w.r.t. x and then w.r.t. y. For that we take strip PQ parallel to the x-axis and moving it from $y = 0$ to $y = 1$.

$\therefore$ The limits of region are $0 \le y \le 1$ & $0 \le x \le \sqrt{1-y^2}$

$\therefore$ Change of order of integration is

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{(1+e^y)\sqrt{1-x^2-y^2}}\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{(1+e^y)} \frac{1}{\sqrt{(1-y^2)-x^2}}\, dx\, dy$$

$$= \int_0^1 \frac{1}{1+e^y} \left[\sin^{-1} \frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{1}{(1+e^y)} \left[\frac{\pi}{2} - 0\right] dy$$

$$= -\frac{\pi}{2} \int_0^1 \frac{-e^{-y}}{e^{-y}+1}\, dy$$

$$= -\frac{\pi}{2} \left[\log(e^{-y}+1)\right]_0^1$$

$$= -\frac{\pi}{2} \left[\log\left(\frac{1}{e}+1\right) - \log 2\right]$$

$$= -\frac{\pi}{2} \left[\log\left(\frac{1+e}{2e}\right)\right]$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{(1+e^y)\sqrt{1-x^2-y^2}}\, dx\, dy = \frac{\pi}{2} \log\left(\frac{2e}{1+e}\right).$$

Triple Integral: If $f(x,y,z)$ is continuous in a region V in three dimensional space with V is divided into n-sub regions $\Delta V_1, \Delta V_2, \dots, \Delta V_r$ then for (x_r, y_r, z_r) lies in ΔV_r, triple integral is denoted by $\iiint_V f(x,y,z)\, dv$ and defined as

$\iiint_V f(x,y,z)\ dv = \lim_{\substack{n\to\infty \\ \Delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\Delta V_r$

==

Volume by triple integration:

Volume of the region V in a three dimensional space is given by

Volume of $V = \iiint_V dv = \iiint_V dxdydz$

==

Ex: Evaluate $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2yz\ dz\,dy\,dx.$

Solution: Let, $I = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2yz\ dz\,dy\,dx$

$= \int_{x=0}^{1}\int_{y=0}^{2} x^2y\left[\frac{z^2}{2}\right]_1^2 dy\,dx$

$= \int_{x=0}^{1}\int_{y=0}^{2} x^2y\left[2-\frac{1}{2}\right] dy\,dx$

$= \frac{3}{2}\int_{x=0}^{1} x^2\left[\frac{y^2}{2}\right]_0^2 dx$

$= \frac{3}{2}\int_{x=0}^{2} x^2[2-0]\ dx$

$= 3\left[\frac{x^3}{3}\right]_0^1$

$\therefore$ I = 1

==

Ex: Evaluate $\int_{y=0}^{3}\int_{x=0}^{2}\int_{z=0}^{1}(x+y+z)\ dz\,dy\,dx.$

Solution: Let $I = \int_{y=0}^{3}\int_{x=0}^{2}\int_{z=0}^{1}(x+y+z)\ dz\,dy\,dx$

$= \int_{y=0}^{3}\int_{x=0}^{2}[(x+y)z+\frac{z^2}{2}]_1^2\ dx\,dy$

$= \int_{y=0}^{3}\int_{x=0}^{2}[x+y+\frac{1}{2}-0]\ dx\,dy$

$= \int_{y=0}^{3}\left[\frac{x^2}{2}+\left(y+\frac{1}{2}\right)x\right]_0^2 dy$

$= \int_{y=0}^{3}[2+2y+1-0]\ dy$

$= \int_{y=0}^{3}[2y+3]\ dy$

$= [y^2+3y]_0^3$

$= 9+9-0$

$\therefore\ I = 18.$

==

Ex: Evaluate $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\ dx\,dy\,dz.$

Solution: Let $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\ dz\, dy\, dx$

$$= \int_0^a \int_0^x [e^{x+y+z}]_0^{x+y}\ dy\, dx$$

$$= \int_0^a \int_0^x [e^{2x+2y} - e^{x+y}]\ dy\, dx$$

$$= \int_0^a \left[\frac{1}{2}e^{2x+2y} - e^{x+y}\right]_0^x\ dx$$

$$= \int_0^a \left[\frac{1}{2}e^{4x} - e^{2x} - \frac{1}{2}e^{2x} + e^x\right]\ dx$$

$$= \int_0^a \left[\frac{1}{2}e^{4x} - \frac{3}{2}e^{2x} + e^x\right]\ dx$$

$$= \left[\frac{1}{8}e^{4x} - \frac{3}{4}e^{2x} + e^x\right]_0^a$$

$$= \left[\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a\right] - \left[\frac{1}{8} - \frac{3}{4} + 1\right]$$

$$= \frac{1}{8}\ [e^{4a} - 6e^{2a} + 8e^a] - \left[\frac{1-6+8}{8}\right]$$

$$\therefore\ I = \frac{1}{8}\ [e^{4a} - 6e^{2a} + 8e^a - 3]$$

==

Ex: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\ dx\, dy\, dz.$

Solution: Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\ dz\, dy\, dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{1-x^2-y^2}}\ dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left(\frac{\pi}{2} - 0\right)\ dy\, dx$$

$$= \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}}\ dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\ dx$$

$$= \frac{\pi}{2}\ \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \frac{\pi}{2}\ \left[0 + \frac{1}{2}\left(\frac{\pi}{2}\right) - 0\right]$$

$$\therefore\ I = \frac{\pi^2}{8}$$

==

Ex: Evaluate $\iiint (x+y+z)\ dx\, dy\, dz$ over the tetrahedron $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution: The region over tetrahedron $x = 0, y = 0, z = 0$ and $x + y + z = 1$ is expressed as $0 \le x \le 1, 0 \le y \le 1 - x$ and $0 \le z \le 1 - x - y$.

$$\therefore \iiint_V (x+y+z)\ dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\ dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y)z + \frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y)(1-x-y) + \frac{(1-x-y)^2}{2} - 0\right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y) - (x+y)^2 + \frac{1}{2} - (x+y) + \frac{1}{2}(x+y)^2\right] dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{1}{2} - \frac{1}{2}(x+y)^2\right] dy\,dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} [1-(x+y)^2]\ dy\,dx$$

$$= \frac{1}{2} \int_0^1 \left[y - \frac{1}{3}(x+y)^3\right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left[1 - x - \frac{1}{3}(1)^3 - 0 + \frac{1}{3}x^3\right] dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{2}{3} - x + \frac{1}{3}x^3\right] dx$$

$$= \frac{1}{2} \left[\frac{2}{3}x - \frac{1}{2}x^2 + \frac{1}{12}x^4\right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12} - 0\right]$$

$$= \frac{1}{2}\left[\frac{8-6+1}{12}\right]$$

$$\therefore \iiint_V (x+y+z)\ dx\,dy\,dz = \frac{1}{8}$$

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Ex: Evaluate $\iiint \frac{1}{(x+y+z+1)^3}\ dx\,dy\,dz$ over the region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$.

Solution: The given region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$

is expressed as $0 \leq x \leq 1, 0 \leq y \leq 1-x$ and $0 \leq z \leq 1-x-y$.

$$\therefore \iiint \frac{1}{(x+y+z+1)^3}\ dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z+1)^{-3}\ dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{-1}{2} \int_0^1 \int_0^{1-x} [2^{-2} - (x+y+1)^{-2}]\ dy\,dx$$

$$= \frac{-1}{2} \int_0^1 \left[\frac{1}{4}y - \frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= \frac{-1}{2} \int_0^1 \left[\frac{1}{4}(1-x) + (2)^{-1} - 0 - (x+1)^{-1}\right] dx$$

$$= \frac{-1}{2} \int_0^1 \left[\frac{1}{4} - \frac{1}{4}x + \frac{1}{2} - \frac{1}{(x+1)}\right] dx$$

$$= \frac{-1}{2}\int_0^1 \left[\frac{3}{4} - \frac{1}{4}x - \frac{1}{(x+1)}\right] dx$$
$$= \frac{-1}{2}\left[\frac{3}{4}x - \frac{1}{8}x^2 - \log(x+1)\right]_0^1$$
$$= \frac{-1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log 2 - 0\right]$$
$$= \frac{-1}{2}\left[\frac{5}{8} - \log 2\right]$$
$$\therefore \iiint \frac{dxdydz}{(x+y+z+1)^3} = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

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Ex: Using triple integration find the volume of a sphere of radius a.

Solution: The equation of sphere of radius a is $x^2 + y^2 + z^2 = a^2$.

The region of volume V of a sphere is expressed as

$-a \le x \le a, -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}$ and $-\sqrt{a^2 - x^2 - y^2} \le z \le \sqrt{a^2 - x^2 - y^2}$

$\therefore$ Volume of sphere $= \iiint_V dx\, dy\, dz$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$
$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$
$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}}\, dy\, dx$$
$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$$
$$= 8\int_0^a\left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$
$$= 8\int_0^a\left[0 + \frac{(a^2-x^2)}{2}\left(\frac{\pi}{2}\right) - 0\right] dx$$
$$= 2\pi\int_0^a (a^2 - x^2)\, dx$$
$$= 2\pi\left[a^2x - \frac{1}{3}x^3\right]_0^a$$
$$= 2\pi\left[a^3 - \frac{1}{3}a^3 - 0\right]$$
$$= \frac{4}{3}\pi a^3 \text{ cubic unit.}$$

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Ex: Find the volume of the region bounded by the co-ordinate planes $(i.e.\, x = 0, y = 0, z = 0)$ and $x + y + z = 1$.

Solution: The volume of the region bounded by the co-ordinates planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

The region of volume V of is expressed as

$0 \le x \le 1, 0 \le y \le 1 - x$ and $0 \le z \le 1 - x - y$

$\therefore V = \iiint_V dx\, dy\, dz = \int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\, dy\, dx$

$$= \int_0^1 \int_0^{1-x} [z]_0^{1-x-y} \; dy\, dx$$
$$= \int_0^1 \int_0^{1-x} [(1-x-y)] \; dy\, dx$$
$$= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2\right]_0^{1-x} \; dx$$
$$= \int_0^1 \left[(1-x)^2 - \frac{1}{2}(1-x)^2 - 0\right] dx$$
$$= \frac{1}{2}\int_0^1 [(1-x)^2] \; dx$$
$$= \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1$$
$$= \frac{-1}{6}[(1-x)^3]_0^1$$
$$= \frac{-1}{6}[0-1]$$
$$\therefore V = \frac{1}{6} \text{ cubic unit.}$$

Ex: Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3, z = 0$.

Solution: The region V bounded by cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3, z = 0$ is expressed as $-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$ and $0 \le z \le 3 - y$.

$\therefore$ Volume $= \iiint_V dx\, dy\, dz$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{3-y} dz\, dy\, dx$$
$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{3-y} \; dy\, dx$$
$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-y) \; dy\, dx$$
$$= 4\int_0^2 \int_0^{\sqrt{4-x^2}} 3\, dy\, dx$$
$$\because \int_{-a}^{a} f(x)\; dx = \begin{cases} 2\int_0^a f(x)\, dx \; if\; f(x) \; is\; even\; function \\ 0 \qquad if\; f(x)\; is\; odd\; function \end{cases}$$
$$= 12\int_0^2 \int_0^{\sqrt{4-x^2}} dy\, dx$$
$$= 12\int_0^2 [y]_0^{\sqrt{4-x^2}} \; dx$$
$$= 12\int_0^2 \sqrt{4-x^2} \; dx$$
$$= 12\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$
$$= 12\left[0 + 2\left(\frac{\pi}{2}\right) - 0\right]$$

$\therefore$ Volume $= 12\pi$ cubic unit.

॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."