Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: F.Y.B.SC SEM.-II

SUBJECT: MTH-203(B): NUMERICAL ANALYSIS

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MTH 203(B): NUMERICAL ANALYSIS

UNIT-I Solution of Algebraic and Transcendental Equations **No. of Periods – 12**

Errors and their computation. Absolute, relative and percentage errors. The Bisection method. The iteration method. The method of false position. Newton-Raphson method.

UNIT-II Interpolation **No. of Periods – 12**

Finite differences: Forward differences, backward differences, central differences, Symbolic relations and separation of symbols. Gauss's forward difference formula. Gauss's backward difference formula. Interpolations with unevenly spaced points. Lagrange's interpolation formula. Inverse Lagrange's Formula.

UNIT-III Curve Fitting **No. of Periods – 11**

Least squares curve fitting procedures. Fitting of straight line. Non-linear curve fitting: power function. Fitting of polynomial of degree two. Fitting of exponential function

UNIT-IV Numerical Solutions of Ordinary Differential Equations **No of Periods – 10**

Numerical solution of first order ODE by Taylor's series, Euler's method and Modified Euler's method. Runge–Kutta methods Runge–Kutta second and fourth order formulae.

Reference books:

1. Introductory Methods of Numerical Analysis, by S. S. Sastry, Prentice Hall India Learning Private Limited; Fifth edition, 2012.
2. Introduction to Numerical Analysis, by Carl-Erik Froberg, Addison-Wesley, Second edition, 1979.
3. Numerical Methods by V.N. Vedamurty and N. Ch. S. N. Iyehgar, Vikas Publishing House, India, 1995.
4. Numerical methods for scientific and engineering computation, by M. K. Jain, S. R. K. Iyenger and R. K. Jain. New Age International Publisher Pvt. Ltd., 1999.

Learning Outcomes:

Student will be able to:

a) understand basic concepts of methods of solutions of equations viz. bisection, iteration, Newton-Raphson methods and method of false position.

b) understand methods of curve fitting viz. Gauss's forward and backward difference formulae and Lagrange's interpolation formula.

c) use of curve fitting such as least square, polynomial and exponential fittings for set of given data.

d) use Taylor's series, Euler's method. Modified Euler's method., Runge Kutta methods for solving ordinary differential equations.

UNIT-1: SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Significant Digit: To write a number x correct to n significant digits write the number from x in which there are n digits from leading non-zero number in x.

Ex. Write a number a = 3.4560712 correct to 2, 4 and 6 significant digits.

Solution: A number a = 3.4560712 correct to 2 significant digits is a = 3.4

A number a = 3.4560712 correct to 4 significant digits is a = 3.446

A number a = 3.4560712 correct to 6 significant digits is a = 3.45607

Ex. Write a number b = 0.007102506 correct to 2, 4 and 6 significant digits.

Solution: A number b = 0.007102506 correct to 2 significant digits is b = 0.0071

A number b = 0.007102506 correct to 4 significant digits is b = 0.007102

A number b = 0.007102506 correct to 6 significant digits is b = 0.00710250

Ex. Write significant digits of numbers

i) 3.57 × 104, ii) 3.570 × 104, iii) 3.5700 × 104,

Solution: i) Significant digit of a number 3.57 × 104 is 3

ii) Significant digit of a number 3.570 × 104 is 4

iii) Significant digit of a number 3.5700 × 104 is 5

Rounded off: To round off a number x to n^{th} decimal place digit, to right from its decimal point. If the digit in the $(n+1)^{th}$ is greater than 4, we take x upto n^{th} decimal place and add 1×10^{-n} i.e. add 1 in n^{th} decimal place digit to get the rounded off value of x, if the digit in the n^{th} place is odd, and add 0 if the digit in the n^{th} place is even. Otherwise we take x as the number formed by leaving its $(n+1)^{th}$ decimal place onwards.

Ex. Round off a number a = 3.4560712 to 1, 2 and 6 decimal place.

Solution: A number a = 3.4560712 rounded to 1 decimal place is a = 3.4

A number a = 3.4560712 rounded to 2 decimal place is a = 3.46

A number a = 3.4560712 rounded to 6 decimal place is a = 3.456071

Truncation: To truncate a number x in its decimal fractional part to n^{th} place is formed by leaving its $(n+1)^{th}$ decimal place onwards.

Truncation Error: If x_1 is truncated number of number x, then $x - x_1$ is called truncated error.

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Ex. Find the truncated error when a number a = 3.4560712 is

i) truncated to 2^{nd} decimal place,

ii) truncated to 5^{th} decimal place,

iii) truncated to 6^{th} decimal place

Solution: i) A number a = 3.4560712 is truncated to 2^{nd} decimal place $a_1 = 3.45$

$\therefore$ Truncated error = $a - a_1 = 3.4560712 - 3.45 = 0.0060712$

ii) A number a = 3.4560712 is truncated to 5^{th} decimal place $a_1 = 3.45607$

$\therefore$ Truncated error = $a - a_1 = 3.4560712 - 3.45607 = 0.0000012$

iii) A number a = 3.4560712 is truncated to 6^{th} decimal place $a_1 = 3.456071$

$\therefore$ Truncated error = $a - a_1 = 3.4560712 - 3.456071 = 0.0000002$

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Error: If a true value of number x is approximated by a number x_1, then $E = \Delta x = x - x_1$ is called an error.

Absolute Error: If a true value of number x is approximated by a number x_1, then $E_a = |E| = |x - x_1|$ is called an absolute error.

Relative Error: If a true value of number x is approximated by a number x_1, then $E_r = \frac{\Delta x}{x} = \frac{x-x_1}{x}$ is called a relative error .

Percentage Error: If a true value of number x is approximated by a number x_1, then $E_p = 100\, E_r = \frac{x-x_1}{x} \times 100$ is called a percentage error.

Remark: If a number x is rounded to N decimal place, then error $E = \Delta x = \frac{1}{2} \times 10^{-N}$

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Ex. If x = 0.64 is corrected to 2 decimal place, then find relative error and percentage error.

Solution: Let x = 0.64 is corrected to 2 decimal place.

$\therefore$ Error $E = \Delta x = \frac{1}{2} \times 10^{-2} = 0.005$

Relative error = $E_r = \frac{\Delta x}{x} = \frac{0.005}{0.64} = 0.0078$

and Percentage error = 100 $E_r = \frac{x-x_1}{x} \times 100 = \frac{0.005}{0.64} \times 100 = 0.78$

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Ex. If $\sqrt{11}$ is approximated by 3, then find its error, absolute error, relative error and percentage error.

Solution: Let x = $\sqrt{11}$ is approximated by $x_1 = 3$.

∴ Error E = Δx = x – x_1 = $\sqrt{11} - 3$

Absolute error = $|E| = |x - x_1| = |\sqrt{11} - 3| = \sqrt{11} - 3 \quad \because \sqrt{11} > 3$

Relative error = $E_r = \frac{\Delta x}{x} = \frac{\sqrt{11} - 3}{\sqrt{11}}$

and Percentage error = 100 $E_r = \frac{x-x_1}{x} \times 100 = \frac{\sqrt{11} - 3}{\sqrt{11}} \times 100$

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Ex. If $\sqrt{11}$ is approximated by 4, then find its error, absolute error, relative error and percentage error.

Solution: Let x = $\sqrt{11}$ is approximated by $x_1 = 4$.

∴ Error E = Δx = x – x_1 = $\sqrt{11} - 4$

Absolute error = $|E| = |x - x_1| = |\sqrt{11} - 4| = 4 - \sqrt{11} \qquad \because \sqrt{11} < 4$

Relative error = $E_r = \frac{\Delta x}{x} = \frac{4-\sqrt{11}}{\sqrt{11}}$

and Percentage error = 100 $E_r = \frac{4-\sqrt{11}}{\sqrt{11}} \times 100$

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Ex. An approximate value of π is given by $\frac{22}{7} = 3.14285714$ and its true value is x = 3.14159265, then find the absolute error and the relative error.

Solution: Let a true value of π, x = 3.14159265 is approximated by

$x_1 = \frac{22}{7} = 3.14285714$

∴ Error E = Δx = x – x_1 = 3.14159265 – 3.14285714 = – 0.00126449

Absolute error = $E_a = |E| = |x - x_1| = |-0.00126449| = 0.00126449$

and Relative error = $E_r = \frac{0.00126449}{3.14159265} = 0.00040250$

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Ex. Three approximate values of $\frac{1}{6}$ are given by 0.16, 0.166 and 0.165. Find the best approximation of these.

Solution: Three approximate values of $\frac{1}{6}$ are given by 0.16, 0.166 and 0.165

∴ Absolute errors are

i) $|x - x_1| = \left|\frac{1}{6} - 0.16\right| = \left|\frac{1}{6} - \frac{8}{50}\right| = \frac{1}{150}$

ii) $|x - x_1| = \left|\frac{1}{6} - 0.166\right| = \left|\frac{1}{6} - \frac{249}{1500}\right| = \frac{1}{1500}$

i) $|x - x_1| = \left|\frac{1}{6} - 0.165\right| = \left|\frac{1}{6} - \frac{33}{200}\right| = \frac{1}{600}$

The best approximation is 0.166 because it has the least error.

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Polynomials: If $a_0, a_1, a_2, \ldots\ldots, a_n$ are real numbers with $a_0 \neq 0$, then $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ is called a general polynomial of variable x of degree n.

Polynomials Equation: If $a_0, a_1, a_2, \ldots\ldots, a_n$ are real numbers with $a_0 \neq 0$ and x is variable, then $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n = 0$ is called a polynomial equation of degree n

Algebraic Equation: An equation of type $y = f(x)$ is called an algebraic equation if it can be expressed as $f_ny_n + f_{n-1}y_{n-1} + \ldots\ldots + f_1y_1 + f_0 = 0$, where f_i is an i^{th} order polynomial in x.

Transcendental Equation: A non-algebraic equation is called a transcendental equation if it involves trigonometric, exponential, logarithmic etc. functions.

e.g. $4\sin x - e^x = 0$ and $\log x^3 - 5\tan x = 0$ are the transcendental equations

Root of an Equation: The value of x which satisfies the given equation $f(x) = 0$ is called root of an equation.

Note: A transcendental equation may have finite or infinite number of real roots or may not have real roots.

Solution of an Equation: The process of finding the roots of an equation $f(x) = 0$ is called solution of that equation.

Remark: i) Geometrically a root of an equation $f(x) = 0$ is that value of x where the graph of $y = f(x)$ cuts the x-axis.

ii) If $f(x)$ is divisible by $(x - a)$, then $x = a$ is the root of $f(x) = 0$.

iii) Every algebraic equation of degree n has only n roots. Which are real or imaginary.

iv) If f(x) is continuous in the interval (a, b) and f(a), f(b) have an opposite signs, then the equation f(x) = 0 has at least one root between x = a and x = b.

v) Every equation of odd degree has at least one real root.

vi) If a polynomial of degree n vanishes for more than n values of x, it must be identically zero.

Methods of Solutions: The methods of solution of the equation includes.

i) Direct Methods, ii) Graphical Methods,

iii) Trial and Error Methods, iv) Iterative Methods,

Remark: 1) In this unit we use four iterative methods-

i) Bisection method, ii) Regula falsi method,

iii) Iteration method, iv) Newton Raphson method,

2) An iterative technique begins with an approximate value of the root which is known as initial guess.

The Bisection method: If f(x) = 0 be the given equation with f(a) < 0 and f(b) > 0, then there exist one root between a and b. In this method we assume that the approximate value of root is $x_0 = \frac{a+b}{2}$.

If $f(x_0) = 0$, then x_0 is the root of equation f(x) = 0.

But if $f(x_0) \neq 0$, then the root lies either between x_0 and a or between x_0 and b.

If $f(x_0) > 0$, then the root lies between x_0 and a.

If $f(x_0) < 0$, then the root lies between x_0 and b.

Denote the new interval by $[a_1, b_1]$ whose length is $\frac{|b-a|}{2}$.

Let $x_1 = \frac{a_1+b_1}{2}$. As before $[a_1, b_1]$ is bisected at x_1 and new interval will be exactly half of previous one.

We repeat the process until the latest interval which contains the root is as small as desired, say ϵ.

Note that the interval width is reduced half of previous one at each step and at the end of n^{th} step, the new interval $[a_n, b_n]$ of length is $\frac{|b-a|}{2^n}$.

$\therefore$ We have $\frac{|b-a|}{2^n} \leq \epsilon$.

Taking log on both sides, we get,

$\log [\frac{|b-a|}{2^n}] \leq \log \epsilon$

i.e. $\log(|b-a|)$-n log 2 $\leq \log \epsilon$

i.e. $\log(|b-a|)$- $\log \epsilon \leq$ n log 2

i.e. nlog 2 $\geq \log \{\frac{|b-a|}{\epsilon}\}$

$\therefore n \geq \frac{\log \{\frac{|b-a|}{\epsilon}\}}{\log 2}$

which gives the number of iterations required to get an accuracy ϵ.

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Note: i) The Bisection method fails if f(x) is not continuous in [a, b],

ii) The Bisection method gives the real root of an equation f(x) = 0.

iii) The Bisection method is always convergent.

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Ex. Find the real root of the equation $x^3 - x - 4 = 0$ by using bisection method, perform five iterations.

Solution: Let $f(x) = x^3 - x - 4 = 0$ be the given equation with

$f(1) = 1 - 1 - 4 = -4 < 0$ and $f(2) = 8 - 2 - 4 = 2 > 0$

∴ The root lies 1 and 2.

i) Take $x_0 = \frac{1+2}{2} = 1.5$

$\therefore f(x_0) = f(1.5) = (1.5)^3 - 1.5 - 4 = -2.125 < 0$

∴ The root lies 1.5 and 2.

ii) Let $x_1 = \frac{1.5+2}{2} = 1.75$

$\therefore f(x_1) = f(1.75) = (1.75)^3 - 1.75 - 4 = -0.390625 < 0$

∴ The root lies 1.75 and 2.

iii) Let $x_2 = \frac{1.75+2}{2} = 1.875$

$\therefore f(x_2) = f(1.875) = (1.875)^3 - 1.875 - 4 = 0.716796875 > 0$

∴ The root lies 1.75 and 1.875

iv) Let $x_3 = \frac{1.75+1.875}{2} = 1.8125$

$\therefore f(x_3) = f(1.8125) = (1.8125)^3 - 1.8125 - 4 = 0.141845703 > 0$

∴ The root lies 1.75 and 1.8125

v) Let $x_4 = \frac{1.75+1.8125}{2} = 1.78125$

be the approximate value of the root upto fifth iteration.

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Ex. Find the real root of the equation $x^3 – x - 1 = 0$ by bisection method, perform five iterations.

Solution: Let $f(x) = x^3 – x - 1 = 0$ be the given equation with

$f(1) = 1 - 1 – 1 = - 1 < 0$ and $f(2) = 8 – 2 - 1 = 5 > 0$

∴ The root lies 1 and 2.

i) Take $x_0 = \frac{1+2}{2} = 1.5$

∴ $f(x_0) = f(1.5) = (1.5)^3 – 1.5 – 1 = 0.875 > 0$

∴ The root lies 1 and 1.5.

ii) Let $x_1 = \frac{1+1.5}{2} = 1.25$

∴ $f(x_1) = f(1.25) = (1.25)^3 – 1.25 – 1 = -0.296875 < 0$

∴ The root lies 1.25 and 1.5.

iii) Let $x_2 = \frac{1.25+1.5}{2} = 1.375$

∴ $f(x_2) = f(1.375) = (1.375)^3 – 1.375 – 1 = 0.224609375 > 0$

∴ The root lies 1.25 and 1.375

iv) Let $x_3 = \frac{1.25+1.375}{2} = 1.3125$

∴ $f(x_3) = f(1.3125) = (1.3125)^3 – 1.3125 – 1 = -0.0515136719 < 0$

∴ The root lies 1.3125 and 1.375

v) Let $x_4 = \frac{1.3125+1.375}{2} = 1.34375$

be the approximate value of the root upto fifth iteration.

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Ex. Find a root of the equation $x^3 – 4x - 9 = 0$ using bisection method, upto fourth iterations.

Solution: Let $f(x) = x^3 – 4x - 9 = 0$ be the given equation with

$f(2) = 8 - 8 – 9 = - 9 < 0$ and $f(3) = 27 – 12 - 9 = 6 > 0$

∴ The root lies 2 and 3.

i) Take $x_0 = \frac{2+3}{2} = 2.5$

$\therefore$ $f(x_0) = f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375 < 0$

$\therefore$ The root lies 2.5 and 3.

ii) Let $x_1 = \frac{2.5+3}{2} = 2.75$

$\therefore$ $f(x_1) = f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.796875 > 0$

$\therefore$ The root lies 2.5 and 2.75.

iii) Let $x_2 = \frac{2.5+2.75}{2} = 2.625$

$\therefore$ $f(x_2) = f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.41210938 < 0$

$\therefore$ The root lies 2.625 and 2.75

iv) Let $x_3 = \frac{2.625+2.75}{2} = 2.6875$

be the approximate value of the root upto fourth iteration.

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Ex. Using bisection method determine a real root of the equation $8x^3 - 2x - 1 = 0$ upto fourth iterations.

Solution: Let $f(x) = 8x^3 - 2x - 1 = 0$ be the given equation with

$f(0) = 0 - 0 - 1 = -1 < 0$ and $f(1) = 8 - 2 - 1 = 5 > 0$

$\therefore$ The root lies 0 and 1.

i) Take $x_0 = \frac{0+1}{2} = 0.5$

$\therefore$ $f(x_0) = f(0.5) = 8(0.5)^3 - 2(0.5) - 1 = -1 < 0$

$\therefore$ The root lies 0.5 and 1.

ii) Let $x_1 = \frac{0.5+1}{2} = 0.75$

$\therefore$ $f(x_1) = f(0.75) = 8(0.75)^3 - 2(0.75) - 1 = 0.875 > 0$

$\therefore$ The root lies 0.5 and 0.75.

iii) Let $x_2 = \frac{0.5+0.75}{2} = 0.625$

$\therefore$ $f(x_2) = f(0.625) = 8(0.625)^3 - 2(0.625) - 1 = -0.296875 < 0$

$\therefore$ The root lies 0.625 and 0.75

iv) Let $x_3 = \frac{0.625+0.75}{2} = 0.6875$

be the approximate value of the root upto fourth iteration.

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Ex. Find a positive root of the equation $xe^x = 1$, which lies between 0 and 1, perform four iterations.

Solution: Let $f(x) = xe^x - 1 = 0$ be the given equation with

$f(0) = 0 – 1 = - 1 < 0$ and $f(1) = e – 1 = 2.718 – 1 = 1.718 > 0$

∴ The root lies 0 and 1.

i) Take $x_0 = \frac{0+1}{2} = 0.5$

∴ $f(x_0) = f(0.5) = 0.5(e^{0.5}) – 1 = - 0.175639365 < 0$

∴ The root lies 0.5 and 1.

ii) Let $x_1 = \frac{0.5+1}{2} = 0.75$

∴ $f(x_1) = f(0.75) = 0.75(e^{0.75}) – 1 = 0.587750012 > 0$

∴ The root lies 0.5 and 0.75.

iii) Let $x_2 = \frac{0.5+0.75}{2} = 0.625$

∴ $f(x_2) = f(0.625) = 0.625(e^{0.625}) – 1 = 0.167653723 > 0$

∴ The root lies 0.5 and 0.625

iv) Let $x_3 = \frac{0.5+0.625}{2} = 0.5625$

be the approximate value of the root upto fourth iteration.

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The Iteration method: If f(x) = 0 be the given equation with f(a) < 0 and f(b) > 0, then there exist one root between a and b. In this method we take an approximate value x_0 of a desired root λ which lies between a and b. Express the equation f(x) = 0 as

$x = \emptyset(x)$ ……(i) such that $|\emptyset'(x)| < 1 \ \forall \ x \in (a, b)$.

Putting $x = x_0$ in equation (i), we get first approximation as $x_1 = \emptyset(x_0)$,

and next successive approximations are obtained as follows

$x_2 = \emptyset(x_1)$, $x_3 = \emptyset(x_2)$ and so on in general $x_r = \emptyset(x_{r-1})$

i.e. $x_n = \emptyset(x_{n-1})$ where n = 1, 2, 3, ……

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Note: i) The successive approximations $x_1, x_2, x_3, ……, x_n$ must converges to desired root λ.

ii) The function $\emptyset(x)$ should be chosen such that the sequence should converges to root. It happens when $|\emptyset'(x)| < 1 \ \forall x \in (a, b)$.

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Ex. Use Iterative method to find the real root of the equation $x^3 + x - 5 = 0$ correct to four significant figures.

Solution: Let $f(x) = x^3 + x - 5 = 0$ be the given equation with

$f(1) = 1 + 1 – 5 = - 3 < 0$ and $f(2) = 8 + 2 - 5 = 5 > 0$

∴ The root lies 1 and 2.

Now $x^3 + x - 5 = 0 \Rightarrow x^3 = 5 – x$ i.e. $x = (5-x)^{1/3} = \emptyset(x)$

where $\emptyset(x) = (5-x)^{1/3}$

$\therefore \emptyset'(x) = -\frac{1}{3}(5-x)^{-2/3} = \frac{-1}{3(5-x)^{2/3}}$

$\therefore |\emptyset'(x)| = \left|\frac{-1}{3(5-x)^{2/3}}\right| < 1 \ \forall\ x \in (1, 2).$

∴ Iteration method is applicable.

∴ Take $x_0 = 1.5$

i) First approximation is

$x_1 = \emptyset(x_0) = \emptyset(1.5) = (5-1.5)^{1/3} = 1.518294486$

ii) Second approximation is

$x_2 = \emptyset(x_1) = \emptyset(1.518294486) = (5-1.518294486)^{1/3} = 1.51564449$

iii) Third approximation is

$x_3 = \emptyset(x_2) = \emptyset(1.51564449) = (5-1.51564449)^{1/3} = 1.516028922$

∴ The approximate value of the root correct to four significant figures is 1.516

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Ex. Find the real root of the equation $x^3 + x^2 - 1 = 0$ which lies between 0 and 1, correct upto four decimal places by calculating three approximations by iteration method.

Solution: Let $f(x) = x^3 + x^2 - 1 = 0$ be the given equation which expressed as

$x^2 (x+1) = 1$ i.e. $x^2 = \frac{1}{x+1}$ i.e. $x = \frac{1}{\sqrt{x+1}} = \emptyset(x)$

where $\emptyset(x) = (x+1)^{-1/2}$

$\therefore \emptyset'(x) = -\frac{1}{2}(x+1)^{-3/2} = \frac{-1}{2(x+1)^{3/2}}$

$\therefore |\emptyset'(x)| = \left|\frac{-1}{2(x+1)^{3/2}}\right| < 1 \ \forall\ x \in (0, 1).$

∴ Iteration method is applicable.

∴ Take $x_0 = 0.75$

i) First approximation is

$x_1 = \emptyset(x_0) = \emptyset(0.75) = \frac{1}{\sqrt{0.75+1}} = 0.7559289$

ii) Second approximation is

$x_2 = \emptyset(x_1) = \emptyset(0.7559289) = \frac{1}{\sqrt{0.7559289+1}} = 0.7546517$

iii) Third approximation is

$x_3 = \emptyset(x_2) = \emptyset(0.7546517) = \frac{1}{\sqrt{0.7546517+1}} = 0.7549263$

∴ The approximate value of the root is 0.7549 up to four decimal places.

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Ex. Find the real root of the equation $x^3 + x - 1 = 0$ which lies between 0 and 1, correct upto four decimal places by calculating three approximations by iteration method.

Solution: Let $f(x) = x^3 + x - 1 = 0$ be the given equation

$f(0) = 0 + 0 – 1 = - 1 < 0$ and $f(1) = 1 + 1 - 1 = 1 > 0$

∴ The root lies 0 and 1.

Now $x^3 + x - 1 = 0 \Rightarrow x (x^2+1) = 1$ i.e. $x = \frac{1}{x^2+1} = \emptyset(x)$

$\therefore \emptyset'(x) = \frac{-2x}{(x^2+1)^2}$

$\therefore |\emptyset'(x)| = \left|\frac{-2x}{(x^2+1)^2}\right| < 1 \ \forall\ x \in (0, 1).$

∴ Iteration method is applicable.

∴ Take $x_0 = 0.5$

i) First approximation is

$x_1 = \emptyset(x_0) = \emptyset(0.5) = \frac{1}{0.25+1} = 0.8$

ii) Second approximation is

$x_2 = \emptyset(x_1) = \emptyset(0.8) = \frac{1}{0.64+1} = 0.609756 \approx 0.61$

iii) Third approximation is

$x_3 = \emptyset(x_2) = \emptyset(0.61) = \frac{1}{0.3721+1} = 0.7288$

∴ The approximate value of the root is 0.7288.

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Ex. Find the root of the equation $2x - \cos x - 3 = 0$ by iteration method.

Take $x_0 = \frac{\pi}{2}$ and calculate two approximations.

Solution: Let $f(x) = 2x - \cos x - 3 = 0$ be the given equation which expressed as

$x = \frac{3+\cos x}{2} = \emptyset(x)$

$\therefore \emptyset'(x) = -\frac{1}{2}\sin x$

$\therefore |\emptyset'(x)| = \left|-\frac{1}{2}\sin x\right| < 1 \ \forall\ x \in R$

∴ Iteration method is applicable.

∴ Take $x_0 = \frac{\pi}{2}$

i) First approximation is

$x_1 = \emptyset(x_0) = \emptyset(\frac{\pi}{2}) = \frac{3+\cos\frac{\pi}{2}}{2} = 1.5$

ii) Second approximation is

$x_2 = \emptyset(x_1) = \emptyset(1.5) = \frac{3+\cos 1.5}{2} = 1.9998$

∴ The approximate value of the root is 1.9998.

==

Ex. Starting woth $x_0 = 0.12$ solve the equation $x = 0.21\sin(0.5 + x)$ by iteration method. Perform third iterations.

Solution: Let $x = 0.21\sin(0.5 + x) = \emptyset(x)$

∴ $\emptyset'(x) = 0.21\cos(0.5 + x)$

∴ $|\emptyset'(x)| = |0.21\cos(0.5 + x)| < 1 \ \forall\ x \in R$

∴ Take $x_0 = 0.12$

i) First approximation is

$x_1 = \emptyset(x_0) = \emptyset(0.12) = 0.21\sin(0.5 + 0.12) = 0.00227237$

ii) Second approximation is

$x_2 = \emptyset(x_1) = \emptyset(0.00227237) = 0.21\sin(0.5 + 0.00227237) = 0.00184$

iii) Third approximation is

$x_3 = \emptyset(x_2) = \emptyset(0.00184) = 0.21\sin(0.5 + 0.00184) = 0.001839$

∴ The approximate value of the root is 0.001839.

==

Ex. Find the square root of 20 correct to three decimal places by using recursion formula $x_{i+1} = \frac{1}{2}(x_i + \frac{20}{x_i})$.

Solution: Given that $x_{i+1} = \frac{1}{2}(x_i + \frac{20}{x_i})$… (1)

Take $x_0 = 4.5$ and $i = 0$ in equation (1), we get,

$x_1 = \frac{1}{2}(x_0 + \frac{20}{x_0}) = \frac{1}{2}(4.5 + \frac{20}{4.5}) = 4.47$

Now take $x_1 = 4.47$ and $i = 1$ in equation (1), we get,

$x_2 = \frac{1}{2}(x_1 + \frac{20}{x_1}) = \frac{1}{2}(4.47 + \frac{20}{4.47}) = 4.472$

∴ The approximate value of $\sqrt{20}$ correct to three decimal places is 4.472.

==

The Method of False Position or Regula Falsi Method:

If f(x) = 0 be the given equation with $f(x_0)$ and $f(x_1)$ have an opposite sign, then the graph y = f(x) crosses the X –axis between these two points x_0 and x_1 and hence the root lies between x_0 and x_1.

The equation of chord joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is

$$\frac{y-f(x_0)}{f(x_0)-f(x_1)} = \frac{x-x_0}{x_0-x_1} \;\ldots\ldots\; (i)$$

The chord intersects the X-axis at y = 0, putting y = 0 in (i), we get,

$$\frac{-f(x_0)}{f(x_1)-f(x_0)} = \frac{x-x_0}{x_1-x_0}$$

i.e. $-\frac{f(x_0)}{f(x_1)-f(x_0)}(x_1-x_0) = x - x_0$

$\therefore x = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1-x_0)$

Hence the second approximation of root is given by

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1-x_0) \ldots\ldots (ii)$$

If $f(x_2)$ and $f(x_0)$ have an opposite sign, then root lies between x_0 and x_2 and replace x_1 by x_2 in (ii) and obtain next approximation x_3.

If $f(x_2)$ and $f(x_1)$ have an opposite sign, then root lies between x_1 and x_2 and replace x_0 by x_2 in (ii) and obtain next approximation x_3.

The process is repeated till the root is obtained to desired accuracy.

==

Ex. Find the real root of the equation $x^3 - 9x + 1 = 0$ lying between 2 and 4 by method of false position to three decimal places

Solution: Let $f(x) = x^3 - 9x + 1 = 0$ be the given equation.

Take $x_0 = 2$ and $x_1 = 4$

f(2) = 8 - 18 + 1 = - 9 < 0 and f(4) = 64 - 36 + 1 = 29 > 0

∴ The root lies 2 and 4.

i) First approximation is

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1-x_0) = 2 - \frac{f(2)}{f(4)-f(2)}(4-2)$$

$$= 2 - \frac{(-9)}{29+9}(2)$$

$$= 2 + \frac{18}{38} = 2.4736842$$

Now $f(x_2) = (2.4736842)^3 - 9(2.4736842) + 1 = -6.1264034 < 0$

∴ The root lies $x_2 = 2.4736842$ and $x_1 = 4$.

ii) Second approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f(x_1)-f(x_2)}(x_1 - x_2) = 2.4736842 - \frac{f(2.4736842)}{f(4)-f(2.4736842)}(4 - 2.4736842)$$

$$= 2.4736842 + \frac{6.1264034}{29+6.1264034}(1.5263158)$$

$$= 2.7398893$$

Now $f(x_3) = (2.7398893)^3 - 9(2.7398893) + 1 = -3.0906725 < 0$

∴ The root lies $x_3 = 2.7398893$ and $x_1 = 4$.

iii) Third approximation is

$$x_4 = x_3 - \frac{f(x_3)}{f(x_1)-f(x_3)}(x_1 - x_3)$$

$$= 2.7398893 - \frac{f(2.7398893)}{f(4)-f(2.7398893)}(4 - 2.7398893)$$

$$= 2.7398893 + \frac{3.0906725}{29+3.0906725}(1.2601107)$$

$$= 2.86125$$

∴ The approximate value of root upto three decimal places is 2.861.

===

Ex. Find the real root of the equation $x^3 - 2x - 5 = 0$ by Regula Falsi method.

Solution: Let $f(x) = x^3 - 2x - 5 = 0$ be the given equation.

As $f(2) = 8 - 4 - 5 = -1 < 0$ and $f(3) = 27 - 6 - 5 = 16 > 0$

∴ The root lies 2 and 3.

∴ Take $x_0 = 2$ and $x_1 = 3$

i) First approximation is

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 - x_0) = 2 - \frac{f(2)}{f(3)-f(2)}(3 - 2)$$

$$= 2 + \frac{1}{16+1}$$

$$= 2 + \frac{1}{17} = 2.059$$

Now $f(x_2) = (2.059)^3 - 2(2.059) - 5 = -0.386 < 0$

∴ The root lies $x_2 = 2.059$ and $x_1 = 3$.

ii) Second approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f(x_1)-f(x_2)}(x_1 - x_2) = 2.059 - \frac{f(2.059)}{f(3)-f(2.059)}(3 - 2.059)$$

$= 2.059 + \frac{0.386}{16+0.386}(0.941)$

$= 2.0812$

Now $f(x_3) = (2.0812)^3 - 2(2.0812) - 5 = -0.1479041 < 0$

∴ The root lies $x_3 = 2.0812$ and $x_1 = 3$.

iii) Third approximation is

$$x_4 = x_3 - \frac{f(x_3)}{f(x_1)-f(x_3)}(x_1 - x_3)$$

$$= 2.0812 - \frac{f(2.0812)}{f(3)-f(2.0812)}(3 - 2.0812)$$

$$= 2.0812 + \frac{0.1479041}{16+0.1479041}(0.9188)$$

$$= 2.0934$$

∴ The approximate value of root is 2.0934.

==

Ex. Find the real root between 1 and 2 of the equation $x^3 - x - 4 = 0$ upto four decimal places using Regula Falsi method.

Solution: Let $f(x) = x^3 - x - 4 = 0$ be the given equation.

As $f(1) = 1 - 1 - 4 = -4 < 0$ and $f(2) = 8 - 2 - 4 = 2 > 0$

∴ The root lies 1 and 2.

∴ Take $x_0 = 1$ and $x_1 = 2$

i) First approximation is

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 - x_0) = 1 - \frac{f(1)}{f(2)-f(1)}(2 - 1)$$

$$= 1 + \frac{4}{2+4}$$

$$= 1.66666$$

Now $f(x_2) = (1.66666)^3 - (1.66666) - 4 = -1.3709 < 0$

∴ The root lies $x_2 = 1.66666$ and $x_1 = 2$.

ii) Second approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f(x_1)-f(x_2)}(x_1 - x_2) = 1.66666 - \frac{f(1.66666)}{f(2)-f(1.66666)}(2 - 1.66666)$$

$$= 1.66666 + \frac{1.3709}{2+1.3709}(0.33334)$$

$$= 1.78048$$

Now $f(x_3) = (1.78048)^3 - (1.78048) - 4 = -0.2197 < 0$

∴ The root lies $x_3 = 1.78048$ and $x_1 = 2$.

iii) Third approximation is

$$x_4 = x_3 - \frac{f(x_3)}{f(x_1)-f(x_3)}(x_1 - x_3)$$

$$= 1.78048 - \frac{f(1.78048)}{f(2)-f(1.78048)}(2 - 1.78048)$$

$$= 1.78048 + \frac{0.2197}{2+0.2197}(0.21952)$$

$$= 1.79447$$

∴ The approximate value of root correct to four decimal places is 1.7944.

==

Ex. Find the real root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places by Regula Falsi method.

Solution: Let $f(x) = x^3 - 4x - 9 = 0$ be the given equation.

As $f(2) = 8 - 8 - 9 = -9 < 0$ and $f(3) = 27 - 12 - 9 = 6 > 0$

∴ The root lies 2 and 3.

∴ Take $x_0 = 2$ and $x_1 = 3$

i) First approximation is

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 - x_0) = 2 - \frac{f(2)}{f(3)-f(2)}(3-2)$$

$$= 2 + \frac{9}{6+9}$$

$$= 2 + \frac{9}{15} = 2.6$$

Now $f(x_2) = (2.6)^3 - 4(2.6) - 9 = -1.824 < 0$

∴ The root lies $x_2 = 2.6$ and $x_1 = 3$.

ii) Second approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f(x_1)-f(x_2)}(x_1 - x_2) = 2.6 - \frac{f(2.6)}{f(3)-f(2.6)}(3 - 2.6)$$

$$= 2.6 + \frac{1.824}{6+1.824}(0.4)$$

$$= 2.693$$

Now $f(x_3) = (2.693)^3 - 4(2.693) - 9 = -0.241694 < 0$

∴ The root lies $x_3 = 2.693$and $x_1 = 3$.

iii) Third approximation is

$$x_4 = x_3 - \frac{f(x_3)}{f(x_1)-f(x_3)}(x_1 - x_3)$$

$= 2.693 - \frac{f(2.693)}{f(3)-f(2.693)}(3-2.693)$

$= 2.0812 + \frac{0.241694}{6+0.241694}(0.307)$

$= 2.705$

∴ The approximate value of root correct to three decimal places is 2.705.

==

Ex. Find the real root of the equation $x^3 - x^2 - 2 = 0$ by Regula Falsi method.

Solution: Let $f(x) = x^3 - x^2 - 2 = 0$ be the given equation.

As $f(1) = 1 - 1 - 2 = -2 < 0$ and $f(2) = 8 - 4 - 2 = 2 > 0$

∴ The root lies 1 and 2.

∴ Take $x_0 = 1$ and $x_1 = 2$

i) First approximation is

$x_2 = x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 - x_0) = 1 - \frac{f(1)}{f(2)-f(1)}(2-1)$

$= 1 + \frac{2}{2+2}$

$= 1.5$

Now $f(x_2) = (1.5)^3 - (1.5)^2 - 2 = -0.875 < 0$

∴ The root lies $x_2 = 1.5$ and $x_1 = 2$.

ii) Second approximation is

$x_3 = x_2 - \frac{f(x_2)}{f(x_1)-f(x_2)}(x_1 - x_2) = 1.5 - \frac{f(1.5)}{f(2)-f(1.5)}(2-1.5)$

$= 1.5 + \frac{0.875}{2+0.875}(0.5)$

$= 1.6522$

Now $f(x_3) = (1.6522)^3 - (1.6522)^2 - 2 = -0.2197 < 0$

∴ The root lies $x_3 = 1.6522$ and $x_1 = 2$.

iii) Third approximation is

$x_4 = x_3 - \frac{f(x_3)}{f(x_1)-f(x_3)}(x_1 - x_3)$

$= 1.6522 - \frac{f(1.6522)}{f(2)-f(1.6522)}(2-1.6522)$

$= 1.6522 + \frac{0.241694}{2+0.241694}(0.3478)$

$= 1.6866$

∴ The approximate value of root is 1.6866.

==

The Newton Raphson Method: Suppose x_0 is an approximate root and $x_1 = x_0 + h$ is a correct root of given equation f(x) = 0, then $f(x_1) = 0$ i.e. $f(x_0 + h) = 0$.

∴ by Taylor's series, we get,

$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$

Neglecting the second and higher order derivatives, we get,

$f(x_0) + h f'(x_0) = 0$

i.e. $h = -\frac{f(x_0)}{f'(x_0)}$ if $f'(x_0) \neq 0$

∴ a better approximation than x_0 is $x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$

∴ Successive approximations $x_2, x_3, \ldots, x_{n+1}$ are given by

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ where $f'(x_n) \neq 0$

This formula is called Newton Rapson Formula.

==

Ex. Find the real root of the equation $x^3 - 2x - 5 = 0$ by Newton Raphson method correct up to three decimal places.

Solution: Let $f(x) = x^3 - 2x - 5 = 0$ be the given equation with $f'(x) = 3x^2 - 2$

As f(2) = 8 - 4 - 5 = - 1 < 0 and f(3) = 27 - 6 - 5 = 16 > 0

∴ The root lies 2 and 3.

∴ We take first approximation $x_1 = 2$ ∵ f(2) < f(3)

By Newton Raphson method-

Second approximation is

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 + \frac{1}{10} = 2.1$

Third approximation is

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{9.261-4.2-5}{13.23-2} = 2.094$

∴ The approximate value of root up to three decimal places is 2.094.

==

Ex. Find the smallest positive root of $x^3 - 5x + 3 = 0$ by Newton Raphson method correct to three decimal places.

Solution: Let $f(x) = x^3 - 5x + 3 = 0$ be the given equation with $f'(x) = 3x^2 - 5$

As f(0) = 0 - 0 + 3 = 3 > 0 and f(1) = 1 - 5 + 3 = -1 < 0

∴ The root lies 0 and 1.

∴ We take first approximation $x_1 = 1 \because f(1) < f(0)$

By Newton Raphson method

Second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{-2} = 0.5$$

Third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{0.125-2.5+3}{0.75-5} = 0.647$$

∴ The approximate value of root to three decimal places is 0.647.

==

Ex. Use Newton Raphson method to find a real root of the equation $x^3 - 3x - 5 = 0$. Perform three iterations.

Solution: Let $f(x) = x^3 - 3x - 5 = 0$ be the given equation with $f'(x) = 3x^2 - 3$

As $f(2) = 8 - 6 - 5 = -3 < 0$ and $f(3) = 27 - 9 - 5 = 13 > 0$

∴ The root lies 2 and 3.

∴ We take first approximation $x_1 = 2 \because f(2) < f(3)$

By Newton Raphson method

Second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-3}{9} = 2.333$$

Third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.333 - \frac{f(2.333)}{f'(2.333)} = 2.333 - \frac{12.698-6.999-5}{13.329} = 2.281$$

∴ The approximate value of root to three decimal places is 2.281.

==

Ex. Find a real root of the equation $xe^x - 1 = 0$, by Newton Raphson method. Take $x_0 = 1$

Solution: Let $f(x) = xe^x - 1 = 0$ be the given equation with $f'(x) = xe^x + e^x = e^x(x + 1)$

As $f(x_0) = f(1) = e - 1 > 0$ and $f'(x_0) = f'(1) = 2e$

Take $x_0 = 1$

By Newton Raphson method

Second approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{e-1}{2e} = \frac{e+1}{2e} = \frac{2.718+1}{2(2.718)} = 0.684$$

Third approximation is

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.684 - \frac{f(0.684)}{f'(0.684)} = 0.684 - \frac{0.684e^{0.684}-1}{2e^{0.684}} = 0.684 - \frac{0.356}{3.963} = 0.5942$

∴ The approximate value of root to three decimal places is 0.594.

==

Ex. Obtain Newton-Raphson formula for square root of N

Solution: Let $x = \sqrt{N}$ ∴ $x^2 = N$ ∴ $f(x) = x^2 - N = 0$ ∴ $f'(x) = 2x$

By Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

is the Newton Raphson formula for square root of N.

==

Ex. Find the square root of N correct to three decimal places by Newton-Raphson formula if N = 10.

Solution: Let Newton Raphson formula for square root of N is

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

Here N = 10

$$\therefore x_{n+1} = \frac{1}{2}\left[x_n + \frac{10}{x_n}\right]$$

Now $x = \sqrt{10}$ ∴ $x^2 = 10$ ∴ $f(x) = x^2 - 10 = 0$ ∴ $f'(x) = 2x$

As $f(x) = x^2 - 10$ ∴ $f(3) = 9 - 10 = -1 < 0$ & $f(4) = 16 - 10 = 6 > 0$

∴ Take $x_0 = 3$

$$x_1 = \frac{1}{2}\left[x_0 + \frac{10}{x_0}\right] = \frac{1}{2}\left[3 + \frac{10}{3}\right] = 3.167$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{10}{x_1}\right] = \frac{1}{2}\left[3.167 + \frac{10}{3.167}\right] = 3.162$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{10}{x_2}\right] = \frac{1}{2}\left[3.162 + \frac{10}{3.162}\right] = 3.162$$

∴ $\sqrt{10} = 3.162$ up to three decimal places.

==

MULTIPLE CHOICE QUESTIONS (MCQ'S)

1) Number 0.007102506 correct to 2 significant digits is ………

A) 0.0071 B) 0.007102 C) 0.00710 D) 0.007

2) Number 0.007102506 correct to 4 significant digits is ………

A) 0.0071 B) 0.007102 C) 0.00710 D) 0.007

3) Number 0.007102506 correct to 5 significant digits is ………

A) 0.0071 B) 0.007102 C) 0.0071025 D) 0.007

4) Number 0.007102506 correct to 6 significant digits is ………

A) 0.00710250 B) 0.007102 C) 0.0071025 D) 0.007

5) Number 3.4560712 is correct to 2 significant digits is ………

A) 3.4 B) 3.45 C) 3.456 D) 3.4560

6) Number 3.4560712 is correct to 3 significant digits is ………

A) 3.4 B) 3.45 C) 3.456 D) 3.4560

7) Number 3.4560712 is correct to 4 significant digits is ………

A) 3.4 B) 3.45 C) 3.456 D) 3.4560

8) Number 3.4560712 is correct to 5 significant digits is ………

A) 3.4560 B) 3.45607 C) 3.4560 D) 3.4560712

9) Number 3.4560712 is correct to 6 significant digits is ………

A) 3.4560 B) 3.45607 C) 3.4560 D) 3.4560712

10) Number 3.4560712 is rounded to 1 decimal places is ………

A) 3.456 B) 3.46 C) 3.4 D) 3.45

11) Number 3.4560712 is rounded to 2 decimal places is ………

A) 3.46 B) 3.4 C) 3.456 D) 3.45

11) Number 3.4560712 is rounded to 3 decimal places is ………

A) 3.4570 B) 3.456 C) 3.457 D) 3.4560

12) Number 3.4560712 is rounded to 5 decimal places is ………

A) 3.4560 B) 3.4560712 C) 3.45608 D) 3.45607

13) $\sqrt{2}$ is an …… number.

A) even B) rational C) irrational D) complex

14) Significant digit of a number 3.57 × 104 is ……

A) 2 B) 3 C) 4 D) 5

15) Significant digit of a number 3.570 × 104 is ……

A) 2 B) 3 C) 4 D) 5

16) Significant digit of a number 3.5700 × 104 is ……

A) 2 B) 3 C) 4 D) 5

17) To truncate a number x in its decimal fractional part to n^{th} place is formed by leaving its …… decimal place onwards.

A) $(n-1)^{th}$ B) n^{th} C) $(n+1)^{th}$ D) None of these

18) If x_1 is truncated number of number x, then $x - x_1$ is called ……error.

A) truncated B) absolute C) relative D) percentage

19) A truncated number of a number 3.4560712 to 2^{nd} decimal place is ……

A) 3.46 B) 3.45 C) 3.456 D) 3.4560

20) A truncated number of a number 3.4560712 to 5^{th} decimal place is ……

A) 3.4561 B) 3.4560 C) 3.45607 D) 3.456071

21) A truncated number of a number 3.4560712 to 6^{th} decimal place is ……

A) 3.4561 B) 3.45607 C) 3.456072 D) 3.456071

22) If 3.45 is truncated number of number 3.4560712 to 2^{nd} decimal place, then truncated error is ……

A) 0.0060712 B) 0.0060712 C) 0.0060712 D) 0.0060712

23) If 3.45607 is truncated number of number 3.4560712 to 5^{th} decimal place, then truncated error is ……

A) 0.0000012 B) 0.0000712 C) 0.00007 D) 0.000071

24) If 3.456071 is truncated number of number 3.4560712 to 6^{th} decimal place, then truncated error is ……

A) 0.0000010 B) 0.0000002 C) 0.000007 D) 0.0000702

25) If the true value x of a quantity is approximated by a value x_1, then $\Delta x = x-x_1$ is called ..….

A) error B) absolute error C) relative error D) percentage error

26) If a number x is approximated by a value x_1, then error Δx = ..….

A) $|x - x_1|$ B) $x - x_1$ C) $x_1 - x$ D)None of these

27) If a number x is approximated by a value x_1, then the absolute error $E_a = |E|$ = ……

A) $|x_1|$ B) $|x|$ C) $x - x_1$ D) $|x - x_1|$

28) If the true value x of a quantity is approximated by a value x_1,

then $E_a = |E| = |x - x_1|$ is called …… error.

A) relative B) absolute C) percentage D) None of these

28) If a number x is approximated by a value x_1, then $E_r = \frac{|x-x_1|}{x}$ is called …… error.

A) relative B) absolute C) percentage D) None of these

29) If a number x is approximated by a value x_1, then the relative error E_r = ……

A) $\frac{|x-x_1|}{x}$ B) $\frac{|x-x_1|}{x_1}$ C) $\frac{x_1}{x}$ D) None of these

30) If the true value x of a quantity is approximated by a value x_1, then $E_p = 100\ E_r$ is called …… error.

A) relative B) absolute C) percentage D) None of these

31) If a number x is approximated by a value x_1, then the percentage error E_p = ……

A) $(x - x_1) \times 100$ B) 100 Er C) $\frac{E_r}{100}$ D) None of these

32) If a number x is rounded to N decimal place, then error $E = \Delta x$ = ……

A) $-\frac{1}{2} \times 10^{-N}$ B) 10^{-N} C) $\frac{1}{2} \times 10^{-N}$ D) None of these

33) If the true value $\sqrt{11}$ is approximated by a value 3, then the absolute error |E| =……

A)3 B) $\sqrt{11}$ C) $\sqrt{11} - 3$ D) $3 - \sqrt{11}$

34) If 4 is the approximate value of $\sqrt{11}$, then the absolute error |E| =

A) 4 B) $\sqrt{11}$ C) $\sqrt{11} - 4$ D) $4 - \sqrt{11}$

35) If the true value $\sqrt{11}$ is approximated by a value 3, then the relative error E_r = ……

A) 3 B) $\frac{\sqrt{11}-3}{3}$ C) $\frac{\sqrt{11}-3}{\sqrt{11}}$ D) $3 - \sqrt{11}$

36) If the true value $\sqrt{11}$ is approximated by a value 4, then the relative error E_r = …

A) 4 B) $\frac{\sqrt{11}-4}{4}$ C) $\frac{4-\sqrt{11}}{\sqrt{11}}$ D) $4 - \sqrt{11}$

37) If 3 is the approximate value of $\sqrt{11}$, then the percentage error E_p = ……

A) $\frac{\sqrt{11}-3}{\sqrt{11}} \times 100$ B) $\frac{\sqrt{11}-3}{3} \times 100$ C) 300 D) $3 - \sqrt{11} \times 100$

38) If 4 is the approximate value of $\sqrt{11}$, then the percentage error E_p = ……

A) $\frac{\sqrt{11}-4}{\sqrt{11}} \times 100$ B) $\frac{4-\sqrt{11}}{\sqrt{11}} \times 100$ C) $\frac{\sqrt{11}-4}{4} \times 100$ D) $4 - \sqrt{11} \times 100$

39) If true value and approximate value of π are 3.14159265 and 3.14285714 respectively, then the absolute error E_a = ……

A) 0.126449 B) 0.0126449 C) 0.00126449 D) 0.000126449

40) If true value and approximate value of π are 3.14159265 and 3.14285714 respectively, then the relative error E_r = ……

A) 0.00004025 B) 0.0004025 C) 0.004025 D) 0.04025

41) If true value and approximate value of π are 3.14159265 and 3.14285714 respectively, then the percentage error E_p = ……

A) 0.00004025 B) 0.0004025 C) 0.004025 D) 0.04025

42) If three approximate values of $\frac{1}{6}$ are given by 0.16, 0.166 and 0.165, then the best approximation is ……

A) 0.16 B) 0.166 C) 0.165 D) None of these

43) An expression of the type $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_{n-1}x + a_n$, where all a_i's are constants, provided $a_0 \neq 0$ and n is positive integer, is called as …… in x of degree n.

A) polynomial B) algebraic equation

C) transcendental equation D) None of these

44) An expression of the type $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_{n-1}x + a_n = 0$, where all a_i's are constants, provided $a_0 \neq 0$ and n is positive integer, is called ……… in x of degree n.

A) an equation B) polynomial

C) transcendental equation D) None of these

45) A non-algebraic equation is called a …… equation, if it involves trigonometric, exponential, logarithmic etc. functions.

A) algebraic B) polynomial C) transcendental D) None of these

46) An equation $4\sin x - e^x = 0$ is …… equation.

A) transcendental B) polynomial C) algebraic D) None of these

47) An equation $\log x^3 - 5\tan x = 0$ is …… equation.

A) polynomial B) transcendental C) algebraic D) None of these

48) The value of x which satisfies the given equation f(x) = 0, is called …… of an equation.

A) factor B) solution C) root D) None of these

49) Statement 'A transcendental equation may have finite or infinite number of real roots

or may not have real roots.' is ……

A) true B) false C) may be true or false D) None of these

50) The process of finding the roots of an equation f(x) = 0 is called ……

A) zeros B) solution C) roots D) None of these

51) Geometrically a root of an equation f(x) = 0 is that value of x where the graph of y = f(x) cuts the …...

A) X-axis B) Y-axis C) Z-axis D) None of these

52) If x = a is the root of f(x) = 0, then f(x) is divisible by ……

A) x B) a C) x - a D) None of these

53) Every algebraic equation of degree n has only … roots. Which are real or imaginary.

A) n B) 2 C) 3 D) None of these

54) Every equation of odd degree has at least one …… root.

A) 0 B) positive C) negative D) real

55) Every cubic equation has at least one …… root.

A) real B) positive C) negative D) None of these

56) If f(x) is continuous in the interval [a, b] and f(a) and f(b) have an opposite signs then by the bisection method first approximate value of the root is…

A) a B) b C) $\frac{a+b}{2}$ D) $\sqrt{ab}$

57) Bisection method is applicable if f(x) is ……

A) factorable B) continuous C) not continuous D) None of these

58) The root of the equation $x^2 – x – 3 = 0$ is lies between ……

A) 0 and 1 B) 1 and 2 C) 2 and 3 D) 3 and 4

59) The root of the equation $x^3 - x – 4 = 0$ is lies between ……

A) 0 and 1 B) 1 and 2 C) 2 and 3 D) 3 and 4

60) The root of the equation $x^3 - 4x – 9 = 0$ is lies between ……

A) 0 and 1 B) 1 and 2 C) 2 and 3 D) 3 and 4

61) The real roots of the equation $8x^3 - 2x – 1 = 0$ are lies between ……

A) 0 and 1 B) 1 and 2 C) 2 and 3 D) 3 and 4

62) In the iteration method if the equation f(x) = 0 is expressed as x = ∅(x), then the successive approximations are given by x_n = …….

A) $\emptyset(x_n)$ B) $\emptyset(x_{n-1})$ C) $\emptyset(x_{n-2})$ D) None of these

63) If f(a) and f(b) have an opposite signs and the equation f(x) = 0 is expressed as $x = \emptyset(x)$, then iteration method is applicable if …… ∀x ∈ (a, b).

A) $|\emptyset'(x)| > 1$ B) $|\emptyset'(x)| = 1$ C) $|\emptyset'(x)| < 1$ D) None of these

64) To find the root of the equation $x^3 + x - 5 = 0$ by iteration method, the equation is expressed as x = …….

A) $5 - x^3$ B) $(5 - x)^{1/3}$ C) $\frac{5}{x^2+1}$ D)None of these

65) To find the root of the equation $x^3 + x - 1 = 0$ by iteration method, the equation is expressed as x = …….

A) $1 - x^3$ B) 1 - x C) $\frac{1}{x^2+1}$ D)None of these

66) To find the root of the equation 2x – cosx – 3 = 0 by iteration method, the equation is expressed as x = …….

A) cosx + 3 B)3 - cosx C)$\frac{3+\cos x}{2}$ D)3 - 2x

67) Root of an equation f(x) = 0 is lies between x_0 and x_1 if $f(x_0)$ and $f(x_1)$ have an……

A) opposite signs B) equal signs C) equal values D) different values

68) By Regula Falsi Method if the root of an equation f(x) = 0 is lies between x_0 and x_1, then first approximate value x_2 = ……

A) $x_0 + \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 - x_0)$ B) $x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 - x_0)$

C) $x_0 - \frac{f(x_0)}{f(x_1)-f(x_0)}(x_1 + x_0)$ D) different values

69) If the root of an equation $x^3 - x^2 - 2 = 0$ is lies between 1 and 2, then by the Method False Position first approximate value x_2 = …

A) 1.5 B) 1.4 C) 1.3 D) 1.2

70) By Newton-Raphson Method, successive approximations of an equation f(x) = 0 are given by the formula x_{n+1} = ……

A) $x_n - f'(x_n)$ B) $x_n - \frac{f'(x_n)}{f(x_n)}$ C) $x_n - \frac{f(x_n)}{f'(x_n)}$ D)None of these

71) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is called ……

A) Newton-Raphson Formula B) Regula-Falsi

C) Bisection Formula D) None of these

72) Newton-Raphson Formula for square root of N is x_{n+1} = ……

A) $\frac{1}{2}[x_n + N]$ B) $\frac{1}{2}[x_n - \frac{N}{x_n}]$ C) $\frac{1}{2}[x_n + \frac{N}{x_n}]$ D)None of these

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UNIT-2: INTERPOLATION

Interpolation: The technique of obtaining the value of a function for any intermediate values of the independent variable is called interpolation.

Shift Operator: Shift operator is denoted by E and defined as Ef(x) = f(x+h).

Note: $E^2f(x) = E[Ef(x)] = Ef(x+h) = f(x+2h)$, Similarly $E^3f(x) = f(x+3h)$ and so on, ingeneral $E^nf(x) = f(x+nh)$, where n is any real number.

Ex. Find i) $E^{1/2}f(x)$, ii) $E^{-1/2}f(x)$, iii) $E^{-1}f(x)$, iv) $E^{-n} f(x)$, v) $E^2\sin x$, vi) $E^{-2}\tan x$.

Solution: Using $E^nf(x) = f(x+nh)$, we get,

i) $E^{1/2}f(x) = f(x+\frac{1}{2}h)$

ii) $E^{-1/2}f(x) = f(x-\frac{1}{2}h)$

iii) $E^{-1}f(x) = f(x-h)$

iv) $E^{-n} f(x) = f(x-nh)$

v) $E^2\sin x = \sin(x+2h)$

vi) $E^{-2}\tan x = \tan(x-2h)$

Forward difference Operator: Forward difference operator is denoted by Δ and defined as $\Delta f(x) = f(x+h) - f(x)$.

Note: i) $\Delta f(x) = f(x+h) - f(x)$ is called first forward difference of f(x) and $\Delta^n f(x)$ is called n^{th} forward difference of f(x).

ii) $\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E - 1)f(x)$

$\therefore \Delta = E - 1$ i.e $E = \Delta + 1$

be the relation between shift operator and forward difference operator.

iii) If n values of f(x) are given then $\Delta^n f(x) = 0$, which used for finding missing term in given data.

Ex. Find i) $\Delta^2 f(x)$, ii) $\Delta^3 f(x)$, iii) $\Delta\log x$, iv) $\Delta\cos(ax+b)$, v) Δe^x, vi) $\Delta^2 e^x$.

Solution: Using $\Delta = E - 1$, we get,

i) $\Delta^2 = (E - 1)^2 = E^2 - 2E + 1$

$\therefore \Delta^2 f(x) = E^2f(x) - 2Ef(x) + f(x)$

$\therefore \Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$

ii) $\Delta^3 = (E-1)^3 = E^3 - 3E^2 + 3E - 1$

$\therefore \Delta^3 f(x) = E^3 f(x) - 3E^2 f(x) + 3Ef(x) - f(x)$

$\therefore \Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$

iii) As $\Delta = E - 1$

$\therefore \Delta \log x = E\log x - \log x$

$\therefore \Delta \log x = \log(x+h) - \log x$

$\therefore \Delta \log x = \log(1+\frac{h}{x})$

iv) As $\Delta = E - 1$

$\therefore \Delta\cos(ax+b) = E\cos(ax+b) - \cos(ax+b)$

$\therefore \Delta\cos(ax+b) = \cos[a(x+h)+b] - \cos(ax+b)$

$\therefore \Delta\cos(ax+b) = \cos(ax+ah+b) - \cos(ax+b)$

v) As $\Delta = E - 1$

$\therefore \Delta e^x = Ee^x - e^x$

$\therefore \Delta e^x = e^{x+h} - e^x$

$\therefore \Delta e^x = e^x(e^h - 1)$

vi) $\Delta^2 = (E-1)^2 = E^2 - 2E + 1$

$\therefore \Delta^2 e^x = E^2 e^x - 2Ee^x + e^x$

$\therefore \Delta^2 e^x = e^{x+2h} - 2e^{x+h} + e^x$

$\therefore \Delta^2 e^x = e^x(e^{2h} - 2e^h + 1)$

$\therefore \Delta^2 e^x = e^x(e^h - 1)^2$

==

Ex. Show that $\Delta\log f(x) = \log[1+\frac{\Delta f(x)}{f(x)}]$.

Proof: Using $\Delta = E - 1$, we get,

$\therefore \Delta\log f(x) = E\log f(x) - \log f(x)$

$\therefore \Delta\log f(x) = \log f(x+h) - \log f(x)$

$= \log[\frac{f(x+h)}{f(x)}]$

$= \log[\frac{Ef(x)}{f(x)}]$

$= \log[\frac{(1+\Delta)f(x)}{f(x)}]$

$= \log[\frac{f(x)+\Delta f(x)}{f(x)}]$

$\therefore \Delta \log f(x) = \log[1+ \frac{\Delta f(x)}{f(x)}]$

Hence proved.

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Ex. Find i) $\Delta(e^{ax}\log bx)$ ii) $\Delta\tan^{-1}x$ iii) $\Delta^2\cos 2x$.

Solution: Using $\Delta = E - 1$, we get,

ii) $\Delta(e^{ax}\log bx) = E(e^{ax}\log bx) - (e^{ax}\log bx)$

$=[(e^{a(x+h)}\log b(x+h)] - (e^{ax}\log bx)$

$= e^{ax}[e^{ah}\log(bx+bh) - \log bx]$

ii) $\Delta\tan^{-1}x = E\tan^{-1}x - \tan^{-1}x$

$= \tan^{-1} x+h) - \tan^{-1}x$

$= \tan^{-1}(\frac{x+h-x}{1+(x+h)x})$

$= \tan^{-1}(\frac{h}{1+hx+x^2})$

iii) As $\Delta^2 = (E-1)^2 = E^2 - 2E + 1$

$\therefore \Delta^2\cos 2x= E^2\cos 2x - 2E\cos 2x + \cos 2x$

$\therefore \Delta^2 \cos 2x = \cos 2(x+2h) - 2\cos 2(x+h) + \cos 2x$

$\therefore \Delta^2 \cos 2x = \cos(2x+4h) - 2\cos(2x+2h) + \cos 2x$

==

Backward difference Operator: Backward difference operator is denoted by ∇ and defined as $\nabla f(x) = f(x) - f(x-h)$.

Note: i) $\nabla f(x) = f(x)- f(x-h)$ is called first backward difference of f(x).

and $\nabla^n f(x)$ is called n^{th} backward difference of f(x).

ii) As $\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1}f(x) = (1 - E^{-1})f(x)$

$\therefore \nabla = 1 - E^{-1}$ i.e. $E^{-1} = 1 - \nabla$ $\therefore E = (1 - \nabla)^{-1}$

be the relation between shift operator and backward difference operator.

==

Ex. Find i) $\nabla^2 f(x)$, ii) $\nabla^3 f(x)$, iii) $\nabla\sin x$, iv) $\nabla\sec x$, v) ∇e^x, vi) $\nabla^2 e^x$.

Solution: Using $\nabla = 1 - E^{-1}$, we get,

i) $\nabla^2 = (1 - E^{-1})^2 = 1 - 2E^{-1} + E^{-2}$

$\therefore \nabla^2 f(x) = f(x) - 2 E^{-1}f(x) + E^{-2}f(x)$

$\therefore \nabla^2 f(x) = f(x) - 2f(x-h) + f(x-2h)$

ii) $\nabla^3 = (1 - E^{-1})^3 = 1 - 3E^{-1} + 3 E^{-2} - E^{-3}$

$\therefore \nabla^3 f(x) = f(x) - 3E^{-1}f(x) + 3E^{-2}f(x) - E^{-3}f(x)$

$\therefore \nabla^3 f(x) = f(x) - 3\, f(x-h) + 3f(x-2h) - f(x-3h)$

iii) As $\nabla = 1 - E^{-1}$

$\therefore \nabla \sin x = \sin x - E^{-1}\sin x$

$\therefore \nabla \sin x = \sin x - \sin(x-h)$

iv) As $\nabla = 1 - E^{-1}$

$\therefore \nabla \sec x = \sec x - E^{-1}\sec x$

$\therefore \nabla \sec x = \sec x - \sec(x-h)$

v) As $\nabla = 1 - E^{-1}$

$\therefore \nabla e^x = e^x - E^{-1}e^x$

$\therefore \nabla e^x = e^x - e^{x-h}$

$\therefore \nabla e^x = e^x\,(1 - e^{-h})$

vi) $\nabla^2 = (1 - E^{-1})^2 = 1 - 2E^{-1} + E^{-2}$

$\therefore \nabla^2 e^x = e^x - 2E^{-1}e^x + E^{-2}e^x$

$\therefore \nabla^2 e^x = e^x - 2e^{x-h} + e^{x-2h}$

$\therefore \nabla^2 e^x = e^x\,(1 - 2e^{-h} + e^{-2h})$

$\therefore \nabla^2 e^x = e^x\,(1 - e^{-h})^2$

==

Ex. Prepare forward and backward difference table for the function y = f(x) to the values x_0, x_1, x_2, x_3, x_4 and x_5.

Solution: i) Forward difference table for the function y = f(x) to the given values is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

ii) Backward difference table for the function y = f(x) to the given values is

x	y = f(x)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

==

Note: We observe that $\Delta f(x_i) = f(x_{i+1}) - f(x_i) = \nabla f(x_{i+1})$ i.e. $\Delta y_i = y_{i+1} - y_i = \nabla y_{i+1}$.

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Central difference Operator: Central difference operator is denoted by δ and defined as

$$\delta f(x) = f(x + \tfrac{h}{2}) - f(x - \tfrac{h}{2}).$$

Note: i) $\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2})$ is called first central difference of f(x).

and $\delta^n f(x)$ is called n^{th} central difference of f(x).

ii) As $\delta f(x) = f(x + \frac{h}{2}) - f(x - \frac{h}{2}) = E^{1/2} f(x) - E^{-1/2} f(x) = (E^{1/2} - E^{-1/2}) f(x)$

$\therefore \delta = E^{1/2} - E^{-1/2}$

be the relation between shift operator and backward difference operator.

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Ex. Find i) $\delta^2 f(x)$, ii) $\delta^3 f(x)$, iii) $\delta^4 f(x)$, iv) δcosecx.

Solution: Using $\delta = E^{1/2} - E^{-1/2}$, we get,

i) $\delta^2 = (E^{1/2} - E^{-1/2})^2 = E - 2 + E^{-1}$

$\therefore \delta^2 f(x) = Ef(x) - 2f(x) + E^{-1} f(x)$

$\therefore \delta^2 f(x) = f(x + h) - 2f(x) + f(x - h)$

ii) $\delta^3 = (E^{1/2} - E^{-1/2})^3 = E^{3/2} - 3E^{1/2} + 3E^{-1/2} - E^{-3/2}$

$\therefore \delta^3 f(x) = E^{3/2} f(x) - 3E^{1/2} f(x) + 3E^{-1/2} f(x) - E^{-3/2} f(x)$

$\therefore \delta^3 f(x) = f(x + \frac{3h}{2}) - 3f(x + \frac{h}{2}) + 3f(x - \frac{h}{2}) - f(x - \frac{3h}{2})$

iii) As $\delta^2 = (E^{1/2} - E^{-1/2})^2 = E - 2 + E^{-1}$

$\therefore \delta^4 = E^2 + 4 + E^{-2} - 4E + 2 - 4E^{-1}$

i.e. $\delta^4 = E^2 - 4E + 6 - 4E^{-1} + E^{-2}$

$\therefore \delta^4 f(x) = E^2 f(x) - 4Ef(x) + 6f(x) - 4E^{-1}f(x) + E^{-2} f(x)$

$\therefore \delta^4 f(x) = f(x + 2h) - 4f(x + h) + 6f(x) - 4f(x - h) + f(x - 2h)$

iv) As $\delta = E^{1/2} - E^{-1/2}$

$\therefore \delta \text{cosec} x = E^{1/2}\text{cosec} x - E^{-1/2}\text{cosec} x$

$\therefore \delta \text{cosec} x = \text{cosec}(x + \frac{h}{2}) - \text{cosec}(x - \frac{h}{2})$

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Ex. Prepare central difference table for the function y = f(x) to the values x_0, x_1, x_2, x_3, x_4 and x_5.

Solution: i) Central difference table for the function y = f(x) to the given values is

x	y = f(x)	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

==

Averaging Operator or Mean Operator: Averaging operator or mean operator is denoted by μ and defined as $\mu f(x) = \frac{1}{2}\ [f(x + \frac{h}{2}) + f(x - \frac{h}{2})]$

Note: i) $\mu f(x) = \frac{1}{2}[f(x + \frac{h}{2}) + f(x - \frac{h}{2})]$ is called first mean difference of f(x).

and $\mu^n f(x)$ is called n^{th} mean difference of f(x).

ii) As $\mu f(x) = \frac{1}{2}[f(x + \frac{h}{2}) + f(x - \frac{h}{2})] = \frac{1}{2}[E^{1/2}f(x) + E^{-1/2}f(x)] = \frac{1}{2}(E^{1/2} + E^{-1/2})f(x)$

$\therefore \mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

be the relation between shift operator and backward difference operator.

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Ex. Find i) $\mu^2 f(x)$, ii) $\mu^3 f(x)$ and iii) $\mu \cot x$

Solution: Using $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$, we get,

i) $\mu^2 = [\frac{1}{2}(E^{1/2} + E^{-1/2})]^2 = \frac{1}{4}(E + 2 + E^{-1})$

$\therefore \mu^2 f(x) = \frac{1}{4}[Ef(x) + 2f(x) + E^{-1}f(x)]$

$\therefore \mu^2 f(x) = \frac{1}{4}[f(x + h) + 2f(x) + f(x - h)]$

ii) $\mu^3 = \frac{1}{8}(E^{1/2} + E^{-1/2})^3 = \frac{1}{8}[E^{3/2} + 3E^{1/2} + 3E^{-1/2} + E^{-3/2}]$

$\therefore \mu^3 f(x) = \frac{1}{8}[E^{3/2}f(x) + 3E^{1/2}f(x) + 3E^{-1/2}f(x) + E^{-3/2}f(x)]$

$\therefore \mu^3 f(x) = \frac{1}{8}[f(x + \frac{3h}{2}) + 3f(x + \frac{h}{2}) + 3f(x - \frac{h}{2}) + f(x - \frac{3h}{2})]$

iv) As $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

$\therefore \mu \cot x = \frac{1}{2}[E^{1/2}\cot x + E^{-1/2}\cot x]$

$\therefore \mu \cot x = \frac{1}{2}[\cot(x + \frac{h}{2}) + \cot(x - \frac{h}{2})]$

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Ex. Using definitions of operators E, Δ, ∇, δ and μ, show that

i) $\Delta = E\nabla$ ii) $\nabla = E^{-1}\Delta$ iii) $\Delta = \nabla(1 - \nabla)^{-1}$ iv) $(E - 1)\nabla^{-1} = 1 + \Delta$

v) $\Delta\nabla = \nabla\Delta = \delta^2$ vi) $\delta^2 E = \Delta^2$ vii) $E^{-1/2} = \mu - \frac{\delta}{2}$ viii) $1 + \mu^2\delta^2 = (1 + \frac{\delta^2}{2})^2$

Proof: i) Consider $E\nabla f(x) = E[\nabla f(x)]$

$= E[f(x) - f(x-h)]$

$= Ef(x) - Ef(x-h)$

$= f(x+h) - f(x)$

$E\nabla f(x) = \Delta f(x)$

$\therefore E\nabla = \Delta$

ii) Consider $E^{-1}\Delta f(x) = E^{-1}[\Delta f(x)]$

$= E^{-1}[f(x+h) - f(x)]$

$= E^{-1}f(x+h) - E^{-1}f(x)$

$= f(x) - f(x-h)$

$E^{-1}\Delta f(x) = \nabla f(x)$

$\therefore E^{-1}\Delta = \nabla$

iii) Consider $\nabla(1 - \nabla)^{-1} = \nabla(E^{-1})^{-1} = \nabla E$

$\therefore \nabla(1 - \nabla)^{-1} = \Delta$ by (i)

iv) Consider $(E - 1)\nabla^{-1} = \Delta\nabla^{-1}$

$= E\nabla\nabla^{-1}$ $\because \Delta = E\nabla$

$= E$

$= 1 + \Delta$

v) Consider $\Delta\nabla = (E - 1)(1 - E^{-1})$

$= E - 1 - 1 + E^{-1}$

$= E - 2 + E^{-1}$

$= (E^{1/2} - E^{-1/2})^2$

$= \delta^2$

Similarly, $\nabla\Delta = \delta^2$

vi) Consider $\delta^2 E = (E^{1/2} - E^{-1/2})^2 E$

$= (E - 2 + E^{-1})E$

$= E^2 - 2E + 1$

$= (E - 1)^2$

$= \Delta^2$

vii) Consider $\mu - \frac{\delta}{2} = \frac{1}{2}(E^{1/2} + E^{-1/2}) - \frac{1}{2}(E^{1/2} - E^{-1/2})$

$= \frac{1}{2}E^{1/2} + \frac{1}{2}E^{-1/2} - \frac{1}{2}E^{1/2} + \frac{1}{2}E^{-1/2}$

$= E^{-1/2}$

viii) $1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2$

$= \frac{1}{2}[2 + E - 2 + E^{-1}]$

$1 + \frac{\delta^2}{2} = \frac{1}{2}(E + E^{-1})$ …… (1)

Consider $1 + \mu^2\delta^2 = 1 + [\frac{1}{2}(E^{1/2} + E^{-1/2})]^2(E^{1/2} - E^{-1/2})^2$

$= 1 + \frac{1}{4}(E - E^{-1})^2$

$= \frac{1}{4}[4 + E^2 - 2 + E^{-2}]$

$= \frac{1}{4}(E^2 + 2 + E^{-2})$

$= [\frac{1}{2}(E + E^{-1})]^2$

$= (1 + \frac{\delta^2}{2})^2$ by (1). Hence proved.

==

Ex. Construct a forward difference table for the function $f(x) = x^3 + 5x - 7$ with $x = -1, 0, 1, 2, 3, 4, 5$ and form the table to obtain f(6) and f(7).

Solution: From given function $f(x) = x^3 + 5x - 7$, we have,

$f(-1) = -1 - 5 - 7 = -13$, $f(0) = -7$, $f(1) = 1 + 5 - 7 = -1$, $f(2) = 8 + 10 - 7 = 11$,

$f(3) = 27 + 15 - 7 = 35$, $f(4) = 64 + 20 - 7 = 77$ and $f(5) = 125 + 25 - 7 = 143$

Let f(6) = a and f(7) = b

The forward difference table is

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-13			
		6		
0	-7		0	
		6		6
1	-1		6	
		12		6
2	11		12	
		24		6
3	35		18	
		42		6
4	77		24	
		66		a-233
5	143		a-209	
		a-143		b-3a+352
6	a		b-2a+143	
		b-a		
7	b			

Here third order forward differences are constant.

$\therefore$ a-233 = 6 i.e. f(6) = a = 239 and b-3a+352 = 6 i.e. f(7) = b = 6-352+717 = 371

===

Ex. Construct a forward difference table from the following values of x and y, show that the third difference are constant.

x	35	36	37	38	39	40	41
y	14.295	14.143	13.986	13.825	13.661	13.495	13.328

Solution: The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
35	14.295			
		-0.152		
36	14.143		-0.005	
		-0.157		0.001
37	13.986		-0.004	
		-0.161		0.001
38	13.825		-0.003	
		-0.164		0.001
39	13.661		-0.002	
		-.0166		0.001
40	13.495		-0.001	
		-0.167		
41	13.328			

Here third order forward differences are constant.

==

Ex. Construct a forward difference table from the following values of x and y, show that the third difference are constant.

x	0.0	0.2	0.4	0.6	0.8
y	1.0	1.1927	1.3894	1.5616	1.7174

Solution: The backward difference table for the given values is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.0	1.0				
		0.1927			
0.2	1.1927		0.0040		
		0.1967		-0.0205	
0.4	1.3894		-0.0245		0.0276
		0.1722		0.0071	
0.6	1.5616		-0.0164		
		0.1558			
0.8	1.7174				

==

Ex. Prove that $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots\ldots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}$

Proof: Consider $u_x - \Delta^n u_{x-n} = u_x - \Delta^n E^{-n} u_x = [1 - \Delta^n E^{-n}]\, u_x$

$$= [1 - \frac{\Delta^n}{E^n}]\, u_x$$

$$= E^{-n}[E^n - \Delta^n]\, u_x$$

$= E^{-n}(E - \Delta)[E^{n-1} + \Delta E^{n-2} + \Delta^2 E^{n-3} + \ldots\ldots + \Delta^{n-1}]\, u_x$

$= E^{-n}(1)[E^{n-1} + \Delta E^{n-2} + \Delta^2 E^{n-3} + \ldots\ldots + \Delta^{n-1}]\, u_x$

$= [E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + \ldots\ldots + \Delta^{n-1} E^{-n}]\, u_x$

$= E^{-1}u_x + \Delta E^{-2}u_x + \Delta^2 E^{-3}u_x + \ldots\ldots + \Delta^{n-1}E^{-n}u_x$

$\therefore u_x - \Delta^n u_{x-n} = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots\ldots + \Delta^{n-1}u_{x-n}$

Hence $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots\ldots + \Delta^{n-1}u_{x-n} + \Delta^n u_{x-n}$ is proved.

==

Ex. Prove that $u_0 + u_1 + u_2 + \ldots\ldots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \ldots + \Delta^n u_0$

Proof: Consider L.H.S. $= u_0 + u_1 + u_2 + \ldots\ldots + u_n$

$= u_0 + Eu_0 + E^2u_0 + \ldots\ldots + E^n u_0$

$= (1 + E + E^2 + \ldots\ldots + E^n)u_0$

$= [\frac{E^{n+1}-1}{E-1}]u_0$

$= [\frac{(1+\Delta)^{n+1}-1}{\Delta}]u_0$

$= [\frac{{}^{n+1}C_1\Delta + {}^{n+1}C_2\Delta^2 + {}^{n+1}C_3\Delta^3 + \cdots + \Delta^{n+1}}{\Delta}]u_0$

$= ({}^{n+1}C_1 + {}^{n+1}C_2\Delta + {}^{n+1}C_3\Delta^2 + \cdots + \Delta^n)\, u_0$

$= {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \ldots + \Delta^n u_0$

= R.H.S.

Hence proved.

==

Ex. Prove that $e^x = (\frac{\Delta^2}{E})e^x \frac{Ee^x}{\Delta^2 e^x}$

Proof: Consider R.H.S.$= (\frac{\Delta^2}{E})e^x \frac{Ee^x}{\Delta^2 e^x}$

$= [\frac{(E-1)^2}{E}]e^x \frac{Ee^x}{(E-1)^2 e^x}$

$= [\frac{E^2-2E+1}{E}]e^x \frac{Ee^x}{(E^2-2E+1)e^x}$

$= (E - 2 + E^{-1})e^x \frac{e^{x+1}}{(E^2e^x - 2Ee^x + e^x)}$

$= (e^{x+1} - 2e^x + e^{x-1}) \frac{e^{x+1}}{(e^{x+2} - 2e^{x+1} + e^x)}$

$= e^x(e - 2 + e^{-1}) \frac{e}{(e^2 - 2e + 1)}$

$= e^x(e^2 - 2e + 1)\ \frac{1}{(e^2 - 2e + 1)}$

$= e^x$ = L.H.S. Hence proved.

==

Ex. Show that $\Delta^p y_k = \nabla^p y_{k+p}$

Proof: Consider L.H.S.= $\Delta^p y_k$

$= (\nabla E)^p y_k \qquad \because \Delta = \nabla E$

$= \nabla^p E^p y_k$

$= \nabla^p y_{k+p}$

= L.H.S.

Hence proved.

==

Ex. Show that i) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ ii) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Proof: i) Consider R.H.S.= $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$= \frac{\nabla E}{\nabla} - \frac{E^{-1}\Delta}{\Delta} \qquad \because \Delta = \nabla E$ and $\nabla = E^{-1}\Delta$

$= E - E^{-1}$

$= (1 + \Delta) - (1 - \nabla)$

$= \Delta + \nabla$

= L.H.S.

Hence proved.

iI) Consider L.H.S.= $\mu\delta$

$= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) \qquad \nabla E$ and $\nabla = E^{-1}\Delta$

$= \frac{1}{2}(E - E^{-1})$

$= \frac{1}{2}[(1 + \Delta) - (1 - \nabla)]$

$= \frac{1}{2}(\Delta + \nabla)$

= R.H.S.

Hence proved.

==

Ex. Show that $\delta[f(x)g(x)] = \mu f(x)\delta g(x) + \mu g(x)\delta f(x)$

Proof: Consider R.H.S. = $\mu f(x)\delta g(x) + \mu g(x)\delta f(x)$

$= \frac{1}{2}(E^{1/2} + E^{-1/2})f(x)\,(E^{1/2} - E^{-1/2})g(x)$

$+ \frac{1}{2}(E^{1/2} + E^{-1/2})g(x)\,(E^{1/2} - E^{-1/2})f(x)$

$= \frac{1}{2}[f(x+\frac{h}{2}) + f(x-\frac{h}{2})][g(x+\frac{h}{2}) - g(x-\frac{h}{2})]$

$\frac{1}{2}[g(x+\frac{h}{2}) + g(x-\frac{h}{2})][f(x+\frac{h}{2}) - f(x-\frac{h}{2})]$

$= \frac{1}{2}f(x+\frac{h}{2})g(x+\frac{h}{2}) - \frac{1}{2}f(x+\frac{h}{2})g(x-\frac{h}{2}) + \frac{1}{2}f(x-\frac{h}{2})g(x+\frac{h}{2}) - \frac{1}{2}f(x-\frac{h}{2})g(x-\frac{h}{2})$

$+\frac{1}{2}g(x+\frac{h}{2})f(x+\frac{h}{2}) - \frac{1}{2}g(x+\frac{h}{2})f(x-\frac{h}{2}) + \frac{1}{2}g(x-\frac{h}{2})f(x+\frac{h}{2}) - \frac{1}{2}g(x-\frac{h}{2})f(x-\frac{h}{2})$

$= f(x+\frac{h}{2})g(x+\frac{h}{2}) - g(x-\frac{h}{2})f(x-\frac{h}{2})$

$= E^{1/2}[f(x)g(x)] - E^{-1/2}[f(x)g(x)]$

$= (E^{1/2} - E^{-1/2})[f(x)g(x)]$

$= \delta[f(x)g(x)]$

= L.H.S.

Hence proved.

==

Ex. Estimate the missing term in the following table

x	0	1	2	3	4
f(x)	1	3	9	?	81

Proof: Four values of f(x) are given

$\therefore \Delta^4 f(x) = 0$

$\therefore (E-1)^4 f(x) = 0$

$\therefore (E^4 - 4E^3 + 6E^2 - 4E + 1)f(x) = 0$

$\therefore E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0$

$\therefore f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$

Putting x = 0, we get,

$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

i.e. $81 - 4f(3) + 6(9) - 4(3) + 1 = 0$

i.e. $81 + 54 - 12 + 1 = 4f(3)$

$\therefore 4f(3) = 124$

$\therefore f(3) = 31$ be the required missing value.

==

Ex. Find the missing terms in the following table

x	1	2	3	4	5
f(x)	7	-	13	21	37

Proof: Four values of f(x) are given

$\therefore \Delta^4 f(x) = 0$

$\therefore (E-1)^4 f(x) = 0$

$\therefore (E^4 - 4E^3 + 6E^2 - 4E + 1)f(x) = 0$

$\therefore E^4f(x) - 4E^3f(x) + 6E^2f(x) - 4Ef(x) + f(x) = 0$

$\therefore f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$

Putting x = 1, we get,

$f(5) - 4f(4) + 6f(3) - 4f(2) + f(1) = 0$

i.e. $37 - 4(21) + 6(13) - 4f(2) + 7 = 0$

i.e. $37 - 84 + 78 + 7 = 4f(2)$

$\therefore 4f(2) = 38$

$\therefore f(2) = 9.5$ be the required missing value.

===

Gauss's forward difference formula: If $p = \frac{x-x_0}{h}$, then $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-4)}{5!}\Delta^5 y_{-2} + \ldots\ldots$ is known as Gauss's forward central difference interpolation formula.

Gauss's backward difference formula: If $p = \frac{x-x_0}{h}$, then $y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-2} + \frac{p(p^2-1)(p+2)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-4)}{5!}\Delta^5 y_{-3} + \ldots\ldots$ is known as Gauss's backward central difference interpolation formula..

===

Ex. Using Gauss forward central difference formula, interpolate at x = 32 given that $f(25) = 0.2707$, $f(30) = 0.3027$, $f(35) = 0.3086$, $f(40) = 0.3794$

Solution: Select $x_0 = 30$, we prepare difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-1} = 25$	$y_{-1} = 0.2707$			
		$\Delta y_{-1} = 0.0320$		
$x_0 = 30$	$y_0 = 0.3027$		$\Delta^2 y_{-1} = 0.0039$	
		$\Delta y_0 = 0.0359$		$\Delta^3 y_{-1} = 0.0010$
$x_1 = 35$	$y_1 = 0.3386$		$\Delta^2 y_0 = 0.0049$	
		$\Delta y_1 = 0.0408$		
$x_2 = 40$	$y_2 = 0.3794$			

We have to interpolate at x = 32.

$\therefore p = \frac{x-x_0}{h} = \frac{32-30}{5} = 0.4$

By Gauss's forward formula.

$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \ldots\ldots$, we get

$y(32) = (0.3027) + 0.4(0.0359) + \frac{0.4(0.4-1)}{2}(0.0039) + \frac{0.4(0.16-1)}{6}(0.0010)$

i.e. $y(32) = 0.3166$

==

Ex. Use Gauss's forward formula to find y_{30}, given that

$y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$, $y_{37} = 15.5154$

Solution: Select $x_0 = 29$, we prepare difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 21$	$y_{-2} = 18.4708$				
		$\Delta y_{-2} = -0.6564$			
$x_{-1} = 25$	$y_{-1} = 17.8144$		$\Delta^2 y_{-2} = -0.0510$		
		$\Delta y_{-1} = -0.7074$		$\Delta^3 y_{-2} = -0.0054$	
$x_0 = 29$	$y_0 = 17.1070$		$\Delta^2 y_{-1} = -0.0564$		$\Delta^4 y_{-2} = -0.0022$
		$\Delta y_0 = -0.7638$		$\Delta^3 y_{-1} = -0.0076$	
$x_1 = 33$	$y_1 = 16.3432$		$\Delta^2 y_0 = -0.0640$		
		$\Delta y_1 = -0.8278$			
$x_2 = 37$	$y_2 = 15.5154$				

We have to interpolate at x = 30.

$\therefore p = \frac{x-x_0}{h} = \frac{30-29}{4} = 0.25$

By Gauss's forward formula.

$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots$, we get

$y(30) = (17.1070) + 0.25(-0.7638) + \frac{0.25(0.25-1)}{2}(-0.0564)$

$$+ \frac{0.25(0.0625-1)}{6}(-0.0054) + \frac{0.25(0.0625-1)(0.25-2)}{24}(-0.0022)$$

i.e. $y(30) = 16.969$

==

Ex. From the following table of values find y when x = 27.5 Using Gauss forward central difference formula.

x	25	26	27	28	29
y	16.195	15.919	15.630	15.326	15.006

Solution: Select $x_0 = 27$, we prepare difference table as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 25$	$y_{-2} = 16.195$				
		$\Delta y_{-2} = -0.276$			
$x_{-1} = 26$	$y_{-1} = 15.919$		$\Delta^2 y_{-2} = -0.013$		
		$\Delta y_{-1} = -0.289$		$\Delta^3 y_{-2} = -0.002$	
$x_0 = 27$	$y_0 = 15.630$		$\Delta^2 y_{-1} = -0.015$		$\Delta^4 y_{-2} = 0.001$
		$\Delta y_0 = -0.304$		$\Delta^3 y_{-1} = -0.001$	
$x_1 = 28$	$y_1 = 15.326$		$\Delta^2 y_0 = -0.016$		
		$\Delta y_1 = -0.320$			
$x_2 = 29$	$y_2 = 15.006$				

We have to interpolate at x = 27.5.

$\therefore p = \frac{x - x_0}{h} = \frac{27.5 - 27}{1} = 0.5$

By Gauss's forward formula.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots \text{, we get}$$

$$y(27.5) = (15.630) + 0.5(-0.304) + \frac{0.5(0.5-1)}{2}(-0.015) + \frac{0.5(0.25-1)}{6}(-0.002)$$

$$+ \frac{0.5(0.25-1)(0.5-2)}{24}(0.001)$$

i.e. y(27.5) = 15.480

==

Ex. Use Gauss's forward central difference formula to find the polynomial of degree four which takes the following values of the function f(x):

x	1	2	3	4	5
f(x)	1	-1	1	-1	1

Solution: Take $x_0 = 3$, here h = 1 ∴ $p = \frac{x-x_0}{h} = \frac{x-3}{1} = x-3$.

We prepare forward difference table as:

x	y=f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 1$	$y_{-2} = 1$				
		$\Delta y_{-2} = -2$			
$x_{-1} = 2$	$y_{-1} = -1$		$\Delta^2 y_{-2} = 4$		
		$\Delta y_{-1} = 2$		$\Delta^3 y_{-2} = -8$	
$x_0 = 3$	$y_0 = 1$		$\Delta^2 y_{-1} = -4$		$\Delta^4 y_{-2} = 16$
		$\Delta y_0 = -2$		$\Delta^3 y_{-1} = 8$	
$x_1 = 4$	$y_1 = -1$		$\Delta^2 y_0 = 4$		
		$\Delta y_1 = 2$			
$x_2 = 5$	$y_2 = 1$				

By Gauss's forward formula.

$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots$, we get

$f(x) = 1+(x-3)(-2) + \frac{(x-3)(x-4)}{2}(-4) + \frac{(x-3)(x^2-6x+8)}{6}(8) + \frac{(x-3)(x^2-6x+8)(x-5)}{24}(16)$

i.e. $f(x) = 1-2x+6-2(x^2-7x+12) + \frac{4}{3}(x^3-6x^2+8x+12-3x^2+18x-24)$

$+ \frac{2}{3}(x^2-8x+15)(x^2-6x+8)$

$= \frac{1}{3}[3-6x+18-6(x^2-7x+12)+4(x^3-9x^2+26x-12)$

$+2(x^4-6x^3+8x^2-8x^3+48x^2-64x+15x^2-90x+120)$

$= \frac{1}{3}[3-6x+18-6x^2+42x-72+4x^3-36x^2+104x-48+2x^4-28x^3+142x^2-64x+15x^2$

$-308x+240)$

$= \frac{1}{3}(2x^4-24x^3+115x^2-232x+141)$

==

Ex. Find the value of $\cos 51^0\ 42'$ by Gauss's backward central difference formula.

Given that:

x	50^0	51^0	52^0	53^0	54^0
cosx:	0.6428	0.6293	0.6157	0.6018	0.5878

Solution: Take $x_0 = 52^0$, here $x = 51^0 42'$ and $h = 1^0$ ∴ $p = \frac{x-x_0}{h} = \frac{51^0 42' - 52^0}{1^0} = -18' = -0.3^0$

We prepare forward difference table as:

x	y= cosx	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 50^0$	$y_{-2} = 0.6428$				
		$\Delta y_{-2} = -0.0135$			
$x_{-1} = 51^0$	$y_{-1} = 0.6293$		$\Delta^2 y_{-2} = -0.0001$		
		$\Delta y_{-1} = -0.0136$		$\Delta^3 y_{-2} = -0.0002$	
$x_0 = 52^0$	$y_0 = 0.6157$		$\Delta^2 y_{-1} = -0.0003$		$\Delta^4 y_{-2} = 0.0004$
		$\Delta y_0 = -0.0139$		$\Delta^3 y_{-1} = 0.0002$	
$x_1 = 53^0$	$y_1 = 0.6018$		$\Delta^2 y_0 = -0.0001$		
		$\Delta y_1 = -0.0140$			
$x_2 = 54^0$	$y_2 = 0.5878$				

By Gauss's backward formula.

$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-2} + \frac{p(p^2-1)(p+2)}{4!}\Delta^4 y_{-2} + \ldots\ldots$, we get

$\cos 51^0\ 42' = 0.6157 + (-0.3)(-0.0136) + \frac{(-0.3)(0.7)}{2}(-0.0003)$

$+ \frac{(-0.3)(-0.91)}{6}(-0.0002) + \frac{(-0.3)(-0.91)(1.7)}{24}(0.0004)$

i.e. $\cos 51^0\ 42' = 0.6198$

==

Ex. From the following table:

Year	1901	1911	1921	1931	1941	1951
Sales in thousands	12	15	20	27	39	52

Find the sales of concern for the year 1936 by Gauss's backward formula. (Mar.2019)

Solution: Take $x_0 = 1931$, here $x = 1936$ and $h = 10$ ∴ $p = \frac{x-x_0}{h} = \frac{1936-1931}{10} = 0.5$.

We prepare forward difference table as:

x=Year	y = Sales in thousands	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-3} = 1901$	$y_{-3} = 12$					
		$\Delta y_{-3} = 3$				
$x_{-2} = 1911$	$y_{-2} = 15$		$\Delta^2 y_{-3} = 2$			
		$\Delta y_{-2} = 5$		$\Delta^3 y_{-3} = 0$		
$x_{-1} = 1921$	$y_{-1} = 20$		$\Delta^2 y_{-2} = 2$		$\Delta^4 y_{-3} = 3$	
		$\Delta y_{-1} = 7$		$\Delta^3 y_{-2} = 3$		$\Delta^5 y_{-3} = -10$
$x_0 = 1931$	$y_0 = 27$		$\Delta^2 y_{-1} = 5$		$\Delta^4 y_{-2} = -7$	
		$\Delta y_0 = 12$		$\Delta^3 y_{-1} = -4$		
$x_1 = 1941$	$y_1 = 39$		$\Delta^2 y_0 = 1$			
		$\Delta y_1 = 13$				
$x_2 = 1951$	$y_2 = 52$					

By Gauss's backward formula.

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-2} + \frac{p(p^2-1)(p+2)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{p(p^2-1)(p^2-4)}{5!}\Delta^5 y_{-3} + \ldots\ldots,\text{ we get}$$

$$y(1936) = 27+(0.5)(7) + \frac{(0.5)(1.5)}{2}(5) + \frac{(0.5)(-0.75)}{6}(3) + \frac{(0.5)(-0.75)(2.5)}{24}(-7)$$
$$+ \frac{(0.5)(-0.75)(-3.75)}{120}(-10)$$

i.e. $y(1936) = 32.344$

$\therefore$ Sales in the year 1936 is approximately 32.344 thousands.

===

Ex. Using Gauss's backward formula estimate the number of persons earning wages between Rs.60 and Rs.70 from the following data:

Wages (Rs.):	Below 40	40-60	60-80	80-100	100-120
No. of persons (in thousands):	250	120	100	70	50

Solution: We prepare forward difference table as:

Below(x)	y= Sales in thousands	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 40$	$y_{-2} = 250$				
		$\Delta y_{-2} = 120$			
$x_{-1} = 60$	$y_{-1} = 370$		$\Delta^2 y_{-2} = -20$		

		$\Delta y_{-1} = 100$		$\Delta^3 y_{-2} = -10$	
$x_0 = 80$	$y_0 = 470$		$\Delta^2 y_{-1} = -30$		$\Delta^4 y_{-2} = 20$
		$\Delta y_0 = 70$		$\Delta^3 y_{-1} = 10$	
$x_1 = 100$	$y_1 = 540$		$\Delta^2 y_0 = -20$		
		$\Delta y_1 = 50$			
$x_2 = 120$	$y_2 = 590$				

Take $x_0 = 80$, here $x = 70$ and $h = 20$ $\therefore$ $p = \frac{x-x_0}{h} = \frac{70-80}{20} = -0.5$.

By Gauss's backward formula.

$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-2} + \frac{p(p^2-1)(p+2)}{4!}\Delta^4 y_{-2} + \ldots\ldots$, we get

$y(70) = 470 + (-0.5)(100) + \frac{(-0.5)(0.5)}{2}(-30) + \frac{(-0.5)(-0.75)}{6}(-10) + \frac{(-0.5)(-0.75)(`1.5)}{24}(20)$

i.e. $y(70) = 423.6$

$\therefore$ the number of persons earning wages between Rs. 60 and Rs. 70

$= 423.6 - 370 = 53.6$ thousands.

==

Lagrange's Interpolation Formula:

If $y = f(x)$ takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots\ldots, x_n$

then $f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots\ldots$

$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-2})(x-x_n)}{(x_{n-1}-x_0)(x_{n-1}-x_1)\ldots(x_{n-1}-x_n)}y_{n-1} + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$

Lagrange's Interpolation formula for unequal intervals.

Proof: Let $y = f(x)$ be a polynomial in x which takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots\ldots, x_n$. There are (n+1) values of f(x) so $(n+1)^{th}$ difference is zero. Thus f(x) is a polynomial in x of degree n. Let this polynomial be

$y = f(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n) + a_1(x-x_0)(x-x_2)\ldots(x-x_n)$

$+ a_2(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n) + \ldots + a_n(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})$

…… (1)

Where $a_0, a_1, a_2, \ldots, a_n$ are constants and can be determined by putting $y = y_0$ when $x = x_0$, $y = y_1$ when $x = x_1$, $y = y_2$ when $x = x_2$ etc.

Now put $y = y_0$ when $x = x_0$ in (1), we get,

$y_0 = a_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n) + 0$

$\therefore a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$

Similarly, by putting $y = y_1$ when $x = x_1$ in (1), we get,

$y_1 = a_1(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n) + 0$

$\therefore a_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}$

proceeding in this we get $a_2, a_3, \dots, a_n$.

By substituting the values of $a_0, a_1, a_2, \dots, a_n$ in (1), we get,

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}y_1 + \dots\dots$$
$$+ \frac{(x-x_0)(x-x_1)\dots(x-x_{n-2})(x-x_n)}{(x_{n-1}-x_0)(x_{n-1}-x_1)\dots(x_{n-1}-x_n)}y_{n-1} + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}y_n$$

==

Ex. Find f(4) from the following table

x	0	1	2	5
y	2	5	7	8

Solution: Given $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$ and $y_0 = 2$, $y_1 = 5$, $y_2 = 7$, $y_3 = 8$.

By Lagrange's Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$\therefore f(4) = \frac{(4-1)(4-2)(4-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(4-0)(4-2)(4-5)}{(1-0)(1-2)(1-5)}(5)$$
$$+ \frac{(4-0)(4-1)(4-5)}{(2-0)(2-1)(2-5)}(7) + \frac{(4-0)(4-1)(4-2)}{(5-0)(5-1)(5-2)}(8)$$
$$= \frac{(3)(2)(-1)(2)}{(-1)(-2)(-5)} + \frac{(4)(2)(-1)(5)}{(1)(-1)(-4)} + \frac{(4)(3)(-1)(7)}{(2)(1)(-3)} + \frac{(4)(3)(2)(8)}{(5)(4)(3)}$$
$$= 1.2 - 10 + 14 + 3.2$$
$$= 8.4$$

==

Ex. Compute the value of f(x) for x = 2.5 from the following table

x	1	2	3	4
y	1	8	27	64

Solution: Given $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$ and $y_0 = 1$, $y_1 = 8$, $y_2 = 27$, $y_3 = 64$.

By Lagrange's Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$\therefore y(2.5) = \frac{(2.5-2)(2.5-3)(2.5-4)}{(1-2)(1-3)(1-4)}(1) + \frac{(2.5-1)(2.5-3)(2.5-4)}{(2-1)(2-3)(2-4)}(8)$$
$$+\frac{(2.5-1)(2.5-2)(2.5-4)}{(3-1)(3-2)(3-4)}(27) + \frac{(2.5-1)(2.5-2)(2.5-3)}{(4-1)(4-2)(4-3)}(64)$$
$$= \frac{(0.5)(-0.5)(-1.5)(1)}{(-1)(-2)(-3)} + \frac{(1.5)(-0.5)(-1.5)(8)}{(1)(-1)(-2)} + \frac{(1.5)(0.5)(-1.5)(27)}{(2)(1)(-1)}$$
$$+ \frac{(1.5)(0.5)(-0.5)(64)}{(3)(2)(1)}$$
$$= -0.0625 + 4.5 + 15.1875 - 4$$
$$= 15.625$$

==

Ex. The values of x and y are given as below

x	5	6	9	11
y	12	13	14	16

Find the value of y at x = 10.

Solution: Given $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$ and $y_0 = 12$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$.

Here the values of x are not equally spaced.

∴ We use Lagrange's Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$\therefore y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$
$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$
$$= \frac{(4)(1)(-1)(12)}{(-1)(-4)(-6)} + \frac{(5)(1)(-1)(13)}{(1)(-3)(-5)} + \frac{(5)(4)(-1)(14)}{(4)(3)(-2)} + \frac{(5)(4)(1)(16)}{(6)(5)(2)}$$
$$= 2 - 4.33 + 11.67 + 5.33$$
$$= 14.67$$

==

Ex. Find the form of function given by

x	3	2	1	-1
y	3	12	15	-21

Solution: Given $x_0 = 3$, $x_1 = 2$, $x_2 = 1$, $x_3 = -1$ and $y_0 = 3$, $y_1 = 12$, $y_2 = 15$, $y_3 = -21$.

By Lagrange's Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(3+1)}(3) + \frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)}(12) + \frac{(x-3)(x-2)(x+1)}{(1-3)(1-2)(1+1)}(15) + \frac{(x-3)(x-2)(x-1)}{(-1-3)(-1-2)(-1-1)}(-21)$$

$$= \frac{(x-2)(x-1)(x+1)}{(1)(2)(4)}(3) + \frac{(x-3)(x-1)(x+1)}{(-1)(1)(3)}(12) + \frac{(x-3)(x-2)(x+1)}{(-2)(-1)(2)}(15) + \frac{(x-3)(x-2)(x-1)}{(-4)(-3)(-2)}(-21)$$

$$= \frac{3}{8}(x-2)(x-1)(x+1) - 4(x-3)(x-1)(x+1) + \frac{15}{4}(x-3)(x-2)(x+1) + \frac{7}{8}(x-3)(x-2)(x-1)$$

$$= \frac{1}{8}[3(x-2)(x-1)(x+1) - 32(x-3)(x-1)(x+1) + 30(x-3)(x-2)(x+1) + 7(x-3)(x-2)(x-1)]$$

$$= \frac{1}{8}[(3x-6)(x^2-1) - (32x-96)(x^2-1) + (30x-90)(x^2-x-2) + (7x-21)(x^2-3x+2)]$$

$$= \frac{1}{8}[3x^3-3x-6x^2+6-32x^3+32x+96x^2-96+30x^3-30x^2-60x-90x^2+90x+180 +7x^3-21x^2+14x-21x^2+63x-42]$$

$$= \frac{1}{8}[8x^3-72x^2+136x+48]$$

$$= x^3 - 9x^2 + 17x + 6$$

==

Ex. Find the cubic Lagrange's interpolating polynomial from the following data

x	0	1	2	5
y	2	3	12	147

Solution: Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$ and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$.

By Lagrange's Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= -\frac{1}{5}(x-1)(x-2)(x-5) + \frac{3}{4}x(x-2)(x-5)$$

$-2x(x-1)(x-5) + \frac{49}{20}x(x-1)(x-2)$

$= \frac{1}{20}[-4(x-1)(x^2-7x+10) + 15x(x^2-7x+10) - 40x(x^2-6x+5) + 49x(x^2-3x+2)]$

$= \frac{1}{20}[-4x^3+28x^2-40x+4x^2-28x+40+15x^3-105x^2+150x-40x^3+240x^2-200x$
$+49x^3-147x^2+98x]$

$= \frac{1}{20}(20x^3+20x^2-20x+40)$

$= x^3 + x^2 - x + 2$

===

Ex. Find the unique polynomial p(x) of degree 2 such that p(1) = 1, p(3) = 27, p(4)= 64 By Lagrange's method of interpolation.

Solution: Given $x_0 = 1$, $x_1 = 3$, $x_2 = 4$ and $y_0 = 1$, $y_1 = 27$, $y_2 = 64$.

By Lagrange's Interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-3)(x-4)}{(1-3)(1-4)}(1) + \frac{(x-1)(x-4)}{(3-1)(3-4)}(27) + \frac{(x-1)(x-3)}{(4-1)(4-3)}(64)$$

$$= \frac{1}{6}(x^2-7x+12) - \frac{27}{2}(x^2-5x+4) + \frac{64}{3}(x^2-4x+3)$$

$$= \frac{1}{6}(x^2-7x+12- 81x^2+405x-324+128x^2-512x+384)$$

$$= \frac{1}{6}(48x^2-114x+72)$$

$$= 8x^2-19x+12$$

===

MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) If E denote the shift operator, then Ef(x) = ……

A) f(x+h) B) f(x-h) C) f(x) D) None of these

2) If E denote the shift operator, then $E^n f(x)$ = ……

A) f(x+nh) B) f(x–nh) C) f(x) D) None of these

3) If E denote the shift operator, then $E^{-n} f(x)$ = ……

A) f(x+nh) B) f(x–nh) C) f(x) D) None of these

4) If E denote the shift operator, then $E^{-1} f(x)$ = ……

A) f(x–h) B) f(x+h) C) f(x) D) None of these

5) If E denote the shift operator, then $E^{1/2} f(x)$ = ……

A) $f(x-\frac{1}{2}h)$ B) $f(x+\frac{1}{2}h)$ C) f(x) D) None of these

6) If E denote the shift operator, then $E^{-1/2}f(x)$ = ……

A) $f(x-\frac{1}{2}h)$ B) $f(x+\frac{1}{2}h)$ C) f(x) D) None of these

7) $E^2\sin x$ =……

A) sin(x–2h) B) sin(x-h) C) sin(x+2h) D) sin(x+h)

8) $E^{-2}\tan x$ =……

A) tan(x–2h) B) tan(x–h) C) tan(x+2h) D) tan(x+h)

9) If Δ denote the forward difference operator, then Δf(x) =………

A) f(x+h) + f(x) B) f(x–h) – f(x) C) f(x+h) – f(x) D) f(x+h)

10) Relation between forward difference operator Δ and shift operator E is ……

A) Δ = 1+ E B) Δ = 1– E C) $\Delta = 1+ E^{-1}$ D) Δ = E – 1

11) Δ logx =……

A) log(x+h) B) $\log(1+\frac{h}{x})$ C) log(x–h) D) $\log(1-\frac{h}{x})$

12) Δ cosx =……

A) cos(x+h) B) cos(x+h) – cosx C) cos(x–h) D) cosx

13) Δe^x = ……

A) $e^x + e^h$ B) $e^x(h-1)$ C) $e^x(e^h-1)$ D) $e^x(e^h+1)$

14) $\Delta\tan^{-1}x$ = ……

A) $\tan^{-1}(\frac{h}{1+hx+x^2})$ B) $\tan^{-1}(\frac{x}{1+hx+x^2})$ C) $\tan^{-1}(\frac{h}{1-hx-x^2})$ D) None of these

15) If Δ denote the forward difference operator, then $\Delta^2 f(x)$ =……

A) f(x+2h) – 2f(x+h) + f(x) B) f(x–h) – f(x)

C) f(x+h) – f(x) D) f(x+2h) + 2f(x+h) + f(x)

16) $\Delta^2 e^x$ = ……

A) $e^x + e^h$ B) $e^x(e^h-1)^2$ C) $e^x(e^h-1)$ D) $e^x(e^h+1)$

17) $\Delta^2\cos 2x$ = ……

A) cos(2x+4h) - cos(2x+2h)+cos2x B) cos(2x+2h) - 2cos(2x+h) + cos2x

C) cos(2x+4h) - 2cos(2x+2h)+cos2x D) None of these

18) If Δ denote the forward difference operator, then $\Delta^3 f(x)$ = ……

A) f(x+2h) –2f(x+h) + f(x) B) f(x+3h) – 3f(x+2h) + 3f(x+h) – f(x)

C) f(x+2h) + 2f(x+h) + f(x) D) f(x+3h) + 3f(x+2h) + 3f(x+h) + f(x)

19) If n values of f(x) are given, then ……

A) $\Delta^n f(x) > 0$ B) $\Delta^n f(x) = 0$ C) $\Delta^n f(x) < 0$ D) $\Delta^n f(x) \neq 0$

20) n^{th} order forward difference of a polynomial $P_n(x)$ of degree n is ……

A) constant B) 0 C) x D) None of these

21) n^{th} order forward difference of a polynomial of degree ≤ (n -1) is ……

A) constant B) 0 C) x D) None of these

22) If ∇ denote the backward difference operator, then $\nabla f(x)$ = ……

A) f(x+h) – f(x) B) f(x) – f(x-h) C) f(x+h) + f(x) D) f(x-h)

23) $\Delta f(x_i)$ = ……

A) $\nabla f(x_{i-1})$ B) $\nabla f(x_i)$ C) $\nabla f(x_{i+1})$ D) $\nabla f(x_{i+2})$

24) Relation between backward difference operator ∇ and shift operator E is ……

A) ∇ = 1+ E B) ∇ = 1– E C) $\nabla = 1 - E^{-1}$ D) ∇ = E – 1

25) If ∇ is backward difference operator and E is shift operator, then E∇ = ……

A) E B) ∇ C) Δ D) E + 1

26) If Δ is forward difference operator and E is shift operator, then $E^{-1}\Delta$ = ……

A) E B) ∇ C) Δ D) E^{-1}

27) If ∇ denote the backward difference operator, then ∇sinx = ……

A) sinx – sin(x–h) B) sin(x+h) – sinx

C) sin(x–h) – sinx D) None of these

28) If ∇ denote the backward difference operator, then ∇secx = ……

A) sec(x–h) – secx B) sec(x+h) – secx

C) secx – sec(x–h) D) None of these

29) ∇e^x = ……

A) $e^x + e^h$ B) $e^x(h-1)$ C) $e^x(1-e^{-h})$ D) $e^x(e^h+1)$

30) If ∇ denote the backward difference operator, then $\nabla^2 f(x)$ =……

A) f(x+h) – f(x) B) f(x) – f(x+h)

C) f(x) – 2f(x–h) + f(x–2h) D) f(x+2h) + 2f(x+h) + f(x)

31) If ∇ denote the backward difference operator, then $\nabla^2 e^x$ = ……

A) $e^x(1 + e^{-h})^2$ B) $e^x(1 - e^{-h})^2$ C) $e^x(e^h - 1)^2$ D) None of these

32) If ∇ denote the backward difference operator, then $\nabla^3 f(x)$ =………

A) f(x+2h) + 2f(x+h) + f(x) B) f(x) – f(x+h) + f(x+2h)

C) f(x) - 2f(x-h) + f(x–2h) D) f(x) – 3f(x–h) + 3f(x–2h) – f(x–3h)

33) If δ denote the central difference operator, then δf(x) = ……

A) $f(x + \frac{h}{2}) - f(x - \frac{h}{2})$ B) $f(x + \frac{h}{2}) + f(x - \frac{h}{2})$

C) f(x + h) – f(x – h) D) f(x - h) - f(x + h)

34) Relation between central difference operator δ and shift operator E is ……

A) $\delta = E^{1/2} - E^{-1/2}$ B) $\delta = E + E^{-1}$ C) $\delta = E - E^{-1}$ D) $\delta = E^{1/2} + E^{-1/2}$

35) If δ denote the central difference operator, then $\delta f(x + \frac{h}{2}) = \delta f_{1/2}$ = ……

A) $f(x + \frac{h}{2}) - f(x - \frac{h}{2})$ B) $f(x + \frac{h}{2}) + f(x - \frac{h}{2})$

C) f(x + h) – f(x) D) f(x - h) - f(x + h)

36) If δ denote the central difference operator, then $\delta f(x + \frac{3h}{2}) = \delta f_{3/2}$ = ……

A) $f(x + \frac{h}{2}) - f(x - \frac{h}{2})$ B) f(x + 2h) - f(x + h)

C) f(x + h) – f(x – h) D) f(x - h) - f(x + h)

37) If Δ and ∇ are forward and backward difference operators respectively, then ∇Δ = Δ∇ = ……

A) E B) δ C) Δ D) δ^2

38) If δ is central difference operator and E is shift operator, then $\delta^2 E$ = ……

A) δ B) Δ C) Δ^2 D) E^{-1}

39) If μ denote an averaging operator, then μf(x) = ……

A) $\frac{1}{2}$ [f(x+h/2) + f(x-h/2)] B) $\frac{1}{2}$ [f(x+h/2) - f(x-h/2)]

C) [f(x+h/2) + f(x-h/2)] D) [f(x+h/2) - f(x-h/2)]

40) Relation between an averaging operator μ and shift operator E is ……

A) $\mu = \frac{1}{2}(E^{1/2} - E^{-1/2})$ B) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

C) $\mu = \frac{1}{2}(E^{-1/2} - E^{1/2})$ D) $\mu = \frac{1}{2}(E - E^{-1})$

41) If μ is an averaging operator and δ is a central difference operator, then $\mu - \frac{\delta}{2}$ = ……

A) $\frac{1}{2}E^{1/2}$ B) $\frac{1}{2}E^{-1/2}$ C) $E^{1/2}$ D) $E^{-1/2}$

42) If Δ and ∇ are forward and backward difference operators respectively, then $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ =..

A) Δ + ∇ B) Δ - ∇ C) ∇ - Δ D) None of these

43)If μ is an averaging operator and δ is a central difference operator, then μδ = ……

A) $\frac{1}{2}(\Delta - \nabla)$ B) $\frac{1}{2}(\nabla - \Delta)$ C) $\frac{1}{2}(\Delta + \nabla)$ D) $E^{-1/2}$

44) If μ is an averaging operator and δ is a central difference operator, then δ[f(x)g(x)] = ……

A) δf(x)μg(x) - μf(x)δg(x) B) δf(x)μg(x) + μf(x)δg(x)

C) δf(x) δg(x) + δf(x)δg(x) D) μf(x)μg(x) + μf(x) μg(x)

45) $y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-4)}{5!}\Delta^5 y_{-2}$ +…… is known as the Gauss …… central difference formula.

A) forward B) backward C) shift D) average

46) $y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\Delta^3 y_{-2} + \frac{p(p^2-1)(p+2)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1)(p^2-4)}{5!}\Delta^5 y_{-3}$ +…… is known as the Gauss …… central difference formula.

A) forward B) backward C) shift D) average

47) If y = f(x) takes the values y_0, y_1, y_2 corresponding to x_0, x_1, x_2 then Lagrange's Interpolation formula for f(x) = ……

A) $\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$ B) $\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0$

C) $\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 - \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 - \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$ D) $\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$

48) Lagrange's Interpolation Formula is used for …… intervals

A) equal B) unequal

C) both equal and unequal D) closed

49) Statement 'Lagrange's Interpolation Formula is used for both equal and unequal intervals' is ……

A) true B) false

C) may or may not be true D) None of these

50) The polynomial p(x) of degree 2 such that p(1) = 1, p(3) = 27, p(4)= 64 is ……

A) $8x^2 - 19x + 12$ B) $8x^2 - 19x - 12$

C) $8x^2 + 19x + 12$ D) None of these

51) The cubic polynomial obtained by using the data

x	0	1	2	5
y	2	3	12	147

is ……

A) $x^3 + x^2 + x + 2$ B) $x^3 + x^2 - x + 2$

C) $x^3 + x^2 - x - 2$ D) None of these

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UNIT-3: CURVE FITTING

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Error of Approximation: Let y = f(x) be a continuous curve fitting the given data (x_i, y_i), i = 1, 2, ……n, then $E_i = y_i - f(x_i)$ is called error of approximation.

Least Square Method: The least square method requires that the sum of squares of the errors at all the points i.e. $\sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n}[y_i - f(x_i)]^2$ must be minimum.

Weierstrass Theorem: If f(x) is a continuous function in the interval [a, b], then to each $\epsilon > 0$, there exists a polynomial P(x) such that $|f(x) - P(x)| < \epsilon \ \forall \ x \in [a, b]$.

==

Linear Curve Fitting-Fitting of Straight Line: Let the data (x_i, y_i), i = 1, 2, ……n be given. We can find least square approximation to the given data in the form

y = f(x) = a + bx …… (1)

Error of approximation = $y_i - f(x_i) = y_i - a - bx_i$

The sum of squares of the errors should be minimum.

i.e. $S(a, b) = \sum_{i=1}^{n}[y_i - a - bx_i]^2 =$ minimum.

Now, S(a, b) is a function of the two variables a and b. By necessary condition

S(a, b) is minimum if $\frac{\partial S}{\partial a} = -2\sum_{i=1}^{n}(y_i - a - bx_i) = 0$

i.e. $\sum_{i=1}^{n}(y_i - a - bx_i) = 0$

i.e. $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i$

i.e. $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$…… (2)

& $\frac{\partial S}{\partial b} = -2\sum_{i=1}^{n} x_i(y_i - a - bx_i) = 0$

i.e. $\sum_{i=1}^{n} x_i(y_i - a - bx_i) = 0$

i.e. $\sum_{i=1}^{n} x_i y_i - ax_i - b{x_i}^2 = 0$

i.e. $\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} b{x_i}^2$

i.e. $\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} {x_i}^2$…… (3)

By given data (x_i, y_i), i = 1, 2, ……n, we can find the sums occurred in equations (2) and (3) and solving these equations, we get values of a and b, putting these values in (1), we approximation of straight line which fit the given data. Equations (2) and (3) are called the normal equations for fitting a straight line to a given data.

==

Ex. Fit a least square straight line approximation to the data

x	0.5	1.0	1.5	2.0	2.5	3.0
y	0.31	0.82	1.29	1.85	2.51	3.02

Find the least squares error.

Solution: Let the straight line be y = f(x) = a + bx …… (1), the number of the given data values are n = 6. The normal equations for fitting a straight line to a given data are

$$\sum_{i=1}^{6} y_i = 6a + b\sum_{i=1}^{6} x_i \ldots\ldots (2)$$

$$\sum_{i=1}^{6} x_i y_i = a\sum_{i=1}^{6} x_i + b\sum_{i=1}^{6} {x_i}^2 \ldots\ldots (3)$$

	x_i	y_i	${x_i}^2$	$x_i y_i$
	0.5	0.31	0.25	0.155
	1.0	0.82	1.00	0.820
	1.5	1.29	2.25	1.935
	2.0	1.85	4.00	3.700
	2.5	2.51	6.25	6.275
	3.0	3.02	9.00	9.060
$\sum$.	10.5	9.80	22.75	21.945

Using these values of sums, from equations (2) and (3), we get,

6a + 10.5b = 9.80 i.e. a + 1.75b = 1.63

& 10.5a + 22.75b = 21.945 i.e. a + 2.167b = 2.09

Solving these equations, we get, a = - 0.28467 and b = 1.096

∴ Required straight line which fit a given data approximately is

y = f(x) = - 0.28467 + 1.096x

Now the least square error is

$$\sum_{i=1}^{n} [y_i - f(x_i)]^2 = \sum_{i=1}^{6} [y_i + 0.28467 - 1.096x_i]^2$$

= 0.002178 + 0.000075 + 0.004807 + 0.003287 + 0.002989 + 0.000278

= 0.013614

===

Ex. Obtain the least square straight line fit the following data

x	0.2	0.4	0.6	0.8	1
y	0.447	0.632	0.775	0.894	1

Find the least squares error.

Solution: Let the straight line be y = a + bx …… (1), the number of the given data values are n = 5. The normal equations for fitting a straight line to a given data are

$$\sum_{i=1}^{5} y_i = 5a + b\sum_{i=1}^{5} x_i \ldots\ldots (2)$$

$$\sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} {x_i}^2 \ldots\ldots (3)$$

	x_i	y	${x_i}^2$	$x_i y_i$
	0.2	0.447	0.25	0.155
	0.4	0.632	1.00	0.082
	0.6	0.775	2.25	1.935
	0.8	0.894	4.00	3.700
	1.0	1.000	6.25	6.275
∑.	3.0	3.748	2.5224	2.200

Using these values of sums, from equations (2) and (3), we get,

5a + 3b = 3.748 i.e. a + 0.6b = 0.7496

& 3a + 2.5224b = 2.2 i.e. a + 0.8408b = 0.73333

Solving these equations, we get, a = 0.3392 and b = 0.684

∴ Required straight line which fit a given data approximately is

y = 0.3392 + 0.684x

Now the least square error is

$\sum_{i=1}^{n}[y_i - f(x_i)]^2 = \sum_{i=1}^{5}[y_i - 0.3392 - 0.684x_i]^2$

= 0.000841 + 0.000369 + 0.000645 + 0.000058 + 0.000538

= 0.00245

==

Ex. Fit a straight line for the following data

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Find the least squares error.

Solution: Let the straight line be y = a + bx …… (1), the number of the given data values are n = 6. The normal equations for fitting a straight line to a given data are

$$\sum_{i=1}^{6} y_i = 6a + b\sum_{i=1}^{6} x_i \ldots\ldots (2)$$

$$\sum_{i=1}^{6} x_i y_i = a\sum_{i=1}^{6} x_i + b\sum_{i=1}^{6} {x_i}^2 \ldots\ldots (3)$$

	x_i	y	${x_i}^2$	$x_i y_i$
	1	2.4	2.4	1
	2	3	6	4
	3	3.6	10.8	9
	4	4	16	16
	6	5	30	36
	8	6	48	64
∑.	24	24	113.2	130

Using these values of sums, from equations (2) and (3), we get,

6a + 24b = 24 i.e. a + 4b = 4

& 24a + 113.2b = 130 i.e. a + 4.4716b = 5.4162

Solving these equations, we get, a = 1.976 and b = 0.506

∴ Required straight line which fit a given data approximately is

y = 1.976 + 0.506x

==

Ex. Using the method of least square fit a straight line to the following data

x	1	2	3	4	5
y	2	4	6	8	10

Find the least squares error.

Solution: Let the straight line be y = a + bx …… (1), the number of the given data values are n = 5. The normal equations for fitting a straight line to a given data are

$\sum_{i=1}^{5} y_i = 5a + b\sum_{i=1}^{5} x_i$…… (2)

$\sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} {x_i}^2$…… (3)

	x_i	y	${x_i}^2$	$x_i y_i$
	1	2	1	2
	2	4	4	8
	3	6	9	18
	4	8	16	32
	5	10	25	50
∑.	15	30	55	110

Using these values of sums, from equations (2) and (3), we get,

5a + 15b = 30 i.e. a + 3b = 6

& 15a + 55b = 110 i.e. a + $\frac{11}{3}$b = $\frac{22}{3}$

Solving these equations, we get, a = 0 and b = 2

∴ Required straight line which fit a given data approximately is y = 2x.

==

Ex. By the method of least squares, find the straight line that best fits the following data

x	1	2	3	4	5
y	14	27	40	55	68

Solution: Let the straight line be y = a + bx …… (1), the number of the given data values are n = 5. The normal equations for fitting a straight line to a given data are

$$\sum_{i=1}^{5} y_i = 5a + b\sum_{i=1}^{5} x_i \text{ …… (2)}$$

$$\sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} {x_i}^2 \text{ …… (3)}$$

	x_i	y	${x_i}^2$	$x_i y_i$
	1	14	1	14
	2	57	4	114
	3	40	9	120
	4	55	16	220
	5	68	25	340
∑.	15	204	55	748

Using these values of sums, from equations (2) and (3), we get,

5a + 15b = 204 i.e. a + 3b = 40.8

& 15a + 55b = 748 i.e. a + $\frac{11}{3}$b = $\frac{748}{15}$

Solving these equations, we get, a = 0 and b = 13.6

∴ Required straight line which fit a given data approximately is y = 13.6x.

==

Non-Linear Curve Fitting-Fitting of Quadratic Equation:

Let the data (x_i, y_i), i = 1, 2, ……n be given. We can find least square approximation to the given data in the form

y = f(x) = a + bx + cx^2 …… (1)

Error of approximation = y_i - f(x_i) = y_i – a - bx_i - cx_i^2

The sum of squares of the errors should be minimum.

i.e. S(a, b, c) = $\sum_{i=1}^{n}[y_i - a - bxi - cx_i^2]^2 =$ minimum.

i.e. S(a, b, c) = $\sum_{i=1}^{n}[a + bxi + cx_i^2 - y_i]^2 =$ minimum.

Now, S(a, b, c) is a function of the three variables a, b and c. By necessary condition if S(a, b, c) is minimum, then

$$\frac{\partial S}{\partial a} = 2\sum_{i=1}^{n}(a + bxi + cx_i^2 - y_i) = 0$$

i.e. $\sum_{i=1}^{n}(a + bxi + cx_i^2 - y_i) = 0$

i.e. $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i + \sum_{i=1}^{n} cx_i^2$

i.e. $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$ …… (2)

$$\frac{\partial S}{\partial b} = 2\sum_{i=1}^{n} x_i(a + bx_i + cx_i^2 - y_i) = 0$$

i.e. $\sum_{i=1}^{n} x_i(a + bx_i + cx_i^2 - y_i) = 0$

i.e. $\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} b\, x_i{}^2 + \sum_{i=1}^{n} cx_i^3$

i.e. $\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i{}^2 + c\sum_{i=1}^{n} x_i^3$ …… (3)

& $\frac{\partial S}{\partial c} = 2\sum_{i=1}^{n} x_i^2(a + bx_i + cx_i^2 - y_i) = 0$

i.e. $\sum_{i=1}^{n} x_i^2(a + bx_i + cx_i^2 - y_i) = 0$

i.e. $\sum_{i=1}^{n} x_i^2 y_i = \sum_{i=1}^{n} ax_i^2 + \sum_{i=1}^{n} b\, x_i{}^3 + \sum_{i=1}^{n} cx_i^4$

i.e. $\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i{}^3 + c\sum_{i=1}^{n} x_i^4$ …… (4)

By given data (x_i, y_i), i = 1, 2, ……n, we can find the sums occurred in equations (2), (3) and (4) and solving these equations, we get values of a, b and c, putting these values in (1), we get equation of parabola which fit the given data approximately. Equations (2), (3) and (4) are called the normal equations for fitting a non-linear curve i.e. quadratic curve i.e. parabola to a given data.

==

Ex. Find the least squares approximation of second degree, to the discrete data

x	-2	-1	0	1	2
y	15	1	1	3	19

Solution: Let the second degree approximation is y = f(x) = a + bx + cx^2 …… (1)

the number of the given data values are n = 5. The normal equations for fitting a second degree curve to a given data are

$$\sum_{i=1}^{5} y_i = 5a + b\sum_{i=1}^{5} x_i + c\sum_{i=1}^{5} x_i{}^2 \text{ …… (2)}$$

$$\sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} x_i{}^2 + c\sum_{i=1}^{5} x_i^3 \text{ …… (3)}$$

$$\& \sum_{i=1}^{5} x_i^2 y_i = a\sum_{i=1}^{5} x_i^2 + b\sum_{i=1}^{5} x_i{}^3 + c\sum_{i=1}^{5} x_i^4 \text{ …… (4)}$$

	x_i	y_i	$x_i{}^2$	$x_i{}^3$	$x_i{}^4$	$x_i y_i$	$x_i^2 y_i$
	-2	15	4	-8	16	-30	60
	-1	1	1	-1	1	-1	1
	0	1	0	0	0	0	0
	1	3	1	1	1	3	3
	2	19	4	8	16	38	76
∑.	0	39	10	0	34	10	140

Using these values of sums, from equations (2), (3) and (4), we get,

5a + 0 + 10c = 39 i.e. a + 2c = $\frac{39}{5}$

0 + 10b + 0 = 10 i.e. b = 1

& $10a + 0 + 34c = 140$ i.e. $a + \frac{17}{5}c = 14$

Solving these equations, we get, $a = -\frac{37}{35}$, b = 1 and $c = \frac{31}{7}$

∴ Required equation of second degree which fit a given data approximately is

$y = -\frac{37}{35} + x + \frac{31}{7}x^2 = \frac{1}{35}(-37+35x + 155x^2)$

==

Ex. Fit a parabola of second degree of the following data

x	0	1	2	3	4
y	1	4	10	17	30

Solution: Let the second degree approximation is $y = f(x) = a + bx + cx^2$ …… (1)

the number of the given data values are n = 5. The normal equations for fitting a second degree curve to a given data are

$$\sum_{i=1}^{5} y_i = 5a + b\sum_{i=1}^{5} x_i + c\sum_{i=1}^{5} {x_i}^2 \text{ …… (2)}$$
$$\sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} {x_i}^2 + c\sum_{i=1}^{5} x_i^3 \text{ …… (3)}$$
$$\& \sum_{i=1}^{5} x_i^2 y_i = a\sum_{i=1}^{5} x_i^2 + b\sum_{i=1}^{5} {x_i}^3 + c\sum_{i=1}^{5} x_i^4 \text{ …… (4)}$$

	x_i	y_i	${x_i}^2$	${x_i}^3$	${x_i}^4$	$x_i y_i$	$x_i^2 y_i$
	0	1	0	0	0	0	0
	1	4	1	1	1	4	4
	2	10	4	8	16	20	40
	3	17	9	27	81	51	153
	4	30	16	64	256	120	480
∑.	10	62	30	100	354	195	677

Using these values of sums, from equations (2), (3) and (4), we get,

5a + 10b + 30c = 62 i.e. a + 2b + 6c = 12.4

10a + 30b + 100c = 195 i.e. a + 3b + 10c = 19.5

& 30a + 100b + 354c = 677 i.e. a + 3.33b + 11.8c = 22.567

Solving these equations, we get, a = 1.2, b = 1.1 and c = 1.5

∴ Required equation parabola of second degree which fit a given data approximately is $y = 1.2 + 1.1x + 1.5x^2$

==

Ex. Fit a parabola of second degree, to the following data

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Solution: Let the second degree approximation is y = f(x) = a + bx + cx^2 …… (1)

the number of the given data values are n = 5. The normal equations for fitting a second degree curve to a given data are

$$\sum_{i=1}^{5} y_i = 5a + b\sum_{i=1}^{5} x_i + c\sum_{i=1}^{5} {x_i}^2 \ldots\ldots (2)$$

$$\sum_{i=1}^{5} x_i y_i = a\sum_{i=1}^{5} x_i + b\sum_{i=1}^{5} {x_i}^2 + c\sum_{i=1}^{5} x_i^3 \ldots\ldots (3)$$

$$\& \sum_{i=1}^{5} x_i^2 y_i = a\sum_{i=1}^{5} x_i^2 + b\sum_{i=1}^{5} {x_i}^3 + c\sum_{i=1}^{5} x_i^4 \ldots\ldots (4)$$

	x_i	y_i	${x_i}^2$	${x_i}^3$	${x_i}^4$	$x_i y_i$	$x_i^2 y_i$
	0	1	0	0	0	0	0
	1	1.8	1	1	1	1.8	1.8
	2	1.3	4	8	16	2.6	5.2
	3	2.5	9	27	81	7.5	22.5
	4	6.3	16	64	256	25.2	100.8
∑.	10	12.9	30	100	354	37.1	130.3

Using these values of sums, from equations (2), (3) and (4), we get,

5a + 10b + 30c = 12.9 i.e. a + 2b + 6c = 2.58

10a + 30b + 100c = 37.1 i.e. a + 3b + 10c = 3.71

& 30a + 100b + 354c = 130.3 i.e. a + 3.33b + 11.8c = 4.34

Solving these equations, we get, a = 1.42, b = -1.07 and c = 0.55

∴ Required equation parabola of second degree which fit a given data approximately is y = 1.42 – 1.07x + $0.55x^2$

==

Fitting an Exponential Curve of Type y = ae^{bx}: Let y = ae^{bx} be the exponential curve which fit the given data (x_i, y_i), i = 1, 2, ……n. First we reduce it to linear form by taking logarithm on both sides as:

log_m y = log_m a + b$x log_m$ e

i.e. Y = A + Bx, which is linear equation having normal equations

$\sum Y$ = nA + B$\sum x$ …… (2)

and $\sum xY$ = A $\sum x$+ B$\sum x^2$ … … . (3),

where, Y = log_m y, A = log_m a and B = $b log_m e$

Solving these equations, we get values of A and B. From this we find values of a and b discussed in two cases.

Case-i) When m = e, then A = log_e a i.e. a = e^A and B = $blog_e\, e$ = b.

Case-ii) When m = 10, then A = log_{10} a i.e. a = 10^A and B = $blog_{10}\, e$ =(0.4343)b

==

Fitting a Logarithmic Curve of Type y = ax^b: Let y = ax^b …… (1)

be the logarithmic curve which fit the given data (x_i, y_i), i = 1, 2, ……n.

First we reduce it to linear form by taking logarithm on both sides as:

log y = log a + b log x

i.e. Y = A + bX, which is linear equation having normal equations

$\sum Y$ = nA + b$\sum X$ …… (2)

and $\sum XY$ = A $\sum X$+ b$\sum X^2$ … … . (3),

where X = log x, Y = log y and A = log a.

Solving these equations, we get values of A= log a i.e. a = e^A and b, putting these values in (1), we approximation of an exponential curve which fit the given data.

Equations (2) and (3) are called the normal equations for fitting a straight line to a given data.

==

Ex. Using the method of least squares fit the non-linear curve of the form y = ae^{bx} to the following data

x	0	2	4
y	5.012	10	31.62

Solution: Let y = ae^{bx} …… (1)

be the exponential curve which fit the given data (x_i, y_i), i = 1, 2, 3.

First we reduce it to linear form by taking logarithm of base 10 on both sides as:

log_{10} y = log_{10} a + bx$log_{10}e$

i.e. Y = A + Bx, which is linear equation having normal equations

$\sum Y$ = 3A + B$\sum x$ …… (2)

and $\sum xY$ = A $\sum x$+ B$\sum x^2$ … … . (3),

where Y = log_{10} y and A = ln a and B = b$log_{10}e$ = 0.4343b

	x	y	Y = log_{10} y	x^2	xY
	0	5.012	0.7	0	0
	2	10	1.0	4	2
	4	31.62	1.5	16	6
∑.	6	--	3.2	20	8

∴ The normal equations become,

3A + 6B = 3.2 and 6A + 20B = 8

Solving these equations, we get,

A = log_{10} a = 0.6667 and B = 0.4343b = 0.2

∴ a = $10^{0.6667}$ = 4.6419 and b = $\frac{0.2}{0.4343}$ = 0.4605

∴ Required exponential curve which fit a given data

approximately is y = 4.6419 $e^{0.4605x}$

==

Ex. Find the best fit of the type y = ae^{bx} to the following data by the method of least squares

x	1	5	7	9	12
y	10	15	12	15	21

Solution: Let y = ae^{bx} …… (1)

be the exponential curve which fit the given data (x_i, y_i), i = 1, 2, ……5.

First we reduce it to linear form by taking logarithm of base 10 on both sides as:

log_{10} y = log_{10} a + bx$log_{10}e$

i.e. Y = A + Bx, which is linear equation having normal equations

$\sum Y$ = 5A + B$\sum x$ …… (2)

and $\sum xY$ = A $\sum x$ + B$\sum x^2$ … … . (3),

where Y = log_{10} y and A = ln a and B = b$log_{10}e$ = 0.4343b

	x	y	Y = log_{10} y	x^2	xY
	1	10	1.0	1	1
	5	15	1.17609	25	5.88045
	7	12	1.07918	49	7.55426
	9	15	1.17609	81	10.58481
	12	21	1.32222	144	15.86664
∑.	34	--	5.75358	300	40.88616

∴ The normal equations become,

5A + 34B = 5.75358 and 34A + 300B = 40.88616

Solving these equations, we get,

A = log_{10} a = 0.97658 and B = 0.4343b = 0.02561

∴ a = $10^{0.97658}$ = 9.47502 and b = $\frac{0.02561}{0.4343}$ = 0.05897

∴ Required exponential curve which fit a given data

approximately is y = 9.47502 $e^{0.05897x}$

Ex. Fit an exponential curve by least squares to the following data

x	1	3	5	7	9
y	5.06	12.09	31.23	82.97	223.74

Solution: Let $y = ae^{bx}$ …… (1)

be the exponential curve which fit the given data (x_i, y_i), i = 1, 2, ……5.

First we reduce it to linear form by taking natural logarithm on both sides as:

$\ln y = \ln a + bx$

i.e. Y = A + bx, which is linear equation having normal equations

$\sum Y = 5A + b\sum x$ …… (2)

and $\sum xY = A\sum x + b\sum x^2$ …….(3),

where Y = ln y and A = ln a

	x	y	Y = ln y	x^2	xY
	1	5.06	1.6214	1	1.6214
	3	12.09	2.4924	9	7.4772
	5	31.23	3.4414	25	17.2070
	7	82.97	4.4185	49	30.9295
	9	223.74	5.4104	81	48.6936
∑.	25	--	17.3841	165	105.9287

∴ The normal equations become,

5A+25b = 17.3841 and 25A + 165b = 105.9287

Solving these equations, we get,

A = ln a = 1.1008 i.e. $a = e^{1.1008}$ = 3.0066 and b = 0.4752

∴ Required exponential curve which fit a given data

approximately is $y = 3.0066\, e^{0.4752x}$

==

Ex. Fit a power curve of the form $y = ax^b$ to the following data

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Solution: Let $y = ax^b$ …… (1)

be a power curve which fit the given data (x_i, y_i), i = 1, 2, ……5.

First we reduce it to linear form by taking logarithm of base 10 on both sides as:

$\log_{10} y = \log_{10} a + b \log_{10} x$

i.e. Y = A + bX, which is linear equation having normal equations

$\sum Y = 5A + b\sum X$ …… (2)

and $\sum XY = A\sum X + b\sum X^2$ ... . (3),

where Y = log_{10} y, X = log_{10} x and A = $log_{10}\, a$

	x	y	X = log_{10} x	Y = log_{10} y	X^2	XY
	1	0.5	0	-0.30103	0	0
	2	2	0.30103	0.30103	0.09062	0.09062
	3	4.5	0.47712	0.65321	0.22764	0.31166
	4	8	0.60206	0.90309	0.36248	0.54371
	5	12.5	0.69897	1.09691	0.48856	0.76671
∑.	--	--	2.07918	2.65321	1.16930	1.71270

∴ The normal equations become,

5A+2.07918b = 2.65321 and 2.07918A + 1.16930b = 1.71270

Solving these equations, we get,

A = log_{10} a = 3.70764 i.e. a = $10^{3.70764}$= 5100.82002 and b = 0.73563

∴ Required power curve which fit a given data approximately is

y = 5100.82002 $x^{0.73563}$

==

Ex. Fit a power curve of the form y = ax^b to the following data

x	2	4	6	8	10
y	0.973	3.839	8.641	15.987	23.794

Solution: Let y = ax^b …… (1)

be a power curve which fit the given data (x_i, y_i), i = 1, 2, ……5.

First we reduce it to linear form by taking natural logarithm on both sides as:

ln y = ln a + b ln x

i.e. Y = A + bX, which is linear equation having normal equations

$\sum Y$ = 5A + b$\sum X$ …… (2)

and $\sum XY = A\sum X + b\sum X^2$ ... . (3),

where Y = ln y, X = ln x and A = ln a

	x	y	X = ln x	Y = ln y	X^2	XY
	2	0.973	0.6931	-0.0274	0.4804	0.0190
	4	3.839	1.3862	1.3452	1.9216	1.8647
	6	8.641	1.7918	2.1565	3.2105	3.8640
	8	15.987	2.0794	2.7718	4.3239	5.7636
	10	23.794	2.3026	3.1694	5.3020	7.2978
∑.	--	--	8.2531	9.4155	15.2384	18.8091

∴ The normal equations become,

5A+8.2531b = 9.4155 and 8.2531A + 15.2384b = 18.8091

Solving these equations, we get,

A = ln a = -1.4134 i.e. a = $e^{-1.4134}$= 0.2433 and b = 2.0225

∴ Required power curve which fit a given data approximately is y = 0.2433 $x^{2.0225}$

==

Ex. Fit a power curve of the form $y = ax^b$ to the following data

x	2	4	7	10	20	40	60	80
y	43	25	18	13	8	5	3	2

Solution: Let $y = ax^b$ …… (1)

be a power curve which fit the given data (x_i, y_i), i = 1, 2, ……8.

First we reduce it to linear form by taking natural logarithm on both sides as:

ln y = ln a + b ln x

i.e. Y = A + bX, which is linear equation having normal equations

$\sum Y$ = 8A + b$\sum X$ …… (2)

and $\sum XY$ = A $\sum X$+ b$\sum X^2$ … … . (3),

where Y = ln y, X = ln x and A = ln a

	x	y	X = ln x	Y = ln y	X^2	XY
	2	43	0.6931	3.7612	0.4804	2.6068
	4	25	1.3862	3.2188	1.9216	4.4619
	7	18	1.9459	2.8904	3.7865	5.6244
	10	13	2.3026	2.5649	5.3020	5.9059
	20	8	2.9957	2.0794	8.9742	6.2292
	40	5	3.6888	1.6094	13.6072	5.9368
	60	3	4.0943	1.0986	16.7633	4.4980
	80	2	4.3820	0.6931	19.2019	3.0372
∑.	--	--	21.4886	17.9158	70.0371	38.3002

∴ The normal equations become,

8A+21.4886b = 17.9158 and 21.4886A + 70.0371b = 38.3002

Solving these equations, we get,

A = ln a = 1. 3816 i.e. a = $e^{1.3816}$ = 79.9658 and b = -0.7975

∴ Required power curve which fit a given data approximately is

y = 79.9658 $x^{-0.7975}$

==

MULTIPLE CHOICE QUESTIONS (MCQ'S)

==

1) If y = f(x) is a continuous curve fitting the given data (x_i, y_i), i = 1, 2, ……n, then the error of approximation is E_i = …….

A) $y_i + f(x_i)$ B) $y_i - f(x_i)$ C) $f(x_i) - y_i$ D) $f(x_i) + y_i$

2) The least square method requires that the sum of squares of the errors at all the points i.e. $\sum_{i=1}^{n} E_i^2 = \sum_{i=1}^{n}[y_i - f(x_i)]^2$ must be ……….

A) minimum B) maximum C) 0 D) 1

3) By …… theorem, if f(x) is a continuous function in the interval [a, b], then to each $\epsilon > 0$, there exists a polynomial P(x) such that $|f(x) - P(x)| < \epsilon \ \forall \ x \in [a, b]$.

A) Rolle's B) Lagrange's C) Weierstrass D) None of these

4) A straight line is a …… curve.

A) linear B) non-linear C) quadratic D) None of these

5) Which of the following is linear curve?

A) parabola B) straight line C) circle D) None of these

6) To fit a given data (x_i, y_i), i = 1, 2, ……n, to a straight line y = f(x) = a + bx, the error of approximation = …….

A) $y_i - a$ B) $y_i - bx_i$ C) $y_i - a - bx_i$ D) None of these

7) By the necessary condition S(a, b) is minimum, if ……

A) $\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$ B) $\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} \neq 0$ C) $\frac{\partial S}{\partial a} \neq \frac{\partial S}{\partial b}$ D) None of these

8) $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$ is one of the normal equation for fitting ……… to a given data (x_i, y_i), i = 1, 2, ……n.

A) a parabola B) a straight line C) an exponential curve D) None of these

9) The normal equations for fitting a linear curve i.e. a straight line y = a + bx to a given data (x_i, y_i), i = 1, 2, ……n are ……

A) $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} {x_i}^2$

B) $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i$

C) $\sum_{i=1}^{n} x_i = na + b\sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} {x_i}^2$

D) None of these

10) From the following data, $\sum x$ = …….

x	1	2	3	4
y	0	3	5	7

A) 1 B) 3 C) 6 D) 10

11) From the following data, $\sum y =$ …….

x	1	2	3	4
y	0	3	5	7

A) 15 B) 30 C) 60 D) 10

12) From the following data, $\sum xy =$ …….

x	1	2	3	4
y	0	3	5	7

A) 15 B) 49 C) 60 D) 10

13) From the following data, $\sum x^2 =$ …….

x	1	2	3	4
y	0	3	5	7

A) 15 B) 49 C) 30 D) 10

14) From the following data, $\sum y^2 =$ …….

x	1	2	3	4
y	0	3	5	7

A) 15 B) 30 C) 60 D) 83

15) From the given data, $\sum x =$ …….

x	0	1	2	3	4
y	1	3	5	7	9

A) 9 B) 10 C) 15 D) 25

16) From the given data, $\sum y =$ …….

x	0	1	2	3	4
y	1	3	5	7	9

A) 25 B) 22 C) 28 D) 20

17) From the given data, $\sum xy =$ ……..

x	0	1	2	3	4
y	1	3	5	7	9

A) 12 B) 16 C) 70 D) 71

18) From the given data, $\sum x^2 =$ ……..

x	0	1	2	3	4
y	1	3	5	7	9

A) 12 B) 30 C) 70 D) 165

19) From the given data, $\sum y^2 =$ ……..

x	0	1	2	3	4
y	1	3	5	7	9

A) 12 B) 30 C) 70 D) 165

20) From the given data, $\sum x =$ …….

x	1	2	3	4	5
y	2	4	6	8	10

A) 9 B) 10 C) 15 D) 25

21) From the given data, $\sum y =$ …….

x	1	2	3	4	5
y	2	4	6	8	10

A) 25 B) 30 C) 28 D) 20

22) From the given data, $\sum xy =$ ……..

x	1	2	3	4	5
y	2	4	6	8	10

A) 100 B) 105 C) 110 D) 120

23) From the given data, $\sum x^2 =$ ……..

x	1	2	3	4	5
y	2	4	6	8	10

A) 12 B) 30 C) 70 D) 55

24) From the given data, $\sum y^2 =$ ……..

x	1	2	3	4	5
y	2	4	6	8	10

A) 120 B) 220 C) 100 D) 165

25) From the given data, $\sum x =$ …….

x	0.5	1.0	1.5	2.0	2.5	3.0
y	0.31	0.82	1.29	1.85	2.51	3.02

A) 9.5 B) 10.5 C) 11.5 D) 12.5

26) From the given data, $\sum y =$ …….

x	0.5	1.0	1.5	2.0	2.5	3.0
y	0.31	0.82	1.29	1.85	2.51	3.02

A) 9.80 B) 10.5 C) 22.75 D) 12.5

27) From the given data, $\sum x^2$ = …….

x	0.5	1.0	1.5	2.0	2.5	3.0
y	0.31	0.82	1.29	1.85	2.51	3.02

A) 9.80 B) 10.5 C) 22.75 D) 12.5

28) Parabola is a …… degree polynomial.

A) second B) first C) third D) forth

29) Parabola is a …… curve.

A) linear B) third degree C) non-linear D) None of these

30) To fit a given data (x_i, y_i), i = 1, 2, ……n, to a parabola $y = f(x) = a + bx + cx^2$, the error of approximation = …….

A) $y_i - a$ B) $a + bx_i + cx_i^2 - y_i$ C) $y_i - a - bx_i$ D) None of these

31) One of normal equation of fitting the curve $y = a + bx + cx^2$ is ……

A) $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$ B) $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$

C) $\sum_{i=1}^{n} y_i = a + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$ D) None of these

32) $\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$ is the one of normal equation of the fitting a curve……

A) $y = a + bx + cx^2$ B) $y = a + bx$ C) $y = ae^{bx}$ D) None of these

33) $y = ae^{bx}$ is the equation of …………

A) circle B) parabola C) ellipse D) an exponential curve.

34) $y = 2e^{3x}$ is the equation of …………

A) linear curve B) an exponential curve

C) logarithmic curve D) None of these

35) $I = I_0 e^{-\alpha t}$ is the equation of …………

A) an exponential curve B) straight line

C) logarithmic curve D) None of these

36) For fitting of an exponential curve of the type $y = ae^{bx}$ to a given data (x_i, y_i), i = 1, 2, ……n, first we reduce the curve into …… form.

A) non-linear B) linear C) logarithmic D) None of these

37) If an exponential curve ae^{bx} is reduced to linear form as Y = A + bx, where Y = ln y and A = ln a, then the normal equations are ……

A) $\sum Y = nA + b\sum x$ and $\sum xY = A\sum x + b\sum x^2$

B) $\sum Y = A + b\sum x$ and $\sum xY = nA + b\sum x^2$

C) $\sum Y = A\sum x + nb$ and $\sum x = A\sum x + b\sum Y$

D) None of these

38) $y = ax^b$ is the equation of …………

A) an exponential curve B) logarithmic curve

C) straight line D) parabola

39) Logarithmic curve is also called …………

A) quadratic curve B) power curve C) straight line D) circle

40) $y = 5x^4$ is the equation of …………

A) circle B) quadratic curve C) straight line D) logarithmic curve

41) For fitting of logarithmic curve of the type $y = ax^b$ to a given data (x_i, y_i), i = 1, 2, ……n, first we reduce the curve into …… form.

A) non-linear B) linear C) exponential D) None of these

42) If Y = ln y, X = ln x and A = ln a, then after reducing logarithmic curve ax^b to linear form Y = A + bx, the normal equations are ……

A) $\sum Y = A + b\sum x$ and $\sum xY = nA + b\sum x^2$

B) $\sum Y = nA + b\sum X$ and $\sum XY = A\sum X + b\sum X^2$

C) $\sum Y = A\sum X + nb$ and $\sum X = A\sum X + b\sum Y$

D) None of these

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UNIT-4: NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Initial Value Problem: First order differential equation $\frac{dy}{dx}$ = f(x, y) with initial condition $y(x_0) = y_0$ is called initial value problem.

Boundary Value Problem: Second or higher order differential equation with conditions at two or more points is called boundary value problem.

Remark: i) Initial value problem may have one solution or more than one solution or no solution.

ii) General solution of differential equation of order n contains n arbitrary constants.

==

Taylor's Series Method: Consider a differential equation with initial condition

$$\frac{dy}{dx} = f(x, y) \;\ldots\ldots\; \text{(i)}$$

$$y(x_o) = y_o \;\ldots\ldots\; \text{(ii)}$$

If y = f(x) is an exact solution of equation (i), then Taylor's series expansion of y(x) about a point $x = x_0$ is

$$y(x) = y_o + (x-x_o)\, y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots\ldots \;\text{(iii)}$$

If the values of y_o, y_0', y_0'', …… are known, then (iii) gives the power series solution of (i).

Derivatives are obtained in following manner,

$y' = f(x, y)$

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\,\frac{\partial f}{\partial y}$$

again differentiating, we get y''' and so on.

==

Remark: Taylor's series method is not very useful to solve practical problems because it contains higher order derivatives of y(x).

==

Ex. By using Taylor's series method, find y(0.1) correct to four decimal places if $\frac{dy}{dx} = x - y^2$ and y(0) = 1.

Solution: Let $\frac{dy}{dx} = x - y^2$ and y(0) = 1 be the given initial value problem.

Here $x_0 = 0$ and $y_0 = 1$.

Taylor's series expansion of y(x) about a point x = 0 is

$$y(x) = y_o + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{(iv)} + \ldots\ldots \text{(i)}$$

Now we find y_o, y_0', y_0'', …… as follows.

As $\frac{dy}{dx} = x - y^2$ i.e. $y' = x - y^2$ $\therefore y_0' = x_o - y_0^2 = 0 - 1^2 = -1$

$y'' = 1 - 2yy'$ $\therefore y_0'' = 1 - 2y_o y_0' = 1 - 2(1)(-1) = 3$

$y''' = -2(y')^2 - 2yy''$ $\therefore y_0''' = -2(y_0')^2 - 2y_o y_0'' = -2(-1)^2 - 2(1)(3) = -8$

$y^{(iv)} = -4y'y'' - 2y'y'' - 2yy''' = -6y'y'' - 2yy'''$

$\therefore y_0^{(iv)} = -6y_0'y_0'' - 2y_o y_0''' = -6(-1)(3) - 2(1)(-8) = 18 + 16 = 34$

and so on. Putting these values in (i), we get,

$$y(x) = 1 + x(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \frac{x^4}{4!}(34) + \ldots\ldots$$

i.e. $y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + \ldots\ldots$

$\therefore y(0.1) = 1 - 0.1 + \frac{3}{2}(0.01) - \frac{4}{3}(0.001) + \frac{17}{12}(0.0001) + \ldots\ldots$

i.e. $y(0.1) \approx 0.9138$

==

Ex. Solve $\frac{dy}{dx} = x^2 - y$ and y(0) = 1 at x = 0.1 and x = 0.2, by using Taylor's series method.

Solution: Let $\frac{dy}{dx} = x^2 - y$ and y(0) = 1 be the given initial value problem.

Here $x_0 = 0$ and $y_0 = 1$.

Taylor's series expansion of y(x) about a point x = 0 is

$$y(x) = y_o + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{(iv)} + \ldots\ldots \text{(i)}$$

Now we find y_o, y_0', y_0'', …… as follows.

As $\frac{dy}{dx} = x^2 - y$ i.e. $y' = x^2 - y$ $\therefore y_0' = x_0^2 - y_o = 0 - 1 = -1$

$y'' = 2x - y'$ $\therefore y_0'' = 2x_0 - y_0' = 0 - (-1) = 1$

$y''' = 2 - y''$ $\therefore y_0''' = 2 - y_0'' = 2 - 1 = 1$

$y^{(iv)} = -y'''$ $\quad \therefore y_0^{(iv)} = -y_0''' = -1$

and so on. Putting these values in (i), we get,

$y(x) = 1 + x(-1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(-1) + \ldots\ldots$

i.e. $y(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \ldots\ldots$

$\therefore y(0.1) = 1 - 0.1 + \frac{1}{2}(0.01) + \frac{1}{6}(0.001) - \frac{1}{24}(0.0001) + \ldots\ldots$

i.e. $y(0.1) \approx 0.9052$

& $\therefore y(0.2) = 1 - 0.2 + \frac{1}{2}(0.04) + \frac{1}{6}(0.008) - \frac{1}{24}(0.0016) + \ldots\ldots$

i.e. $y(0.1) \approx 0.8212$

==

Ex. Use Taylor's series method to obtain y at x = 0.2 if $\frac{dy}{dx} = 2y + 3e^x$, y (0) = 0.

Solution: Let $\frac{dy}{dx} = 2y + 3e^x$, y(0) = 0 be the given initial value problem.

Here $x_0 = 0$ and $y_0 = 0$.

Taylor's series expansion of y(x) about a point x = 0 is

$y(x) = y_o + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{(iv)} + \ldots\ldots$ (i)

Now we find $y_o, y_0', y_0'', \ldots\ldots$ as follows.

As $\frac{dy}{dx} = 2y + 3e^x$ i.e. $y' = 2y + 3e^x$ $\quad \therefore y_0' = 2y_o + 3e^{x_0} = 2(0) + 3e^0 = 0 + 3 = 3$

$y'' = 2y' + 3e^x$ $\quad \therefore y_0'' = 2y_0' + 3e^{x_0} = 2(3) + 3e^0 = 9$

$y''' = 2y'' + 3e^x$ $\quad \therefore y_0''' = 2y_0'' + 3e^{x_0} = 2(9) + 3e^0 = 21$

$y^{(iv)} = 2y''' + 3e^x$ $\quad \therefore y_0^{(iv)} = 2y_0''' + 3e^{x_0} = 2(21) + 3e^0 = 45$

and so on. Putting these values in (i), we get,

$y(x) = 0 + x(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45) + \ldots\ldots$

i.e. $y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots\ldots$

$\therefore y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \ldots\ldots$

i.e. $y(0.2) \approx 0.8110$

==

Ex. Using Taylor's series method, find the solution of an initial value problem $\frac{dy}{dx} = x + y$, $y(1) = 0$ at $x = 1.1$ with $h = 0.1$

Solution: Let $\frac{dy}{dx} = x + y$, $y(1) = 0$ be the given initial value problem.

Here $x_0 = 1$ and $y_0 = 0$.

Taylor's series expansion of y(x) about a point x = 1 is

$$y(x) = y_o + (x-1)y_0' + \frac{(x-1)^2}{2!} y_0'' + \frac{(x-1)^3}{3!} y_0''' + \frac{(x-1)^4}{4!} y_0^{(iv)} + \ldots\ldots \text{(i)}$$

Now we find $y_o, y_0', y_0'', \ldots\ldots$ as follows.

As $\frac{dy}{dx} = x + y$ i.e. $y' = x + y$ $\quad \therefore y_0' = x_o + y_o = 1 + 0 = 1$

$y'' = 1 + y'$ $\quad \therefore y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y''$ $\quad \therefore y_0''' = y_0'' = 2$

$y^{(iv)} = y'''$ $\quad \therefore y_0^{(iv)} = y_0''' = 2$

and so on. Putting these values in (i), we get,

$$y(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(2) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(2) + \ldots\ldots$$

i.e. $y(x) = (x-1) + (x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 + \ldots\ldots$

$\therefore y(1.1) = (0.1) + (0.1)^2 + \frac{1}{3}(0.1)^3 + \frac{1}{12}(0.1)^4 + \ldots\ldots$

i.e. $y(1.1) \approx 0.1103$

==

Ex. Using Taylor's series method, find the value of y a x = 1.02 given that $\frac{dy}{dx} = xy - 1$, $y(1) = 2$.

Solution: Let $\frac{dy}{dx} = xy - 1$, $y(1) = 2$ be the given initial value problem.

Here $x_0 = 1$ and $y_0 = 2$.

Taylor's series expansion of y(x) about a point x = 1 is

$$y(x) = y_o + (x-1)y_0' + \frac{(x-1)^2}{2!} y_0'' + \frac{(x-1)^3}{3!} y_0''' + \frac{(x-1)^4}{4!} y_0^{(iv)} + \ldots\ldots \text{(i)}$$

Now we find $y_o, y_0', y_0'', \ldots\ldots$ as follows.

As $\frac{dy}{dx} = xy - 1$ $\quad \therefore y' = xy - 1$ $\quad \therefore y_0' = x_o y_o - 1 = 1(2) - 1 = 1$

$y'' = y + xy'$ $\quad \therefore y_0'' = y_o + x_o y_0' = 2 + 1(1) = 3$

$y''' = y' + y' + xy'' = 2y' + xy''$ $\therefore y_0''' = 2y_0' + x_o y_0'' = 2(1) + 1(3) = 5$

$y^{(iv)} = 2y'' + y'' + xy''' = 3y'' + xy'''$ $\therefore y_0^{(iv)} = 3y_0'' + x_o y_0''' = 3(3)+1(5) = 14$

and so on. Putting these values in (i), we get,

$y(x) = 2 + (x-1)(1) + \frac{(x-1)^2}{2!}(3) + \frac{(x-1)^3}{3!}(5) + \frac{(x-1)^4}{4!}(14) + \ldots\ldots$

i.e. $y(x) = 2 + (x-1) + \frac{3}{2}(x-1)^2 + \frac{5}{6}(x-1)^3 + \frac{7}{12}(x-1)^4 + \ldots\ldots$

$\therefore y(1.02) = 2 + (0.02) + \frac{3}{2}(0.02)^2 + \frac{5}{6}(0.02)^3 + \frac{7}{12}(0.02)^4 + \ldots\ldots$

i.e. $y(1.02) \approx 2.0206$

==

Ex. Using Taylor's series method, find the value of y at x = 0.1 given that if $\frac{dy}{dx} = 3x + y^2$ and y(0) = 1. Take h = 0.1

Solution: Let $\frac{dy}{dx} = 3x + y^2$ and y(0) = 1 be the given initial value problem.

Here $x_0 = 0$ and $y_0 = 1$.

Taylor's series expansion of y(x) about a point x = 0 is

$y(x) = y_o + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{(iv)} + \ldots\ldots$ (i)

Now we find $y_o, y_0', y_0'', \ldots\ldots$ as follows.

As $\frac{dy}{dx} = 3x + y^2$ i.e. $y' = 3x + y^2$ $\therefore y_0' = 3x_o + y_0^2 = 0 + 1^2 = 1$

$y'' = 3 + 2yy'$ $\therefore y_0'' = 3 + 2y_o y_0' = 3 + 2(1)(1) = 5$

$y''' = 2(y')^2 + 2yy''$ $\therefore y_0''' = 2(y_0')^2 + 2y_o y_0'' = 2(1)^2 + 2(1)(5) = 12$

$y^{(iv)} = 4y'y'' + 2y'y'' + 2yy''' = 6y'y'' + 2yy'''$

$\therefore y_0^{(iv)} = 6y_0'y_0'' + 2y_o y_0''' = 6(1)(5) + 2(1)(12) = 30 + 24 = 54$

and so on. Putting these values in (i), we get,

$y(x) = 1 + x(1) + \frac{x^2}{2!}(5) + \frac{x^3}{3!}(12) + \frac{x^4}{4!}(54) + \ldots\ldots$

i.e. $y(x) = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4 + \ldots\ldots$

$\therefore y(0.1) = 1 + 0.1 + \frac{5}{2}(0.01) + 2(0.001) + \frac{9}{4}(0.0001) + \ldots\ldots$

i.e. $y(0.1) \approx 1.1272$

==

Euler's Method: Consider a differential equation with initial condition

$\frac{dy}{dx} = f(x, y)$ …… (i)

$y(x_o) = y_o$ …… (ii)

Let $x = x_r = x_0 + rh$, where r = 1, 2, 3, …… and suppose y = f(x) be the solution of equation (i).

Consider the Taylor's series expansion of $y(x_0 + h)$ about a point $x = x_0$ as

$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \ldots\ldots$ (iii)

For very small value of h, we neglect h^2 and higher order terms, we get,

$y(x_0 + h) = y(x_0) + hy'(x_0)$

i.e. $y_1 = y_0 + hf(x_0, y_0)$

Repeated application of above formula gives

$y_2 = y_1 + hf(x_1, y_1)$

$y_3 = y_2 + hf(x_2, y_2)$

and so on, ingeneral,

$y_{n+1} = y_n + hf(x_n, y_n)$

Which is called the Euler's formula.

==

Remark: Euler's method is not useful to solve practical problems because process is very slow to obtain reasonable accuracy as we have to take h very small.

==

Ex. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition y = 1 at x = 0, find y approximately at x = 0.1 in five steps, using Euler's method.

Solution: Let $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition y = 1 at x = 0.

Here $x_0 = 0$ and $y_0 = 1$, to find y at x = 0.1 in five steps we take

$h = \frac{x-x_0}{5} = \frac{0.1-0}{5} = 0.02$ i.e. we have to find y at x = 0.02, 0.04, 0.06, 0.08 and 0.1 by using Euler's formula $y_{n+1} = y_n + hf(x_n, y_n)$ as

$y_1 = y(0.02) = y_0 + hf(x_0, y_0) = 1 + (0.02)\left(\frac{1-0}{1+0}\right) = 1.02$

$y_2 = y(0.04) = y_1 + hf(x_1, y_1) = 1.02 + (0.02)\left(\frac{1.02-0.02}{1.02+0.02}\right) = 1.0392$

$y_3 = y(0.06) = y_2 + hf(x_2, y_2) = 1.0392 + (0.02)\left(\frac{1.0392-0.04}{1.0392+0.04}\right) = 1.0577$

$y_4 = y(0.08) = y_3 + hf(x_3, y_3) = 1.0577 + (0.02)\left(\frac{1.0577-0.06}{1.0577+0.06}\right) = 1.0738$

& $y_5 = y(0.1) = y_4 + hf(x_4, y_4) = 1.0738 + (0.02)\left(\frac{1.0738-0.08}{1.0738+0.08}\right) = 1.0910$

i.e. value of y at x = 0.1 is 1.0910.

==

Ex. Given $\frac{dy}{dx} = -y$ with initial condition $y = 1$ at $x = 0$, find y(0.04) with h = 0.01, using Euler's method.

Solution: Let $\frac{dy}{dx} = -y$ with initial condition $y = 1$ at $x = 0$.

Here $x_0 = 0$ and $y_0 = 1$, to find y(0.04) with h = 0.01

i.e. we have to find y at x = 0.01, 0.02, 0.03 and 0.04

by using Euler's formula $y_{n+1} = y_n + hf(x_n, y_n)$ as

$y_1 = y(0.01) = y_0 + hf(x_0, y_0) = 1 + (0.01)(-1) = 0.99$

$y_2 = y(0.02) = y_1 + hf(x_1, y_1) = 0.99 + (0.01)(-0.99) = 0.9801$

$y_3 = y(0.03) = y_2 + hf(x_2, y_2) = 0.9801 + (0.01)(-0.9801) = 0.9703$

& $y_4 = y(0.04) = y_3 + hf(x_3, y_3) = 0.9703 + (0.01)(-0.9703) = 0.9606$

i.e. value of y at x = 0.04 is 0.9606.

==

Ex. Apply Euler's method to find the initial value problem $\frac{dy}{dx} = x + y$, y = 0 when x = 0, find y at x = 1.0 taking h = 0.2 .

Solution: Let $\frac{dy}{dx} = x + y$ with y = 0 at x = 0.

Here $x_0 = 0$ and $y_0 = 0$, to find y at x = 1.0 with h = 0.2

i.e. we have to find y at x = 0.2, 0.4, 0.6, 0.8 and 1.0

by using Euler's formula $y_{n+1} = y_n + hf(x_n, y_n)$ as

$y_1 = y(0.2) = y_0 + hf(x_0, y_0) = 0 + (0.2)(0 + 0) = 0$

$y_2 = y(0.4) = y_1 + hf(x_1, y_1) = 0 + (0.2)(0.2 + 0) = 0.04$

$y_3 = y(0.6) = y_2 + hf(x_2, y_2) = 0.04 + (0.2)(0.4 + 0.04) = 0.128$

$y_4 = y(0.8) = y_3 + hf(x_3, y_3) = 0.128 + (0.2)(0.6 + 0.128) = 0.2736$

& $y_5 = y(1.0) = y_4 + hf(x_4, y_4) = = 0.2736 + (0.2)(0.8 + 0.2736) = 0.48832$

i.e. value of y at x = 1.0 is 0.48832.

==

Modified Euler's Method: Consider a differential equation

$\frac{dy}{dx}$ = f(x, y) …… (i)

$y(x_o) = y_o$ …… (ii)

From (i), we have dy = f(x, y)dx

$\therefore \int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x,y)dx$

$\therefore y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x,y)dx$

$\therefore y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x,y)dx$

$\therefore y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$ by Trapezoidal Rule, where $h = x_1 - x_0$

By taking alternative procedure as,

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$ where $y_1^{(0)} = y_0 + hf(x_0, y_0)$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

and so on, ingeneral,

$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})]$

which is called the modified Euler's formula.

Where n = 0, 1, 2, 3, …. and $y_1^{(n)}$ is the n^{th} approximation to y_1.

==

Remark: We stop modified Euler's method when accuracy $\left|y_1^{(n+1)} - y_1^{(n)}\right| < \varepsilon$ (error tolerance) is obtained. We repeat the computation to find the approximations $y(x_0 + 2h) \approx y_2$, $y(x_0 + 3h) \approx y_3$, etc.

==

Ex. Solve $\frac{dy}{dx} = 1 - y$, y(0) = 0, by using modified Euler's method to find y(0.1) correct upto four decimal places.

Solution: Let $\frac{dy}{dx} = 1 - y$ with y(0) = 0.

Here $x_0 = 0$ and $y_0 = 0$, to find y(0.1) with $h = x - x_0 = 0.1 – 0 = 0.1$

by using modified Euler's formula $y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})]$ as

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

where $y_1^{(0)} = y_0 + hf(x_0, y_0) = 0 + (0.1)(1 - 0) = 0.1$

$\therefore y_1^{(1)} = 0 + \frac{(0.1)}{2}[(1- 0) + (1 - 0.1)] = 0.095$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = 0 + \frac{(0.1)}{2}[(1- 0) + (1 - 0.095)] = 0.09525$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})] = 0 + \frac{(0.1)}{2}[(1- 0) + (1-0.09525)] = 0.0952375$

Now $\left|y_1^{(3)} - y_1^{(2)}\right| < 0.000125$

$\therefore y_1 \approx y_1^{(3)} = 0.0952375$

$\therefore y_1 = y(0.1) \approx 0.0952$ is correct upto four decimal places.

==

Ex. Find the value of y when x = 0.05 correct to four decimals, given that y(0) = 1, and $y' = x^2 + y$, using modified Euler's method. Take h = 0.05.

Solution: Let $y' = x^2 + y$ with y(0) = 1.

Here $x_0 = 0$ and $y_0 = 1$, to find y(0.05) with h = 0.05

by using modified Euler's formula $y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})]$ as

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

where $f(x_0, y_0) = x_0^2 + y_0 = 0^2 + 1 = 1$, $x_1 = x_0 + h = 0 + 0.05 = 0.05$ and

$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.05)(1) = 1.05$

$\therefore y_1^{(1)} = 1 + \frac{(0.05)}{2}[1 + (0.05)^2 + 1.05] = 1.0513125$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{(0.05)}{2}[1 + (0.05)^2 + 1.0513125] = 1.0513453$

Now $\left|y_1^{(2)} - y_1^{(1)}\right| < 0.0000328$

$\therefore y_1 \approx y_1^{(2)} = 1.0513453$

$\therefore y_1 = y(0.05) \approx 1.0513$ is correct upto four decimals.

==

Ex. Given y′ = x + y, y(1) = 1. by using modified Euler's method, find y at x = 1.2 correct to 4 decimals.

Solution: Let y′ = x + y, y(1) = 1.

Here x_0 = 1 and y_0 = 1, to find y at x = 1.2 with h = x - x_0 = 1.2 – 1 = 0.2

by using modified Euler's formula $y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})]$ as

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

where $f(x_0, y_0) = x_0 + y_0 = 1 + 1 = 2$, $x_1 = x_0 + h = 1 + 0.2 = 1.2$ and

$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.2)(2) = 1.4$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$\therefore y_1^{(1)} = 1 + \frac{(0.2)}{2}[2 + (1.2 + 1.4)] = 1.46$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{(0.2)}{2}[2 + (1.2 + 1.46)] = 1.466$

$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + \frac{(0.2)}{2}[2 + (1.2 + 1.466)] = 1.4666$

$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$

$= 1 + \frac{(0.2)}{2}[2 + (1.2 + 1.4666)] = 1.46666$

Now $\left|y_1^{(4)} - y_1^{(3)}\right| < 0.00006$

$\therefore y_1 \approx y_1^{(4)} = 1.46666$

$\therefore y_1 = y(1.2) \approx 1.4666$ is correct upto 4 decimals.

==

Ex. Use modified Euler's method to solve $y' = \frac{y}{2}$, y(0) = 1, find y at x = 0.1 correct to 4 decimals.

Solution: Let $y' = \frac{y}{2}$, y(0) = 1.

Here x_0 = 0 and y_0 = 1, to find y at x = 0.1 with h = x - x_0 = 0.1 – 0 = 0.1

by using modified Euler's formula $y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})]$ as

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

where $f(x_0, y_0) = \frac{y_0}{2} = \frac{1}{2} = 0.5$, $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and

$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.1)(0.5) = 1.05$

$\therefore y_1^{(1)} = 1 + \frac{(0.1)}{2}[\ 0.5 + (\frac{1.05}{2})] = 1.05125$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{(0.1)}{2}[0.5 + (\frac{1.05125}{2})] = 1.05128$

Now $\left|y_1^{(2)} - y_1^{(1)}\right| < 0.00003$

$\therefore y_1 \approx y_1^{(2)} = 1.05128$

$\therefore y_1 = y(0.1) \approx 1.0512$ is correct upto 4 decimals.

==

Runge-Kutta Second Order Method:

Consider the formula

$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$ …… (1)

Now replace y_1 by $y_1 = y_0 + hf(x_0, y_0)$ on the right hand side, we get,

$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))]$

i.e. $y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + hf(x_1, y_0 + hf(x_0, y_0))]$

By setting $k_1 = hf(x_o , y_0)$ and $k_2 = hf(x_1, y_0 + k_1)$, we get,

$y_1 = y_o + \frac{1}{2}[k_1 + k_2]$

Repeating the process, we get,

$y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$y_3 = y_2 + \frac{1}{2}[k_1 + k_2]$

and so on, ingeneral,

$y_{n+1} = y_n + \frac{1}{2}[k_1 + k_2]$

where $k_1 = hf(x_n , y_n)$ and $k_2 = hf(x_{n+1}, y_n + k_1)$.

==

Remark: Runge Kutta methods are more accurate and having great practical importance. These methods are widely used to find the numerical solutions of linear and non-linear ordinary differential equations.

==

Ex. Given $\frac{dy}{dx} = y - x$ with y(0) = 2, find y(0.1) and y(0.2) using Runge-Kutta second order method with h = 0.1

Solution: Given $\frac{dy}{dx} = f(x, y) = y - x$ with y(0) = 2 i.e. $x_0 = 0$ and $y_0 = 2$

To find $y(0.1) \approx y_1$ by using Runge-Kutta second order method, we have the data

$x_0 = 0$, $y_0 = 2$, h = 0.1 and $x_1 = 0.1$

$\therefore k_1 = hf(x_0, y_0) = (0.1)(y_0 - x_0) = (0.1)(2 - 0) = 0.2$ and

$k_2 = hf(x_1, y_0 + k_1) = h(y_0 + k_1 - x_1) = (0.1)(2 + 0.2 - 0.1) = 0.21$

$\therefore y_1 = y_o + \frac{1}{2}[k_1 + k_2] = 2 + \frac{1}{2}[0.2 + 0.21] = 2.205$

i.e. $y(0.1) \approx y_1 = 2.205$

Again to find $y(0.2) \approx y_2$ by using Runge-Kutta second order method, we have the data $x_1 = 0.1$, $y_1 = 2.205$, h = 0.1 and $x_2 = 0.2$

$\therefore k_1 = hf(x_1, y_1) = (0.1)(y_1 - x_1) = (0.1)(2.205 - 0.1) = 0.2105$ and

$k_2 = hf(x_2, y_1 + k_1) = h(y_1 + k_1 - x_2) = (0.1)(2.205 + 0.2105 - 0.2) = 0.22155$

$\therefore y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 2.205 + \frac{1}{2}[0.2105 + 0.22155] = 2.421025$

i.e. $y(0.2) \approx y_2 = 2.421$

==

Ex. Use Runge-Kutta method to find y when x = 1.2 in steps of 0.1 given that $\frac{dy}{dx} = x^2 + y^2$ and y(1) = 1.5

Solution: Given $\frac{dy}{dx} = f(x, y) = x^2 + y^2$ and y(1) = 1.5 i.e. $x_0 = 1$ and $y_0 = 1.5$

We have to find y when x = 1.2 in steps of 0.1 i.e. h = 0.1, $x_1 = 1.1$ and $x_2 = 1.2$

To find $y(1.1) \approx y_1$ by using Runge-Kutta second order method, we have the data

$x_0 = 1$, $y_0 = 1.5$, h = 0.1 and $x_1 = 1.1$

$\therefore k_1 = hf(x_0, y_0) = (0.1)(x_0^2 + y_0^2) = (0.1)(1 + 2.25) = 0.325$ and

$k_2 = hf(x_1, y_0 + k_1) = h[x_1^2 + (y_0 + k_1)^2]$

$= (0.1)[(1.1)^2 + (1.5 + 0.325)^2] = 0.4540625$

$\therefore y_1 = y_o + \frac{1}{2}[k_1 + k_2] = 1.5 + \frac{1}{2}[0.325 + 0.4540625] = 1.8895$

i.e. $y(0.1) \approx y_1 = 1.8895$

Again to find $y(1.2) \approx y_2$ by using Runge-Kutta second order method, we have the data $x_1 = 1.1$, $y_1 = 1.8895$, h = 0.1 and $x_2 = 1.2$

$\therefore k_1 = hf(x_1, y_1) = (0.1)(x_1^2+y_1^2) = (0.1)(1.21 + 3.5702) = 0.4780$ and

$k_2 = hf(x_2, y_1 + k_1) = h[x_2^2+(y_1 + k_1)^2]$

$= (0.1)[(1.2)^2+(1.8895 + 0.4780)^2] = 0.7045$

$\therefore y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 1.8895+ \frac{1}{2}[0.4780 + 0.7045] = 2.4808$

i.e. $y(0.2) \approx y_2 = 2.4808$

==

Ex. Use Runge-Kutta method to find y when x = 0.5 and 1. Given that $\frac{dy}{dx} = \frac{y}{2}$ and y(0) = 1

Solution: Given $\frac{dy}{dx} = f(x, y) = \frac{y}{2}$ and y(0) = 1 i.e. $x_0 = 0$ and $y_0 = 1$

We have to find y when x = 0.5 and 1 i.e. h = 0.5, $x_1 = 0.5$ and $x_2 = 1$

To find $y(0.5) \approx y_1$ by using Runge-Kutta second order method, we have the data

$x_0 = 0$, $y_0 = 1$, h = 0.5 and $x_1 = 0.5$

$\therefore k_1 = hf(x_0, y_0) = (0.5)(\frac{y_0}{2}) = (0.5)(\frac{1}{2}) = 0.25$ and

$k_2 = hf(x_1, y_0 + k_1) = h(\frac{y_0+k_1}{2}) = (0.5)(\frac{1+0.25}{2}) = 0.3125$

$\therefore y_1 = y_o + \frac{1}{2}[k_1 + k_2] = 1 + \frac{1}{2}[0.25+ 0.3125] = 1.28125$

i.e. $y(0.1) \approx y_1 = 1.28125$

Again to find $y(1) \approx y_2$ by using Runge-Kutta second order method, we have the data $x_1 = 0.5$, $y_1 = 1.28125$, h = 0.5 and $x_2 = 1$

$\therefore k_1 = hf(x_1, y_1) = (0.5)(\frac{y_1}{2}) = (0.5)(\frac{1.28125}{2}) = 0.3203$ and

$k_2 = hf(x_2, y_1 + k_1) = h(\frac{y_1+k_1}{2}) = (0.5)(\frac{1.28125+0.3203}{2}) = 0.4004$

$\therefore y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 1.28125+ \frac{1}{2}[0.3203 + 0.4004] = 1.6416$

i.e. $y(0.2) \approx y_2 = 1.6416$

==

Fourth Order Runge-Kutta Method: To find the solution of initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ by fourth order Runge-Kutta method.

The fourth order Runge-Kutta formula is

$y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$,

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = hf(x_0 + h, y_0 + k_3)$

Ingeneral $y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = hf(x_n + h, y_n + k_3)$

==

Remark: Runge Kutta fourth order method is more accurate than Euler's modified method. This method agrees with Taylor's series solution upto the term h^4.

==

Ex. Given $y' = x^2 - y$ with $y(0) = 1$, find $y(0.1)$ and $y(0.2)$ using Runge-Kutta method of fourth order.

Solution: Given $y' = f(x, y) = x^2 - y$ with $y(0) = 1$ i.e. $x_0 = 0$ and $y_0 = 1$

To find $y(0.1) \approx y_1$ by using Runge-Kutta method of fourth order, we have the data $x_0 = 0$, $y_0 = 1$, $h = 0.1$ and $x_1 = 0.1$

$\therefore k_1 = hf(x_0, y_0) = (0.1)(x_0^2 - y_0) = (0.1)(0 - 1) = -0.1$ and

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(0.05, 0.95) = (0.1)(0.0025 - 0.95) = -0.09475$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = hf(0.05, 0.952625)$

$= (0.1)(0.0025 - 0.952625) = -0.0950125$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0.1, 0.9049875)$

$= (0.1)(0.01 - 0.9049875) = -0.08949875$

$\therefore y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[-0.1 + 2(-0.09475) + 2(-0.0950125) - 0.08949875] = 0.9052$

i.e. $y(0.1) \approx y_1 = 0.9052$

Again to find $y(0.2) \approx y_2$ by using Runge-Kutta method of fourth order, we have the data $x_1 = 0.1$, $y_1 = 0.9052$, $h = 0.1$ and $x_2 = 0.2$

$\therefore$ $k_1 = hf(x_1, y_1) = (0.1)(x_1^2 - y_1) = (0.1)(0.01 - 0.9052) = -0.08952$ and

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = hf(0.15, 0.86044)$

$= (0.1)(0.0225 - 0.86044) = -0.083794$

$k_3 = hf(x_0 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = hf(0.15, 0.863303)$

$= (0.1)(0.0225 - 0.863303) = -0.0840803$

$k_4 = hf(x_0 + h, y_1 + k_3) = hf(0.2, 0.82112)$

$= (0.1)(0.04 - 0.82112) = -0.078112$

$\therefore$ $y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 0.9052 + \frac{1}{6}[-0.08952 + 2(-0.083794) + 2(-0.0840803) - 0.078112] = 0.8213$

i.e. $y(0.2) \approx y_2 = 0.8213$

===

Ex. Using Runge-Kutta fourth order method find y(0.1), given that $\frac{dy}{dx} = xy + y^2$ with y(0) = 1,

Solution: Given $\frac{dy}{dx} = f(x, y) = xy + y^2$ with y(0) = 1 i.e. $x_0 = 0$ and $y_0 = 1$

To find $y(0.1) \approx y_1$ by using Runge-Kutta fourth order method, we have the data

$x_0 = 0$, $y_0 = 1$, h = 0.1 and $x_1 = 0.1$

$\therefore$ $k_1 = hf(x_0, y_0) = (0.1)(x_0 y_0 + y_0^2) = (0.1)(0 + 1) = 0.1$ and

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(0.05, 1.05) = (0.1)[(0.05)(1.05) + (1.05)^2] = 0.1155$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = hf(0.05, 1.05775)$

$= (0.1)[(0.05)(1.10775) + (1.05775)^2] = 0.1171723$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0.1, 1.1171723)$

$= (0.1)[(0.1)(1.1171723) + (1.1171723)^2] = 0.1359791$

$\therefore$ $y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0.1 + 2(0.1155) + 2(0.1171723) + 0.1359791] = 1.1168873$

i.e. $y(0.1) \approx y_1 = 1.1168873$

===

Ex. Using Runge-Kutta fourth order method find y(0.1), given that $\frac{dy}{dx} = x + y^2$ with y(0) = 1,

Solution: Given $\frac{dy}{dx} = f(x, y) = x + y^2$ with y(0) = 1 i.e. $x_0 = 0$ and $y_0 = 1$

To find $y(0.1) \approx y_1$ by using Runge-Kutta fourth order method, we have the data

$x_0 = 0$, $y_0 = 1$, h = 0.1 and $x_1 = 0.1$

$\therefore k_1 = hf(x_0, y_0) = (0.1)(x_0 + y_0^2) = (0.1)(0 + 1) = 0.1$ and

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(0.05, 1.05) = (0.1)[(0.05) + (1.05)^2] = 0.11525$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = hf(0.05, 1.057625)$

$= (0.1)[(0.05) + (1.057625)^2] = 0.11685$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0.1, 1.11685)$

$= (0.1)[(0.1) + (1.11685)^2] = 0.134735$

$\therefore y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_2 + k_4]$

$= 1 + \frac{1}{6}[0.1 + 2(0.11525) + 2(0.11685) + 0.134735] = 1.1165$

i.e. $y(0.1) \approx y_1 = 1.1165$

==

Ex. Apply Runge-Kutta fourth order method, to find the approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

Solution: Given $\frac{dy}{dx} = f(x, y) = x + y$ and y = 1 when x = 0 i.e. $x_0 = 0$ and $y_0 = 1$

To find $y(0.2) \approx y_1$ by using Runge-Kutta fourth order method, we have the data

$x_0 = 0$, $y_0 = 1$, h = 0.2 and $x_1 = 0.2$

$\therefore k_1 = hf(x_0, y_0) = (0.2)(x_0 + y_0) = (0.2)(0 + 1) = 0.2$ and

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(0.1, 1.1) = (0.2)(0.1 + 1.1) = 0.24$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = hf(0.1, 1.12)$

$= (0.2)[(0.1 + 1.12) = 0.244$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0.2, 1.244)$

$= (0.2)[(0.2 + 1.244) = 0.2888$

$\therefore y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0.2 + 2(0.24) + 2(0.244) + 0.2888] = 1.2428$

i.e. $y(0.1) \approx y_1 = 1.2428$

==

Ex. Using Runge-Kutta fourth order method obtain y when x = 1.1, given that y = 1.2 when x = 1 and y satisfies the equation $\frac{dy}{dx} = 3x + y^2$.

Solution: Given $\frac{dy}{dx} = f(x, y) = 3x + y^2$ with y = 1.2 when x = 1 i.e. $x_0 = 1$ and $y_0 = 1.2$

To find $y(1.1) \approx y_1$ by using Runge-Kutta fourth order method, we have the data

$x_0 = 1$, $y_0 = 1.2$, h = 0.1 and $x_1 = 1.1$

$\therefore k_1 = hf(x_0, y_0) = hf(1, 1.2) = (0.1)[3(1) + (1.2)^2]\,) = (0.1)(3 + 1.44) = 0.444$ and

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = hf(1.05, 1.422) = (0.1)[3(1.05) + (1.422)^2] = 0.5172$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = hf(1.05, 1.4586)$

$= (0.1)[3(1.05) + (1.4586)^2] = 0.5278$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(1.1, 1.7278)$

$= (0.1)[3(1.1) + (1.7278)^2] = 0.6285$

$\therefore y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_2 + k_4]$

$= 1.2 + \frac{1}{6}[\,0.444 + 2(0.5172) + 2(0.5278) + 0.6285] = 1.72708$

i.e. $y(0.1) \approx y_1 = 1.72708$

==

MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) First order differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_o) = y_o$ is known as ……

A) exact equation B) initial value problem

C) linear equation D) boundary value problem

2) $\frac{dy}{dx} = f(x, y)$ with ………… is known as initial value problem.

A) $y(x_1) = y_o$ B) $y(x_o) = y_1$ C) $y(x_o) = y_o$ D) $y(x_1) = y_1$

3) Second or higher order differential equation with conditions at two or more points is called ……

A) boundary value problem B) initial value problem

C) linear equation D) None of these

4) General solution of differential equation of order n contains … arbitrary constants.

A) n-1 B) n C) n+1 D) 0

5) Taylor's series expansion of y(x) about a point $x = x_0$ is ……

A) $y(x) = (x-x_o)y'_o + (x-x_o)^2/2!y''_o + \ldots$ B) $y(x) = y_o + (x_o)y'_o + (x_o)^2/2!\ y''_o + \ldots$

C) $y(x) = y_o + (x-x_o)y'_o + (x-x_o)^2/2!\ y''_o + \ldots$ D) None of these

6) $y(x) = y_o + (x-x_o)y'_o + (x-x_o)^2/2!\ y''_o + \ldots\ldots$is the ……….series for y(x).

A) Taylor's B) Maclaurin's C) Laurents's D) None of these

7) In Taylors series if we take $x_0 = 0$, then it said to be …… series.

A) Cauchy's B) Maclaurin's C) Laurent's D) None of these

8) If $y' = x + y$, then $y'' = \ldots\ldots\ldots\ldots$

A) 1 B) $1 + y'$ C) $x + y'$ D) $x + 1$

9) If $y' = 2x + y$, then $y'' = \ldots\ldots\ldots\ldots$

A) 2 B) $2 + y'$ C) $2x + y'$ D) $2x +1$

10) If $y' = 2y + 3e^x$, then $y'' = \ldots\ldots\ldots\ldots$

A) $2y + 3e^x$ B) $2y' + e^x$ C) $2y' + 3e^x$ D)$2y' + 9e^x$

11) If $y' = 3y + 5e^x$, then $y'' = \ldots\ldots\ldots\ldots$

A) $3y + 5e^x$ B) $3y' + 5e^x$ C) $3y' + e^x$ D) $3y' + 25e^x$

12) If $y'' = 1 + y'$, then $y''' = \ldots\ldots\ldots\ldots$

A) 0 B) 1 C) y' D) y''

13) If $y'' = 10 + y'$, then $y''' = \ldots\ldots\ldots\ldots$

A) 0 B) y'' C) y' D) 10

14) If $\frac{dy}{dx} = x - y^2$ and y(0) = 1, then $y_0' = \ldots\ldots\ldots\ldots$

A) -1 B) 1 C) 2 D) -2

15) If $\frac{dy}{dx} = x - y^2$ and y(0) = 1, then $y_0'' = \ldots\ldots\ldots\ldots$

A) -1 B) 1 C) 3 D) -3

16) If $\frac{dy}{dx} = x + y$ and y(1) = 0, then $y_0' = \ldots\ldots\ldots\ldots$

A) -1 B) 1 C) 2 D) -2

17) If $\frac{dy}{dx} = x + y$ and y(1) = 0, then $y_0'' = \ldots\ldots\ldots\ldots$

A) -1 B) 1 C) 3 D) 2

18) If $\frac{dy}{dx} = x + y$ and y(1) = 0 , then $y_0''' = \ldots\ldots\ldots\ldots$

A) -1 B) 1 C) 3 D) 2

19) If $\frac{dy}{dx} = x^2 - y$ and y(0) = 1, then $y_0' = \ldots\ldots\ldots\ldots$

A) -1 B) 1 C) 2 D) -2

20) If $\frac{dy}{dx}$ = $x^2 - y$ and y(0) = 1, then y_0'' = …………

A) -1 B) 1 C) 2 D) -3

21) If $\frac{dy}{dx}$ = $2y + 3e^x$ and y(0) = 0, then y_0' = …………

A) -1 B) 1 C) 2 D) 3

22) If $\frac{dy}{dx}$ = $2y + 3e^x$ and y(0) = 0, then y_0'' = …………

A) -1 B) 9 C) 2 D) 0

23) If $\frac{dy}{dx}$ = xy - 1 and y(1) = 2, then y_0' = …………

A) -1 B) 1 C) 2 D) 3

24) If $\frac{dy}{dx}$ = xy - 1 and y(1) = 2, then y_0'' = …………

A) 0 B) 1 C) 3 D) 2

25) If $\frac{dy}{dx}$ = $3x + y^2$ and y(0) = 1, then y_0' = …………

A) -1 B) 1 C) 2 D) 3

26) If $\frac{dy}{dx}$ = $3x + y^2$ and y(0) = 1, then y_0'' = …………

A) 5 B) 1 C) 2 D) 3

27) By Eulers formula y_{n+1} = ………

A) $y_n + hf(x_n, y_n)$ B) $y_o + hf(x_n, y_n)$ C) $y_n + hf(x_o, y_n)$ D) $y_n + hf(x_o, y_o)$

28) For the initial value problem y' = x + y, y = 0 when x = 0. For h = 0.2 by Eulers formula the value of y_1 = ……

A) -1 B) 0 C) 0.2 D) None of these

29) For the initial value problem y' = x + y, y = 0 when x = 0. For h = 0.2 by Eulers formula the value of y_2 = ……

A) 0 B) 0.02 C) 0.03 D) 0.04

30) For the initial value problem y' = x + y, y = 0 when x = 0. For h = 0.2 by Euler's formula the value of y_3 = ……

A) 0.12 B) 0.128 C) 0.1 D) 0.2

31) For the initial value problem y' = x + 2y, y = 0 when x = 0. For h = 0.2 by Euler's formula the value of y_1 = ……

A) -1 B) 0 C) 0.2 D) None of these

32) For the initial value problem y' = x + 2y, y = 0 when x = 0. For h = 0.2 by Euler's formula the value of y_2 = ……

A) 0 B) 0.02 C) 0.03 D) 0.04

33) For the initial value problem y' = x + 2y, y = 0 when x = 0. For h = 0.2 by Eulers formula the value of y_3 = ……

A) 0.12 B) 0.136 C) 0.1 D) 0.2

34) By Modified Euler's Method $y_1^{(n+1)}$ = ………

A) $y_o + h/2[f(x_o, y_o) + f(x_1, y_1^{(n)})]$ B) $y_o + h[f(x_o, y_o) + f(x_1, y_1^{(n)})]$

C) $y_o + h/3[f(x_o, y_o) + f(x_1, y_1^{(n)})]$ D) $y_o + h/4[f(x_o, y_o) + f(x_1, y_1^{(n)})]$

35) If y(0) = 1 and $y' = x^2 + y$ then $f(x_o, y_o)$ = ……. for h = 0.05

A) 0 B) 2 C) 1 D) None of these

36) If y(0) = 1 and $y' = x^2 + y$ then by modified Euler's method $y_1^{(0)}$ = …….

A) 1 B) 1.05 C) 1.1 D) 1.15

37) If y(0) = 1 and $y'= x^2 -4y$ then $f(x_o, y_o)$ = ……. for h = 0.05

A) 0 B) 2 C) -4 D) None of these

38) If y(0) = 1 and $y' = x^2 + y$ then by modified Euler's method $y_1^{(0)}$ = …….

A) 1 B) 1.05 C) 1.1 D) 1.15

39) By Runge-Kutta Second Order Method $y_1 = y_o + \frac{1}{2}[k_1 + k_2]$, where k_1 = ……….

A) $f(x_o, y_o)$ B) $hf(x_o, y_1)$ C) $hf(x_1, y_o)$ D) $hf(x_o, y_o)$

40) By Runge-Kutta Second Order Method $y_1 = y_o + \frac{1}{2}[k_1 + k_2]$, where k_2 = ……….

A) $hf(x_o + h, y_o + k_1)$ B) $f(x_o + h, y_1 + k_1)$

C) $hf(x_1 + h, y_o + k_1)$ D) $f(x_1 + h, y_1 + k_1)$

41) If $y' = x^2 + y^2$ with y(1) = 1.5 and h = 0.1 then by Runge-Kutta Second Order Method k_1 = ……….

A) 0.1 B) 1.5 C) 0.325 D) 0

42) If $y' = x^2 + y^2$ with y(1) = 1.5 and h = 0.1 then by Runge-Kutta Second Order Method k_2 = ……….

A) 0.4540625 B) 1.540625 C) 0.625 D) 0

43) If $\frac{dy}{dx} = y - x$ with y(0) = 2 and h = 0.1 then by Runge-Kutta Second Order Method k_1 = ……….

A) 0.1 B) 0.2 C) 0.3 D) 0.4

44) If $\frac{dy}{dx} = y - x$ with y(0) = 2 and h = 0.1 then by Runge-Kutta Second Order Method k_2 = ……….

A) 0.21 B) 0.22 C) 0.23 D) 0.24

45) If $\frac{dy}{dx} = \frac{y}{2}$ with y(0) = 1 and h = 0.5 then by Runge-Kutta Second Order Method k_1 = ……….

A) 0.15 B) 0.25 C) 0.35 D) 0.45

46) If $\frac{dy}{dx} = \frac{x}{y}$ with y(1) = 2 and h = 0.2 then by Runge-Kutta Second Order Method k_1 = ……….

A) 0.1 B) 0.2 C) 0.3 D) 0.4

47) Fourth order Runge-Kutta formula for the solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ is given by y_1 = …………

A) $y_0 + \frac{1}{2}(k_1 + 2k_2 + 2k_3 + k_4)$ B) $y_0 + \frac{1}{2}(k_1 + k_2 + k_3 + k_4)$

C) $y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ D) $y_0 + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$

48) In Fourth order Runge-Kutta formula k_1 = …………

A) $f(x_o, y_o)$ B) $hf(x_o, y_1)$ C) $hf(x_1, y_o)$ D) $hf(x_o, y_o)$

49) If $y' = x^2 - y$ with y(0) = 1 and h = 0.1, then by Runge-Kutta Fourth Order Method k_1 = ……….

A) 0.1 B) – 0.1 C) 0.2 D) – 0.2

50) If $\frac{dy}{dx} = xy + y^2$ with y(0) = 1 and h = 0.1, then by Runge-Kutta Fourth Order Method k_1 = ……….

A) 0.1 B) – 0.1 C) 0.2 D) – 0.2

51) If $\frac{dy}{dx} = x + y^2$ with y(0) = 1 and h = 0.1, then by Runge-Kutta Fourth Order Method k_1 = ……….

A) 0.1 B) – 0.1 C) 0.2 D) – 0.2

52) If $\frac{dy}{dx} = x + y$ with y = 1 when x = 0 and h = 0.2, then by Runge-Kutta Fourth Order Method k_1 = ……….

A) 0.1 B) – 0.1 C) 0.2 D) – 0.2

53) If $\frac{dy}{dx} = 3x + y^2$ with y = 1.2 when x = 1 and h = 0.1, then by Runge-Kutta Fourth Order Method k_1 = ……….

A) 0.14 B) 0.444 C) 0.222 D) 0.2

54) In Fourth order Runge-Kutta formula k_2 = …………

A) $hf(x_o + \frac{h}{2}, y_o + \frac{k_1}{2})$ B) $hf(x_o + \frac{h}{2}, y_o + \frac{k_2}{2})$

C) $hf(x_o + h, y_o + k_3)$ D) None of these

55) In Fourth order Runge-Kutta formula k_3 = …………

A) $hf(x_o + \frac{h}{2}, y_o + \frac{k_1}{2})$ B) $hf(x_o + \frac{h}{2}, y_o + \frac{k_2}{2})$

C) $hf(x_o + h, y_o + k_3)$ D) None of these

56) In Fourth order Runge-Kutta formula k_4 = …………

A) $hf(x_o + \frac{h}{2}, y_o + \frac{k_1}{2})$ B) $hf(x_o + \frac{h}{2}, y_o + \frac{k_2}{2})$

C) $hf(x_o + h, y_o + k_3)$ D) None of these

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."