Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: F.Y.B.SC SEM.-II

SUBJECT: MTH-201: ORDINARY DIFFERENTIAL EQUATIONS

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MTH 201: ORDINARY DIFFERENTIAL EQUATIONS

Unit-I Differential Equations of First Order and First Degree **No. of Hours: 8**

a) Partial derivatives of first order.

b) Exact differential equations. Condition for exactness.

c) Integrating factor.

d) Rules for finding integrating factors.

e) Linear differential equations.

f) Bernoulli's Equation. Equation reducible to linear form.

Unit-II Differential Equations of First Order and Higher Degree **No. of Hours: 7**

a) Differential equations of first order and higher degree.

b) Equation solvable for p.

c) Equation solvable for y.

d) Equation solvable for x.

e) Clairaut's form.

Unit-III Linear Differential Equations with Constant Coefficients **No. of Hours: 8**

a) Linear differential equations with constant coefficients.

b) Complementary functions.

c) Particular integrals of $f(D)y = X$, where $X = e^{ax}$, $\cos(ax)$, $\sin(ax)$, x^n , $e^{ax}V$, xV with usual notations.

Unit-IV Linear Differential Equations with Variable Coefficients **No. of Hours: 7**

a) Homogeneous linear differential equations (Cauchy's differential equations).

b) Example of Homogeneous linear differential equations.

c) Equations reducible to homogeneous linear differential equations (Legendre's equations)

d) Example of Equations reducible to homogeneous linear differential equations

Reference Books:

1. Introductory Course in Differential Equations, by D. A. Murray, Orient Congman (India) 1967.

2. Differential Equations, by G. F. Simmons, Tata McGraw Hill, 1972.

Learning Outcomes:

After successful completion of this course, the student will be able to:

a) understand basic concepts in differential equations

b) understand method of solving differential equations

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UNIT-1: DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

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Partial Derivatives: 1) Let f(x,y) be a real valued function. If $\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$ is exists and finite, then this limit is called partial derivative of f(x, y) w.r.t.x and it is denoted by $f_x(x, y)$ or $\frac{\partial f}{\partial x}$.

2) Let f(x,y) be a real valued function. If $\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$ is exists and finite, then this limit is called partial derivative of f(x, y) w.r.t.y and it is denoted by $f_y(x, y)$ or $\frac{\partial f}{\partial y}$.

Remark: 1) Partial derivative of f(x, y) w.r.t.x at point (a, b) is given by

$$f_x(a, b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$$

2) Partial derivative of f(x, y) w.r.t.y at point (a, b) is given by

$$f_y(a, b) = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}$$

3) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are called first order partial derivatives.

4) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x\partial y}$, $\frac{\partial^2 f}{\partial y\partial x}$ and $\frac{\partial^2 f}{\partial y^2}$ are called second order partial derivatives.

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Ex. If $u = xy + e^x$ then find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x\partial y}$

Solution: Let $u = xy + e^x$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xy + e^x) = y + e^x$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(xy + e^x) = x + 0 = x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(y + e^x) = 0 + e^x = e^x$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(x) = 0$$

$$\therefore \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x) = 1$$

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Ex. If $u = x^3 + y^3 + 3xy$ then find $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial^2 u}{\partial x\partial y}$

Solution: Let $u = x^3 + y^3 + 3xy$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 + 3xy) = 3x^2 + 0 + 3y = 3x^2 + 3y$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3 + 3xy) = 0 + 3y^2 + 3x = 3y^2 + 3x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + 3y) = 6x + 0 = 6x$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(3y^2 + 3x) = 6y + 0 = 6y$$

$$\therefore \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2 + 3x) = 0 + 3 = 3$$

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Ex. If $u = e^x \sin xy$ then find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ at (0, 0).

Solution: Let $u = e^x \sin xy$

$$\therefore \frac{\partial u}{\partial x} = e^x \sin xy + ye^x \cos xy$$

$$\text{and } \frac{\partial u}{\partial y} = xe^x \cos xy$$

∴ At point (0, 0).

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

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Ex. If $u = x^2y + y^2z + z^2x$ then find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at (1, 1, 1).

Solution: Let $u = x^2y + y^2z + z^2x$

$$\therefore \frac{\partial u}{\partial x} = 2xy + 0 + z^2 = 2xy + z^2$$

$$\frac{\partial u}{\partial y} = x^2 + 2yz + 0 = x^2 + 2yz$$

$$\text{and } \frac{\partial u}{\partial z} = 0 + y^2 + 2zx = y^2 + 2zx$$

∴ At point (1, 1, 1).

$$\frac{\partial u}{\partial x} = 2 + 1 = 3, \quad \frac{\partial u}{\partial y} = 1 + 2 = 3 \text{ and } \frac{\partial u}{\partial z} = 1 + 2 = 3$$

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Ex. If $u = x^3z + y^2x - 2yz$ then find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ at (1, 2, 3).

Solution: Let $u = x^3z + y^2x - 2yz$

$$\therefore \frac{\partial u}{\partial x} = 3x^2z + y^2 - 0 = 3x^2z + y^2$$

$$\frac{\partial u}{\partial y} = 0 + 2yx - 2z = 2yx - 2z$$

$$\text{and } \frac{\partial u}{\partial z} = x^3 + 0 - 2y = x^3 - 2y$$

∴ At point (1, 2, 3).

$$\frac{\partial u}{\partial x} = 9 + 4 = 13$$

$$\frac{\partial u}{\partial y} = 4 - 6 = -2$$

$$\text{and } \frac{\partial u}{\partial z} = 1 - 4 = -3$$

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Ex. If $u = \log(\tan x + \tan y + \tan z)$, show that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Proof: Let $u = \log(\tan x + \tan y + \tan z)$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x + 0 + 0) = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x(\sec^2 x)}{\tan x + \tan y + \tan z} = \frac{2\sin x\cos x}{\tan x + \tan y + \tan z}\left(\frac{1}{\cos^2 x}\right) = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} = \frac{2\tan x}{\tan x + \tan y + \tan z}$$

Similarly $\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z}$

and $\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}$

∴ By adding, we get,

$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$

$$= \frac{2\tan x}{\tan x + \tan y + \tan z} + \frac{2\tan y}{\tan x + \tan y + \tan z} + \frac{2\tan z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z}$$

$$= 2$$

Hence proved.

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Ex. If $u = (x^2 + y^2 + z^2)^{-1/2}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u$

Proof: Let $u = (x^2 + y^2 + z^2)^{-1/2}$

∴ $\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x + 0 + 0)$

∴ $x\frac{\partial u}{\partial x} = -x^2 (x^2 + y^2 + z^2)^{-3/2}$

Similarly $y\frac{\partial u}{\partial y} = -y^2 (x^2 + y^2 + z^2)^{-3/2}$

and $z\frac{\partial u}{\partial z} = -z^2 (x^2 + y^2 + z^2)^{-3/2}$

∴ By adding, we get,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-3/2} = -(x^2 + y^2 + z^2)^{-1/2} = -u$

Hence proved.

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Ex. If $u = \log(x^3 + y^3 - x^2y - xy^2)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2}{x+y}$

Proof: Let $u = \log(x^3 + y^3 - x^2y - xy^2)$

∴ $\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 - x^2y - xy^2}(3x^2 + 0 - 2xy - y^2) = \frac{3x^2 - 2xy - y^2}{x^3 + y^3 - x^2y - xy^2}$

and $\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 - x^2y - xy^2}(0 + 3y^2 + 0 - x^2 - 2xy) = \frac{3y^2 - 2xy - x^2}{x^3 + y^3 - x^2y - xy^2}$

∴ By adding, we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{3x^2 - 2xy - y^2 + 3y^2 - 2xy - x^2}{x^3 + y^3 - x^2y - xy^2}$$

$$= \frac{2x^2 - 4xy + 2y^2}{x^3 + y^3 - x^2y - xy^2}$$

$$= \frac{2(x^2 - 2xy + y^2)}{(x+y)(x^2 - 2xy + y^2)}$$

∴ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{2}{x+y}$

Hence proved.

Differential equation: An equation which contains the terms of derivatives is called differential equation.

Order of Differential equation: An order of the highest ordered derivatives occurring in the equation is called the order of a differential equation.

Degree of Differential equation: Power of the highest ordered derivative occurring in the differential equation when it is free from radical signs and fractional indices is called the degree of a differential equation.

Homogeneous Function: A function f(x, y) is said to be homogeneous function of degree 'n' if $f(x, y) = x^nF(\frac{y}{x})$

Differential equation of First Order and First Degree: If M and N are functions of variables x and y then Mdx + Ndy = 0 is called differential equation of first order and first degree.

Homogeneous Differential equation: If M and N are homogeneous functions of variables x and y of same degree then Mdx + Ndy = 0 or $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ is called homogeneous differential equation.

Exact Differential equation: A differential equation of type Mdx + Ndy = 0 is called exact differential equation if there exist a function u(x, y) such that Mdx + Ndy = du.

e.g. As $2xy^2dx + 2x^2ydy = d(x^2y^2)$

$\therefore$ $2xy^2dx + 2x^2ydy = 0$ is an exact differential equation.

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Theorem: A necessary and sufficient condition for differential equation Mdx + Ndy = 0 to be exact is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof: Necessary condition: Suppose Mdx + Ndy = 0 is exact.

$\therefore$ there exist a function u(x, y) such that Mdx + Ndy = du …(1)

But by total differentiation $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$…(2)

From equation (1) and (2), we have

$M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$

But $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Sufficient condition: Suppose Mdx + Ndy = 0 be a differential equation such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Let us define $f(x, y) = \int_{y-constant}^{\cdot} Mdx$

$\therefore M = \frac{\partial f}{\partial x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$

$\therefore \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$ since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial^2 f}{\partial x \partial y} = 0$

$\therefore \frac{\partial}{\partial x}(N - \frac{\partial f}{\partial y}) = 0$

Integrating both sides w.r.t. x, keeping y constant, we get,

$N - \frac{\partial f}{\partial y} = g(y)$, a function of y only.

$\therefore N = \frac{\partial f}{\partial y} + g(y)$

$\therefore Mdx + Ndy = \frac{\partial f}{\partial x}dx + [\frac{\partial f}{\partial y} + g(y)]dy$

$= d[f + \int g(y)dy]$

$\therefore$ Mdx + Ndy = 0 is an exact differential equation.

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Remark: A general solution of exact differential equation Mdx + Ndy = 0 is

$$\int_{y-constant} Mdx + \int (terms\ in\ N\ not\ containing\ x)dy] = c$$

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Ex. Solve $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$

Solution: Let $(2x^3 + 3y)dx + (3x + y - 1)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = 2x^3 + 3y$ and $N = 3x + y - 1$

$\therefore \frac{\partial M}{\partial y} = 3$ and $\frac{\partial N}{\partial x} = 3$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation is exact .

$\therefore$ It's general solution is

$\int_{y-constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy] = c$

$\therefore \int_{y-constant} (2x^3 + 3y)dx + \int (y - 1)dy] = c$

$\therefore \frac{1}{2}x^4 + 3xy + \frac{1}{2}y^2 - y = c.$

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Ex. Solve $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0$

Solution: Let $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = y^2 - 2xy + 6x$ and $N = -x^2 + 2xy - 2$

$\therefore \frac{\partial M}{\partial y} = 2y - 2x$ and $\frac{\partial N}{\partial x} = -2x + 2y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Given differential equation is exact.

∴ It's general solution is

$\int_{y-constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy$]= c

$\therefore \int_{y-constant} (y^2 - 2xy + 6x)dx + \int (-2)dy$]= c

$\therefore xy^2 - x^2y + 3x^2 - 2y = c.$

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Ex. Solve $3yx^2dx + (x^3 + 8y)dy = 3ydx + 3xdy$ given that for x = 0, y = 1.

Solution: Let $3yx^2dx + (x^3 + 8y)dy = 3ydx + 3xdy$

i.e. $(3yx^2 - 3y)dx + (x^3 + 8y - 3x)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = 3yx^2 - 3y$ and $N = x^3 + 8y - 3x$

$\therefore \frac{\partial M}{\partial y} = 3x^2 - 3$ and $\frac{\partial N}{\partial x} = 3x^2 - 3$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Given differential equation is exact .

∴ It's general solution is

$\int_{y-constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy$]= c

$\therefore \int_{y-constant} (3yx^2 - 3y)dx + \int (8y)dy$]= c

$\therefore yx^3 - 3xy + 4y^2 = c$

Given that for x = 0, y = 1

∴ 0 - 0 + 4 = c

∴ c = 4

∴ Particular solution of given equation is

$yx^3 - 3xy + 4y^2 = 4$

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Ex. Solve $(\sin x.\cos y+e^{2x})dx+(\cos x.\sin y+\tan y)dy = 0$

Solution: Let $(\sin x.\cos y+e^{2x})dx+(\cos x.\sin y+\tan y)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = \sin x.\cos y+e^{2x}$ and $N = \cos x.\sin y+\tan y$

$\therefore \frac{\partial M}{\partial y} = -\sin x.\sin y$ and $\frac{\partial N}{\partial x} = -\sin x.\sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Given differential equation is exact .

∴ It's general solution is

$\int_{y-constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy$]= c

$\therefore \int_{y-constant} (\sin x.\cos y + e^{2x})dx + \int \tan y\, dy$]= c

$\therefore -\cos x.\cos y + \frac{1}{2} e^{2x} + \log \sec y = c.$

Ex. Solve $(\cos 2y - 3x^2y^2)dx+(\cos 2y - 2x\sin 2y - 2x^3y)dy = 0$

Solution: Let $(\cos 2y - 3x^2y^2)dx+(\cos 2y - 2x\sin 2y - 2x^3y)dy = 0$ be the given differential equation, comparing it with $Mdx + Ndy = 0$, we get,

$M = \cos 2y - 3x^2y^2$ and $N = \cos 2y - 2x\sin 2y - 2x^3y$

$\therefore \frac{\partial M}{\partial y} = - 2\sin 2y - 6x^2y$ and $\frac{\partial N}{\partial x} = - 2\sin 2y - 6x^2y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ Given differential equation is exact .

∴ It's general solution is

$\int_{y-constant} Mdx + \int (terms\ of\ N\ not\ containing\ x)dy] = c$

$\therefore \int_{y-constant}(\cos 2y - 3x^2y^2)dx + \int \cos 2y dy] = c$

$\therefore x.\cos 2y - x^3y^2 + \frac{1}{2}\sin 2y = c.$

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Integrating Factor: A function u(x, y) is said to be an integrating factor (I.F.) of non-exact differential equation $Mdx + Ndy = 0$ if $Mudx + Nudy = 0$ is exact.

Rules of finding I.F.:

Rule-I: If the differential equation $Mdx + Ndy = 0$ is homogeneous then $\frac{1}{Mx+Ny}$ is an I.F. if $Mx + Ny \neq 0$

Proof: Let $Mdx + Ndy = 0$ is homogeneous differential equation.

∴ M and N are homogeneous functions of same degree say n.

∴ By Eulers theorem

$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM$ and $x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN$ ......(1)

Given that $Mx + Ny \neq 0$

∴ Multipling by $\frac{1}{Mx+Ny}$ to given equation, we get,

$\frac{M}{Mx+Ny}dx + \frac{N}{Mx+Ny}dy = 0$

i.e. $M_1dx + N_1dy = 0$ where $M_1 = \frac{M}{Mx+Ny}$, $N_1 = \frac{N}{Mx+Ny}$

$\therefore \frac{\partial M_1}{\partial y} = \frac{(Mx+Ny)\frac{\partial M}{\partial y} - M(x\frac{\partial M}{\partial y} + N + y\frac{\partial N}{\partial y})}{(Mx+Ny)^2} = \frac{Ny\frac{\partial M}{\partial y} - MN - My\frac{\partial N}{\partial y}}{(Mx+Ny)^2}$

and $\frac{\partial N_1}{\partial x} = \frac{(Mx+Ny)\frac{\partial N}{\partial x} - N(M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x})}{(Mx+Ny)^2} = \frac{Mx\frac{\partial N}{\partial x} - MN - Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}$

$\therefore \frac{\partial M_1}{\partial y} - \frac{\partial N_1}{\partial x} = \frac{Ny\frac{\partial M}{\partial y} - MN - My\frac{\partial N}{\partial y} - Mx\frac{\partial N}{\partial x} + MN + Nx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}$

$= \frac{N(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}) - M(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y})}{(Mx+Ny)^2}$

$= \frac{N(nM) - M(nN)}{(Mx+Ny)^2}$ by (1)

$= 0$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore M_1dx + N_1dy = 0$ is exact.

$\therefore \frac{1}{Mx+Ny}$ is an I.F.of given equation is proved.

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Ex. Solve (x + y)dx + (y - x)dy = 0

Solution: Let (x + y)dx+(y - x)dy = 0 be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

M = x + y and N = y - x

$\therefore \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But the given differential equation is homogeneous with

$Mx + Ny = (x+y)x + (y-x)y = x^2 + yx + y^2 - xy = x^2 + y^2 \neq 0$

$\therefore$ I.F. $= \frac{1}{Mx+Ny} = \frac{1}{x^2+y^2}$

Multiplying given equation by $\frac{1}{x^2+y^2}$, we get,

$\frac{x+y}{x^2+y^2}dx + \frac{y-x}{x^2+y^2}dy = 0$ which is exact

∴ It's general solution is

$\int_{y-constant}^{\cdot} \frac{x+y}{x^2+y^2}dx + \int 0dy = c$

$\therefore \int_{y-constant}^{\cdot} \frac{x}{x^2+y^2}dx + y\int_{y-constant}^{\cdot} \frac{1}{x^2+y^2}dx = c$

$\therefore \frac{1}{2}\int_{y-constant}^{\cdot} \frac{2x}{x^2+y^2}dx + y\int_{y-constant}^{\cdot} \frac{1}{x^2+y^2}dx = c$

$\therefore \frac{1}{2}log(x^2+y^2) + y.\frac{1}{y}\tan^{-1}(\frac{x}{y}) = c$

i.e $\frac{1}{2}log(x^2+y^2) + \tan^{-1}(\frac{x}{y}) = c$

Ex. Solve $(xy - y^2)dx - x^2dy = 0$

Solution: Let $(xy - y^2)dx - x^2dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = xy - y^2$ and $N = -x^2$

$\therefore \frac{\partial M}{\partial y} = x - 2y$ and $\frac{\partial N}{\partial x} = -2x$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But the given differential equation is homogeneous with

$Mx + Ny = (xy - y^2)x + (-x^2)y = x^2y - y^2x - x^2y = -xy^2 \neq 0$

$\therefore$ I.F. $= \frac{1}{Mx+Ny} = \frac{1}{-xy^2}$

Multiplying given equation by $\frac{-1}{xy^2}$, we get,

$\frac{-1}{xy^2}(xy - y^2)dx + \frac{1}{xy^2}x^2dy = 0$

i.e. $(\frac{1}{x} - \frac{1}{y})dx + \frac{x}{y^2}dy = 0$ which is exact

∴ It's general solution is

$\int_{y-constant}^{\cdot} (\frac{1}{x} - \frac{1}{y})dx + \int 0dy = c$

$\therefore \log x - \frac{x}{y} = c$

Ex. Solve $x^2ydx - (x^3+y^3)dy = 0$

Solution: Let $x^2ydx - (x^3+y^3)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = x^2y$ and $N = - x^3 - y^3$

$\therefore \frac{\partial M}{\partial y} = x^2$ and $\frac{\partial N}{\partial x} = -3x^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But the given differential equation is homogeneous with

$Mx + Ny = (x^2y)x + (- x^3 - y^3)y = x^3y - x^3y - y^4 = -y^4 \neq 0$

$\therefore$ I.F. $= \frac{1}{Mx+Ny} = \frac{1}{-y^4}$

Multiplying given equation by $\frac{-1}{y^4}$, we get,

$\frac{-1}{y^4}x^2ydx + \frac{1}{y^4}(x^3+y^3)dy = 0$

i.e. $(-\frac{x^2}{y^3})dx + (\frac{x^3}{y^4} + \frac{1}{y})\, dy = 0$ which is exact

∴ It's general solution is

$\int_{y-constant}(-\frac{x^2}{y^3})dx + \int \frac{1}{y}dy = c$

$\therefore (-\frac{x^3}{3y^3}) + \log y = c$

==

Rule-II: If the differential equation Mdx + Ndy = 0 is of type $f_1(xy)ydx+f_2(xy)xdy= 0$ then $\frac{1}{Mx-Ny}$ is an I.F. if $Mx - Ny \neq 0$

Proof: Let given differential equation Mdx + Ndy = 0 is of type

$f_1(xy)ydx+f_2(xy)xdy= 0$

Given that $Mx - Ny \neq 0$

$\therefore$ Multipling by $\frac{1}{Mx-Ny}$ to given equation, we get,

$\frac{M}{Mx-Ny}dx + \frac{N}{Mx-Ny}dy = 0$

i.e. $M_1dx + N_1dy = 0$ where $M_1 = \frac{M}{Mx-Ny} = \frac{f_1(xy)y}{f_1(xy)yx - f_2(xy)xy} = \frac{f_1}{x(f_1-f_2)}$ and

$N_1 = \frac{N}{Mx+Ny} = = \frac{f_2(xy)x}{f_1(xy)yx - f_2(xy)xy} = \frac{f_2}{y(f_1-f_2)}$

$\therefore \frac{\partial M_1}{\partial y} = \frac{1}{x}[\frac{(f_1-f_2)(xf'_1) - f_1(xf'_1 - xf'_2)}{(f_1-f_2)^2} = \frac{-f_2f'_1 + f_1f'_2}{(f_1-f_2)^2} = \frac{f_1f'_2 - f_2f'_1}{(f_1-f_2)^2}$

and $\frac{\partial N_1}{\partial x} = \frac{1}{y}[\frac{(f_1-f_2)(yf'_2) - f_2(yf'_1 - yf'_2)}{(f_1-f_2)^2} = \frac{f_1f'_2 - f_2f'_1}{(f_1-f_2)^2}$

$\therefore \frac{\partial M_1}{\partial y} - \frac{\partial N_1}{\partial x} = \frac{f_1f'_2 - f_2f'_1 - f_1f'_2 + f_2f'_1}{(f_1-f_2)^2}$

$= 0$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore M_1dx + N_1dy = 0$ is exact.

$\therefore \frac{1}{Mx-Ny}$ is an I.F.of given equation is proved.

==

Ex. Solve $y(xy+1)dx + (x^2y^2+xy+1)xdy = 0$

Solution: Let $y(xy+1)dx + (x^2y^2+xy+1)xdy = 0$ be the given differential equation of type $f_1(xy)ydx+f_2(xy)xdy= 0$ with $M = (xy+1)y$ and $N = (x^2y^2+xy+1)x$

$\therefore Mx-Ny =(xy+1)yx-(x^2y^2+xy+1)xy=x^2y^2+xy-x^3y^3-x^2y^2-xy=-x^3y^3\neq 0$

$\therefore$ I.F. $= \frac{1}{Mx-Ny} = \frac{1}{-x^3y^3}$

Multiplying given equation by $\frac{-1}{x^3y^3}$, we get,

$\frac{-1}{x^3y^3}y(xy+1)dx - \frac{1}{x^3y^3}(x^2y^2+xy+1)xdy = 0$

i.e. $(-\frac{1}{x^2y} - \frac{1}{x^3y^2})dx + (-\frac{1}{y} - \frac{1}{xy^2} - \frac{1}{x^2y^3})\,dy = 0$ which is exact

$\therefore$ It's general solution is

$\int_{y-constant}(-\frac{1}{x^2y} - \frac{1}{x^3y^2})dx + \int(-\frac{1}{y})dy = c$

$\therefore \frac{1}{xy} + \frac{1}{2x^2y^2} - \log y = c$

==

Ex. Solve $(x^2y^2+4xy+2)xdy + (x^2y^2+5xy+2)ydx = 0$

Solution: Let $(x^2y^2+4xy+2)xdy + (x^2y^2+5xy+2)ydx = 0$ be the given differential equation of type $f_1(xy)ydx+f_2(xy)xdy= 0$ with

$M = (x^2y^2+5xy+2)y$ and $N = (x^2y^2+4xy+2)x$

$\therefore Mx-Ny = (x^2y^2+5xy+2)yx-(x^2y^2+4xy+2)xy$

$= x^3y^3+5x^2y^2+2xy-x^3y^3-4x^2y^2-2xy = x^2y^2\neq 0$

$\therefore$ I.F. $= \frac{1}{Mx-Ny} = \frac{1}{x^2y^2}$

Multiplying given equation by $\frac{1}{x^2y^2}$, we get,

$\frac{1}{x^2y^2}(x^2y^2+5xy+2)ydx + \frac{1}{x^2y^2}(x^2y^2+4xy+2)xdy = 0$

i.e. $(y + \frac{5}{x} + \frac{2}{x^2y})dx + (x + \frac{4}{y} + \frac{2}{xy^2})\,dy = 0$ which is exact

$\therefore$ It's general solution is

$\int_{y-constant}(y + \frac{5}{x} + \frac{2}{x^2y})dx + \int(\frac{4}{y})dy = c$

$\therefore xy+5\log x- \frac{2}{xy} +4\log y = c$

$\therefore xy- \frac{2}{xy} +\log(x^5y^4) = c$

==

Ex. Solve $(\frac{1}{x} + y)dx + (\frac{1}{y} - x)dy = 0$

Solution: Let $(\frac{1}{x} + y)dx + (\frac{1}{y} - x)dy = 0$ i.e. $(1+xy)ydx + (1- xy)xdy = 0$ be the given differential equation of type $f_1(xy)ydx+f_2(xy)xdy= 0$ with

$M = (1+xy)y$ and $N = (1- xy)x$

$\therefore Mx-Ny =(1+xy)yx-(1- xy)xy= xy+x^2y^2-xy+x^2y^2=2x^2y^2\neq 0$

$\therefore$ I.F. $= \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$

Multiplying given equation by $\frac{1}{2x^2y^2}$, we get,

$\frac{1}{2x^2y^2}(1+xy)ydx + \frac{1}{2x^2y^2}(1-xy)xdy = 0$

i.e. $(\frac{1}{2x^2y} + \frac{1}{2x})dx + (\frac{1}{2xy^2} - \frac{1}{2y})\,dy = 0$ which is exact

$\therefore$ It's general solution is

$\int_{y-constant}(\frac{1}{2x^2y} + \frac{1}{2x})dx + \int(-\frac{1}{2y})dy = c_1$

$\therefore \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c_1$ i.e. $\log(\frac{x}{y}) - \frac{1}{xy} = c$ where $2c_1 = c$

==

Rule-III: If $\frac{1}{N}[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}]$ is a function of x alone, say f(x) then $e^{\int f(x)dx}$ I.F. of equation Mdx + Ndy = 0.

Proof: Given differential equation is Mdx + Ndy = 0 ……(1)

such that $\frac{1}{N}[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = f(x)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + Nf(x)$ ……(2)

$\therefore$ Multipling by $e^{\int f(x)dx}$ to given equation, we get,

$e^{\int f(x)dx}Mdx + e^{\int f(x)dx}Ndy = 0$

i.e. $M_1dx + N_1dy = 0$ where $M_1 = e^{\int f(x)dx}M$ and $N_1 = e^{\int f(x)dx}N$

$\therefore \frac{\partial M_1}{\partial y} = e^{\int f(x)dx}\frac{\partial M}{\partial y}$ and

$\frac{\partial N_1}{\partial x} = e^{\int f(x)dx}\frac{\partial N}{\partial x} + f(x)e^{\int f(x)dx}N = e^{\int f(x)dx}[\frac{\partial N}{\partial x} + Nf(x)]$

$\therefore \frac{\partial N_1}{\partial x} = e^{\int f(x)dx}\frac{\partial M}{\partial y}$ by(2)

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore M_1dx + N_1dy = 0$ is exact.

$\therefore e^{\int f(x)dx}$ is an I.F.of given equation is proved.

==

Ex. Solve $(x - y^2)dx + 2xydy = 0$

Solution: Let $(x - y^2)dx + 2xydy = 0$ be the given differential equation,

comparing it with Mdx + Ndy = 0, we get,

$M = x - y^2$ and $N = 2xy$

$\therefore \frac{\partial M}{\partial y} = -2y$ and $\frac{\partial N}{\partial x} = 2y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation is not exact.

But $\frac{1}{N}[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = \frac{1}{2xy}[-2y - 2y] = \frac{-2}{x} = f(x)$

∴ I.F. = $e^{\int f(x)dx} = e^{-2\log x} = x^{-2} = \frac{1}{x^2}$

Multiplying given equation by $\frac{1}{x^2}$, we get,

$\frac{1}{x^2}(x - y^2)dx + \frac{1}{x^2}(2xy)dy = 0$

i.e. $(\frac{1}{x} - \frac{y^2}{x^2})dx + \frac{2y}{x}dy = 0$ which is exact

∴ It's general solution is

$\int_{y-constant}(\frac{1}{x} - \frac{y^2}{x^2})dx + \int 0dy = c$

∴ $\log x + \frac{y^2}{x} = c$

==

Ex. Solve $(x^2 + y^2 + x)dx + xydy = 0$

Solution: Let $(x^2 + y^2 + x)dx + xydy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = x^2 + y^2 + x$ and $N = xy$

∴ $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = y$

∴ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But $\frac{1}{N}[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = \frac{1}{xy}[2y - y] = \frac{1}{x} = f(x)$

∴ I.F. = $e^{\int f(x)dx} = e^{\log x} = x$

Multiplying given equation by x, we get,

$x(x^2 + y^2 + x)dx + x(xy)dy = 0$

i.e. $(x^3 + xy^2 + x^2)dx + x^2y\,dy = 0$ which is exact

∴ It's general solution is

$\int_{y-constant}(x^3 + xy^2 + x^2)dx + \int 0dy = c_1$

∴ $\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 = c_1$

i.e. $3x^4 + 6x^2y^2 + 4x^3 = c$ where $c = 12c_1$

==

Ex. Solve $(2y^2 + 3xy - 2y + 6x)dx + (x^2 + 2xy - x)dy = 0$

Solution: Let $(2y^2 + 3xy - 2y + 6x)dx + (x^2 + 2xy - x)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = 2y^2 + 3xy - 2y + 6x$ and $N = x^2 + 2xy - x$

∴ $\frac{\partial M}{\partial y} = 4y + 3x - 2$ and $\frac{\partial N}{\partial x} = 2x + 2y - 1$

∴ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But $\frac{1}{N}[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}] = \frac{1}{(x^2 + 2xy - x)}[4y + 3x - 2 - 2x - 2y + 1]$

$$= \frac{(x+2y-1)}{x(x+2y-1)} = \frac{1}{x} = f(x)$$

∴ I.F. = $e^{\int f(x)dx} = e^{\log x} = x$

Multiplying given equation by x , we get,

$x(2y^2 + 3xy - 2y + 6x)dx + x(x^2 + 2xy - x)dy = 0$

i.e. $(2xy^2 + 3x^2y - 2xy + 6x^2)dx + (x^3 + 2x^2y - x^2)\,dy = 0$ which is exact

∴ It's general solution is

$$\int_{y-constant}(2xy^2 + 3x^2y - 2xy + 6x^2)dx + \int 0\,dy = c$$

∴ $x^2y^2 + x^3y - x^2y + 2x^3 = c$

==

Rule-IV: If $\frac{1}{M}[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}]$ is a function of y alone, say f(y) then $e^{\int f(y)dy}$ I.F. of equation Mdx + Ndy = 0.

Proof: Given differential equation is Mdx + Ndy = 0 ……(1)

such that $\frac{1}{M}[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}] = f(y)$

∴ $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} + Mf(y)$ ……(2)

∴ Multipling by $e^{\int f(y)dy}$ to given equation, we get,

$e^{\int f(y)dy}Mdx + e^{\int f(y)dy}Ndy = 0$

i.e. $M_1dx + N_1dy = 0$ where $M_1 = e^{\int f(y)dy}M$ and $N_1 = e^{\int f(y)dy}N$

∴ $\frac{\partial M_1}{\partial y} = e^{\int f(y)dy}\frac{\partial M}{\partial y} + f(y)e^{\int f(y)dy}M = e^{\int f(y)dy}[\frac{\partial M}{\partial y} + Mf(y)]$

∴ $\frac{\partial M_1}{\partial y} = e^{\int f(y)dy}\frac{\partial N}{\partial x}$ by (2)

and $\frac{\partial N_1}{\partial x} = e^{\int f(y)dy}\frac{\partial N}{\partial x}$

∴ $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

∴ $M_1dx + N_1dy = 0$ is exact.

∴ $e^{\int f(y)dy}$ is an I.F.of given equation is proved.

==

Ex. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

Solution: Let $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = y^4 + 2y$ and $N = xy^3 + 2y^4 - 4x$

∴ $\frac{\partial M}{\partial y} = 4y^3 + 2$ and $\frac{\partial N}{\partial x} = y^3 - 4$

∴ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But $\frac{1}{M}[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}] = \frac{1}{(y^4+2y)}[y^3 - 4 - 4y^3 - 2]$

$$= \frac{(-3y^3-6)}{y(y^3+2)} = \frac{-3}{y} = f(y)$$

$\therefore$ I.F. = $e^{\int f(y)dy} = e^{-3\log y} = y^{-3} = \frac{1}{y^3}$

Multiplying given equation by $\frac{1}{y^3}$, we get,

$\frac{1}{y^3}(y^4 + 2y)dx + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$

i.e. $(y + \frac{2}{y^2})dx + (x + 2y - \frac{4x}{y^3})\, dy = 0$ which is exact

$\therefore$ It's general solution is

$\int_{y-constant} (y + \frac{2}{y^2})dx + \int 2y dy = c$

$\therefore$ $(y + \frac{2}{y^2})x + y^2 = c$

==

Ex. Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

Solution: Let $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ be the given differential equation, comparing it with Mdx + Ndy = 0, we get,

$M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$

$\therefore \frac{\partial M}{\partial y} = 12x^2y^3 + 2x$ and $\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation is not exact.

But $\frac{1}{M}[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}] = \frac{1}{(3x^2y^4 + 2xy)}[6x^2y^3 - 2x - 12x^2y^3 - 2x]$

$= \frac{(-6x^2y^3 - 4x)}{y(3x^2y^3 + 2x)} = \frac{-2}{y} = f(y)$

$\therefore$ I.F. = $e^{\int f(y)dy} = e^{-2\log y} = y^{-2} = \frac{1}{y^2}$

Multiplying given equation by $\frac{1}{y^2}$, we get,

$\frac{1}{y^2}(3x^2y^4 + 2xy)dx + \frac{1}{y^2}(2x^3y^3 - x^2)dy = 0$

i.e. $(3x^2y^2 + \frac{2x}{y})dx + (2x^3y - \frac{x^2}{y^2})\, dy = 0$ which is exact

$\therefore$ It's general solution is

$\int_{y-constant} (3x^2y^2 + \frac{2x}{y})dx + \int 0 dy = c$

$\therefore$ $x^3y^2 + \frac{x^2}{y} = c$

==

Linear Differential Equation: A differential equation of type $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x alone is called linear differential equation.

Method of Solving Linear Differential Equation:

Let $\frac{dy}{dx} + Py = Q$ i.e. $(Py - Q)dx + dy = 0$……(1) be a linear differential equation

$M = (Py - Q)$ and $N = 1$

Where P and Q are functions of x alone.

$\therefore \frac{\partial M}{\partial y}$ = P and $\frac{\partial N}{\partial x}$ = 0

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ Given differential equation is not exact.

But $\frac{1}{N}[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}]$ = P = f(x)

∴ I.F. = $e^{\int f(x)dx} = e^{\int Pdx}$

Multiplying equation (1) by $e^{\int Pdx}$, we get,

$e^{\int Pdx}(Py - Q)\, dx + e^{\int Pdx} dy = 0$

which is exact

∴ It's general solution is

$\int_{y-constant} e^{\int Pdx}(Py - Q)\, dx + \int 0dy = c$

∴ $y \int e^{\int Pdx} Pdx - \int e^{\int Pdx} Qdx = c$

∴ $ye^{\int Pdx} = \int e^{\int Pdx} Qdx + c$

be the general solution of linear differential equation.

==

Remark: If linear differential equation is of type $\frac{dx}{dy}$ + Px = Q

where P and Q are functions of y alone, then it's G. S. is $xe^{\int Pdy} = \int e^{\int Pdy} Qdy + c$

==

Ex. Solve $\frac{dy}{dx}$ + 2ytanx = sinx

Solution: Let $\frac{dy}{dx}$ + 2ytanx = sinx be the given differential equation, which is linear differential equation, with P = 2tanx and Q = sinx

∴ I.F. = $e^{\int Pdx} = e^{\int 2tanxdx} = = e^{2logsecx} = sec^2x$

∴ General solution of given equaion is

$ye^{\int Pdx} = \int e^{\int Pdx} Qdx + c$

i.e. $ysec^2x = \int sec^2x.\, sinxdx + c$

$= \int secx.\, tanxdx + c$

∴ $ysec^2x = secx + c$

==

Ex. Solve $\frac{dy}{dx} + \frac{y}{tanx}$ = 2sinx.cosx, given that y = 0 when x = $\frac{\pi}{2}$

Solution: Let $\frac{dy}{dx} + \frac{y}{tanx}$ = 2sinx.cosx be the given differential equation, which is linear differential equation,

with P = $\frac{1}{tanx}$ = cotx and Q = 2sinx.cosx

∴ I.F. = $e^{\int Pdx} = e^{\int cotxdx} = = e^{logsinx}$ = sinx

∴ General solution of given equaion is

$ye^{\int Pdx} = \int e^{\int Pdx} Qdx + c$

i.e. $ysinx = \int sinx.\, 2sinx.\, cosxdx + c$

$= 2\int \sin^2x.\cos x dx + c$

$\therefore$ ysinx = $\frac{2}{3}\sin^3 x + c$

Given that y = 0 when x = $\frac{\pi}{2}$

$\therefore$ 0 = $\frac{2}{3}$ + c i.e. c = - $\frac{2}{3}$

$\therefore$ Particular solution of given equaion is

y.sinx = $\frac{2}{3}\sin^3 x - \frac{2}{3}$

==

Ex. Solve x.cosx. $\frac{dy}{dx}$ + (xsinx+cosx)y = 1

Solution: Let x.cosx. $\frac{dy}{dx}$ + (xsinx+cosx)y = 1

i.e. $\frac{dy}{dx} + (\tan x + \frac{1}{x})$ y = $\frac{\sec x}{x}$ be the given differential equation,

which is linear differential equation, with P = $\tan x + \frac{1}{x}$ and Q = $\frac{\sec x}{x}$

Now$\int$Pdx = $\int(\tan x + \frac{1}{x})dx$ = logsecx+logx=logxsecx

$\therefore$ I.F. = $e^{\int Pdx} = e^{\log x\sec x}$ = xsecx

$\therefore$ General solution of given equaion is

$ye^{\int Pdx} = \int e^{\int Pdx}Qdx + c$

i.e. yxsecx = $\int x\sec x.\frac{\sec x}{x}dx + c$

$= \int \sec^2 x\, dx + c$

$\therefore$ xysecx = tanx + c

==

Ex. Solve $\frac{dy}{dx} + x^2y = x^5$

Solution: Let $\frac{dy}{dx} + x^2y = x^5$ be the given differential equation,

which is linear differential equation, with P = x^2 and Q = x^5

Now$\int$Pdx = $\int x^2dx = \frac{1}{3}x^3$

$\therefore$ I.F. = $e^{\int Pdx} = e^{\frac{1}{3}x^3}$

$\therefore$ General solution of given equaion is

$ye^{\int Pdx} = \int e^{\int Pdx}Qdx + c$

i.e. $ye^{\frac{1}{3}x^3} = \int e^{\frac{1}{3}x^3}.x^5dx + c$

In integration put $\frac{1}{3}x^3 = t$ $\therefore x^3 = 3t \therefore 3x^2dx = 3dt$ i.e.$x^2dx = dt$

$\therefore ye^{\frac{1}{3}x^3} = \int e^t.3tdt + c$

$= 3[te^t - \int e^tdt] + c$

$= 3[te^t - e^t] + c$

$= e^t[3t - 3] + c$

$\therefore ye^{\frac{1}{3}x^3} = e^{\frac{1}{3}x^3}(x^3 - 3) + c$

Ex. Solve $(1+x^2)\frac{dy}{dx} + 2xy - 1 = 0$

Solution: Let $(1+x^2)\frac{dy}{dx} + 2xy - 1 = 0$

i.e. $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{1+x^2}$ be the given differential equation,

which is linear differential equation, with $P = \frac{2x}{1+x^2}$ and $Q = \frac{1}{1+x^2}$

Now $\int Pdx = \int\frac{2x}{1+x^2}dx = \log(1+x^2)$

$\therefore$ I.F. $= e^{\int Pdx} = e^{\log(1+x^2)} = 1+x^2$

$\therefore$ General solution of given equaion is

$ye^{\int Pdx} = \int e^{\int Pdx}Qdx + c$

i.e. $y(1+x^2) = \int(1+x^2).\frac{1}{1+x^2}dx + c$

$= \int dx + c$

$\therefore y(1+x^2) = x + c$

==

Bernoulli's Differential Equation: A differential equation of type $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x alone is called Bernoulli's differential equation.

Method of Solving Bernoulli's Differential Equation:

Consider the Bernoulli's equation

$$\frac{dy}{dx} + Py = Qy^n \quad \ldots\ldots(1)$$

where P and Q are functions of x alone

Multiplying equation (1) by y^{-n} we get,

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Put $y^{1-n} = v \quad \therefore (1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$ i.e. $y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$

$\therefore \frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q$

$\therefore \frac{dv}{dx} + (1-n)Pv = (1-n)Q$

$\therefore \frac{dv}{dx} + P_1v = Q_1$

Which is linear differential equation, where $P_1 = (1-n)P$ and $Q_1 = (1-n)Q$

$\therefore$ I.F. $= e^{\int P_1dx}$

$\therefore$ General solution of given equaion is

$$ve^{\int P_1dx} = \int e^{\int P_1dx}Q_1dx + c$$

i.e. $y^{1-n}e^{(1-n)\int Pdx} = \int e^{(1-n)\int Pdx}(1-n)Qdx + c$

==

Remark: 1) Bernoulli's differential equation may be is of type $\frac{dx}{dy} + Px = Qx^n$

where P and Q are functions of y alone.

2) If given differential equation is of type $f'(y)\frac{dy}{dx} + Pf(y) = Q$

where P and Q are functions of x alone, then to reduce it into lineardifferential equation by putting f(y) = v and then solve.

3) If given differential equation is of type $f'(x)\frac{dx}{dy} + Pf(x) = Q$ where P and Q are functions of y alone, then to reduce it into lineardifferential equation by putting f(x) = v and then solve.

==

Ex. Solve $\frac{dy}{dx} + xy = x^3y^3$

Solution: Let $\frac{dy}{dx} + xy = x^3y^3$ be the given differential equation, which is in the form of Bernoulli's equation.

Multiplying equation (1) by y^{-3} we get,

$$y^{-3}\frac{dy}{dx} + xy^{-2} = x^3$$

Put $y^{-2} = v$ $\therefore -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$ i.e. $y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx}$

$\therefore \frac{-1}{2}\frac{dv}{dx} + xv = x^3$

$\therefore \frac{dv}{dx} - 2xv = -2x^3$

Which is linear differential equation, with P = -2x and Q= $-2x^3$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int(-2x)dx} = e^{-x^2}$

$\therefore$ General solution of given equaion is

$$ve^{\int Pdx} = \int e^{\int Pdx} Q dx + c$$

i.e. $y^{-2}\, e^{-x^2} = \int e^{-x^2}(-2x^3)dx + c$

$= \int e^{-x^2}(-x^2)(2x dx) + c$

In integration put $-x^2 = t$ $\therefore$ $-2xdx= dt$ $\therefore$ $2xdx= -dt$

$\therefore y^{-2}\, e^{-x^2} = \int e^t . t(-dt) + c$

$= -\int e^t . tdt + c$

$= -[te^t - \int e^t dt] + c$

$= -[te^t - e^t] + c$

$= -e^t(t - 1) + c$

$= e^t(1-t) + c$

$\therefore y^{-2}\, e^{-x^2} = e^{-x^2}(1+x^2) + c$

i.e. $1 = (1+x^2)\, y^2 + cy^2e^{x^2}$

==

Ex. Solve $xy - \frac{dy}{dx} = y^3e^{-x^2}$

Solution: Let $xy - \frac{dy}{dx} = y^3e^{-x^2}$ i.e. $\frac{dy}{dx} - xy = -y^3e^{-x^2}$ be the given differential equation, which is in the form of Bernoulli's equation.

Multiplying equation (1) by y^{-3} we get,

$$y^{-3}\frac{dy}{dx} - xy^{-2} = -e^{-x^2}$$

Put $y^{-2} = v$ $\quad \therefore -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$ $\quad$ i.e.$y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx}$

$\therefore \frac{-1}{2}\frac{dv}{dx} - xv = -e^{-x^2}$

$\therefore \frac{dv}{dx} + 2xv = 2e^{-x^2}$

Which is linear differential equation, with P = 2x and Q= $2e^{-x^2}$

$\therefore$ I.F. $= e^{\int Pdx} = e^{\int(2x)dx} = e^{x^2}$

$\therefore$ General solution of given equaion is

$$ve^{\int Pdx} = \int e^{\int Pdx}Qdx + c$$

i.e. $y^{-2}\, e^{x^2} = \int e^{x^2}(2e^{-x^2})dx + c$

$\qquad = 2x + c$

$\therefore\ e^{x^2} = 2xy^2 + cy^2$

==

Ex. Solve $\frac{dy}{dx}$ - ytanx+y^2secx = 0

Solution: Let $\frac{dy}{dx}$ - ytanx+y^2secx = 0 i.e. $\frac{dy}{dx}$ - ytanx = -y^2secx be the given differential equation, which is in the form of Bernoulli's equation. Multiplying equation (1) by y^{-2} we get,

$$y^{-2}\frac{dy}{dx} - y^{-1}\tan x = -\sec x$$

Put $y^{-1} = v$ $\quad \therefore -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$ $\quad$ i.e.$y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$

$\therefore -\frac{dv}{dx} - v\tan x = -\sec x$

$\therefore \frac{dv}{dx} + v\tan x = \sec x$

Which is linear differential equation, with P = tanx and Q = secx

$\therefore$ I.F. $= e^{\int Pdx} = e^{\int(\tan x)dx} = e^{log secx} = \sec x$

$\therefore$ General solution of given equation is

$$ve^{\int Pdx} = \int e^{\int Pdx}Qdx + c$$

i.e. y^{-1} secx= $\int \sec x(\sec x)dx + c$

$\qquad = \int \sec^2 x dx + c$

$\therefore\ y^{-1}$ secx = tanx + c

$\therefore$ secx = (tanx + c)y

==

Ex. Solve tany $\frac{dy}{dx}$ + tanx = cosy $\cos^2$x

Solution: Let tany $\frac{dy}{dx}$ + tanx = cosy$\cos^2$x i.e.secytany $\frac{dy}{dx}$ + secytanx = $\cos^2$x be the given differential equation in the form f'(y)$\frac{dy}{dx}$ + Pf(y) = Q with f(y) = secy.

$\therefore$ Put f(y) = v i.e. secy = v $\qquad \therefore$ secytany $\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dv}{dx}$ + vtanx = $\cos^2$x

Which is linear differential equation in v and x, with P = tanx and Q = $\cos^2 x$

∴ I.F. $= e^{\int Pdx} = e^{\int \tan x dx} = e^{\log secx}$ = secx

∴ General solution of given equaion is

$ve^{\int Pdx} = \int e^{\int Pdx} Qdx + c$

i.e. $secy$ secx $= \int$ secx$(\cos^2 x)$dx + c

$= \int$ cosxdx + c

∴ $secy$ secx = sinx + c

==

Ex. Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Solution: Let $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ i.e. $\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$

be the given differential equation in the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$

with f(y) = siny.

∴ Put f(y) = v i.e. siny = v   ∴ $\cos y \frac{dy}{dx} = \frac{dv}{dx}$

∴ $\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^x$

Which is linear differential equation in v and x, with $P = \frac{-1}{1+x}$ and $Q = (1+x)e^x$

∴ I.F. $= e^{\int Pdx} = e^{\int \frac{-1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{(1+x)}$

∴ General solution of given equaion is

$ve^{\int Pdx} = \int e^{\int Pdx} Qdx + c$

i.e. $\frac{\sin y}{(1+x)} = \int \frac{1}{(1+x)}(1+x)e^x dx + c$

$= \int e^x dx + c$

∴ $\frac{\sin y}{(1+x)} = e^x + c$

==

MULTIPLE CHOICE QUESTIONS

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1) If $\lim_{h\to 0} \frac{f(x+h, y)-f(x, y)}{h}$ is exists, then it is denoted by …

A) $f_x(x, y)$ B) $f_y(x, y)$ C) $f_x(a, b)$ D) $f_y(a, b)$

2) If $\lim_{k\to 0} \frac{f(x, y+k)-f(x, y)}{k}$ is exists, then it is denoted by …

A) $f_x(x, y)$ B) $f_y(x, y)$ C) $f_x(a, b)$ D) $f_y(a, b)$

3) Partial derivative of f(x, y) w.r.t.x at point (a, b) is given by $f_x(a, b)$ = …

A) $\lim_{k\to 0} \frac{f(a, b+k)-f(a, b)}{k}$ B) $\lim_{h\to 0} \frac{f(a+h, b)-f(a, b)}{k}$

C) $\lim_{h\to 0} \frac{f(a+h, b)-f(a, b)}{h}$ D) $\lim_{k\to 0} \frac{f(a, b+k)-f(a, b)}{h}$

4) Partial derivative of f(x, y) w.r.t.y at point (a, b) is given by $f_y(a, b)$ = …

A) $\lim_{k\to 0} \frac{f(a, b+k)-f(a, b)}{k}$ B) $\lim_{h\to 0} \frac{f(a+h, b)-f(a, b)}{h}$

C) $\lim_{h \to 0} \frac{f(a+h,\ b)-f(a,\ b)}{k}$ D) $\lim_{k \to 0} \frac{f\ (a,\ b+k)-f\ (a,\ b)}{h}$

5) If u = e^x sinxy then $\frac{\partial u}{\partial x}$ at (0, 0) is …

A) -1 B) 1 C) 0 D) $\frac{\pi}{2}$

6) If u = e^x sinxy then $\frac{\partial u}{\partial y}$ at (0, 0) is …

A) -1 B) 1 C) 0 D) $\frac{\pi}{2}$

7) If u = $x^2y + y^2z + z^2x$ then $\frac{\partial u}{\partial x}$ at (1, 1, 1) is …

A) 5 B) 4 C) 3 D) 2

8) If u = $x^2y + y^2z + z^2x$ then $\frac{\partial u}{\partial y}$ at (1, 1, 1) is …

A) 5 B) 4 C) 3 D) 2

9) If u = $x^2y + y^2z + z^2x$ then $\frac{\partial u}{\partial z}$ at (1, 1, 1) is …

A) 5 B) 4 C) 3 D) 2

10) If u = $x^3z + y^2x - 2yz$ then $\frac{\partial u}{\partial x}$ at (1, 2, 3) is …

A) 11 B) 12 C) 13 D) 14

11) If u = $x^3z + y^2x - 2yz$ then $\frac{\partial u}{\partial y}$ at (1, 2, 3) is …

A) 1 B) -2 C) 13 D) 4

12) If u = $x^3z + y^2x - 2yz$ then $\frac{\partial u}{\partial z}$ at (1, 2, 3) is …

A) -3 B) -2 C) 13 D) 4

13) If u = xy + e^x then $\frac{\partial u}{\partial x}$ is …

A) xy + e^x B) y + e^x C) x D) 0

14) If u = xy + e^x then $\frac{\partial u}{\partial y}$ is …

A) xy + e^x B) y + e^x C) x D) 0

15) If u = xy + e^x then $\frac{\partial^2 u}{\partial x^2}$ is …

A) xy + e^x B) y + e^x C) e^x D) 0

16) If u = xy + e^x then $\frac{\partial^2 u}{\partial y^2}$ is …

A) 0 B) y + e^x C) x D) y

17) If u = xy + e^x then $\frac{\partial^2 u}{\partial x \partial y}$ is …

A) 0 B) 1 C) x D) y

18) If u = xy + e^x then $\frac{\partial^2 u}{\partial y \partial x}$ is …

A) 0 B) 1 C) y+e^x D) e^x

19) If u = log(tanx + tany + tanz) then sin2x $\frac{\partial u}{\partial x}$ + sin2y $\frac{\partial u}{\partial y}$ + sin2z $\frac{\partial u}{\partial z}$ = …

A) -1 B) 0 C) 1 D) 2

20) If $u = (x^2 + y^2 + z^2)^{-1/2}$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \ldots$

A) -u B) u C) 0 D) 1

21) If $u = \log(x^3 + y^3 - x^2y - xy^2)$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \ldots$

A) $\frac{-2}{x+y}$ B) $\frac{2}{x+y}$ C) $\frac{1}{x+y}$ D) 0

22) If M and N are the functions of variables x, y then a differential equation Mdx + Ndy = 0 is called …… differential equation

A) first order and first degree B) first order and higher degree

C) second order and first degree D) None of these

23) If M and N are the homogeneous functions of variables x, y of same degree then a differential equation Mdx + Ndy = 0 is called …… differential equation

A) non-homogeneous B) exact

C) homogeneous D) None of these

24) A differential equation Mdx + Ndy = 0 is exact, if there exist function u(x, y) such that ….

A) Mdx + Ndy = x B) Mdx + Ndy = y

C) Mdx + Ndy = u D) Mdx + Ndy = du

25) A differential equation Mdx + Ndy = 0 is exact if …….

A) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ B) $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$ C) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial x}$ D) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

26) A differential equation Mdx + Ndy = 0 is homogeneous differential equation then I.F. is ….

A) $\frac{1}{Mx-Ny}$ B) $\frac{1}{Mx+Ny}$ C) $\frac{1}{My-Nx}$ D) $\frac{1}{My+Nx}$

27) A differential equation Mdx + Ndy = 0 is of type $f_1(xy)ydx + f_2(xy)xdy = 0$ then I.F. is ….

A) $\frac{1}{Mx-Ny}$ B) $\frac{1}{Mx+Ny}$ C) $\frac{1}{My-Nx}$ D) $\frac{1}{My+Nx}$

28) If $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$ is a function of x alone, say f(x) then I.F. of equation Mdx + Ndy = 0 is ….

A) $e^{\int f(y)dy}$ B) $e^{\int f(x)dx}$ C) $e^{\int f(z)dz}$ D) $e^{\int f(x)dy}$

29) If $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$ is a function of y alone, say f(y) then I.F. of equation Mdx + Ndy = 0 is ….

A) $e^{\int f(y)dy}$ B) $e^{\int f(x)dx}$ C) $e^{\int f(z)dz}$ D) $e^{\int f(y)dx}$

30) A differential equation of type $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x alone is called …….

A) non-linear differential equation B) homogeneous differential equation

C) linear differential equation D) Bernoulli's equation

31) A differential equation of type $\frac{dx}{dy}$ + Px = Q where P and Q are functions of y alone is called …….

A) non-linear differential equation B) homogeneous differential equation
C) linear differential equation D) Bernoulli's equation

32) A differential equation of type $\frac{dy}{dx}$ + Py = Q.y^n where P and Q are functions of x alone is called …….

A) non-linear differential equation B) homogeneous differential equation
C) linear differential equation D) Bernoulli's equation

33) A differential equation of type $\frac{dx}{dy}$ + Px = Q.x^n where P and Q are functions of y alone is called …….

A) non-linear differential equation B) homogeneous differential equation
C) linear differential equation D) Bernoulli's equation

34) I.F. of a linear differential equation of type $\frac{dy}{dx}$ + Py = Q where P and Q are functions of x alone is …..

A) $e^{\int P\,dy}$ B) $e^{\int P\,dx}$ C) $e^{\int P\,dz}$ D) $e^{\int Q\,dx}$

35) I.F. of a linear differential equation of type $\frac{dx}{dy}$ + Px = Q where P and Q are functions of y alone is …..

A) $e^{\int P\,dy}$ B) $e^{\int P\,dx}$ C) $e^{\int P\,dz}$ D) $e^{\int Q\,dy}$

36) I.F. of differential equation (x + y)dx + (y - x) dy = 0 is …..

A) $\frac{1}{x^2+y^2}$ B) $\frac{1}{x^2-y^2}$ C) $\frac{1}{x+y}$ D) 1

37) I.F. of differential equation $\frac{dy}{dx}$ + 2y tanx = sinx is …..

A) $\sec^2 x$ B) logsecx C) tanx D) sinx

38) I.F. of differential equation $\frac{dy}{dx} + \frac{y}{x} = x^3$ is …..

A) x^3 B) x^2 C) x D) $\frac{1}{x}$

39) To solve a differential equation of type f'(y) $\frac{dy}{dx}$ + Pf(y) = Q where P and Q are functions of x alone, we put …..

A) f(x) = v B) f'(x) = v C) f(y) = v D) f'(y) = v

40) To solve a differential equation of type f'(x) $\frac{dx}{dy}$ + Pf(x) = Q where P and Q are functions of y alone, we put …..

A) f(x) = v B) f'(x) = v C) f(y) = v D) f'(y) = v

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UNIT-2: DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

Definition: An equation $F(x, y, p) = p^n + A_1p^{n-1} + A_2p^{n-2} + \ldots + A_{n-1}p + A_n = 0$ is called differential equations of first order and higher degree.

Where $A_1, A_2, \ldots, A_{n-1}, A_n$ are functions of x and y and $p = \frac{dy}{dx}$

Equation Solvable for p:

An equation $F(x, y, p) = p^n + A_1p^{n-1} + A_2p^{n-2} + \ldots + A_{n-1}p + A_n = 0$ is said to be solvable for p if it factorized into n linear factors.

Method of finding the solution of equation solvable for p:

Let an equation $F(x, y, p) = p^n + A_1p^{n-1} + A_2p^{n-2} + \ldots + A_{n-1}p + A_n = 0$ is solvable for p.

$\therefore$ F(x, y, p) is factorized into n linear factors say

$F(x, y, p) = (p-f_1)(p-f_2)\ldots\ldots(p-f_n)$ where $f_1, f_2, \ldots\ldots, f_n$ are functions of x, y

$\therefore$ From given equation, we have,

$(p-f_1)(p-f_2)\ldots\ldots(p-f_n) = 0$

$\Rightarrow p-f_1 = 0$ or $p-f_2 = 0$ or $\ldots\ldots$, $p-f_n = 0$

$\Rightarrow \frac{dy}{dx} - f_1 = 0, \frac{dy}{dx} - f_2 = 0, \ldots\ldots, \frac{dy}{dx} - f_n = 0$

$\Rightarrow \frac{dy}{dx} = f_1(x, y), \frac{dy}{dx} = f_2(x, y), \ldots\ldots, \frac{dy}{dx} = f_n(x, y)$

be the differential equations of first order and first degree.

Solving these equations, we get general solutions as

$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots.., \phi_n(x, y, c_n) = 0.$

As order of given equation is one,

$\therefore$ replace arbitrary constants $c_1, c_2, \ldots\ldots, c_n$ by single constant c.

$\phi_1(x, y, c) = 0, \phi_2(x, y, c) = 0, \ldots.., \phi_n(x, y, c_n) = 0$

be the general solutions of linear factors.

$\therefore$ General solution of given equation is

$\phi_1(x, y, c)\phi_2(x, y, c)\ldots\phi_n(x, y, c) = 0.$

Ex. Solve $p^2 - 8p + 12 = 0$

Solution: Let $p^2 - 8p + 12 = 0$

i.e. $(p - 2)(p - 6) = 0$

be the given differential equation, which is solvable for p

$\therefore p - 2 = 0$ or $p - 6 = 0$

i.e. $\frac{dy}{dx} - 2 = 0$ or $\frac{dy}{dx} - 6 = 0$

$\therefore dy = 2dx$ or $dy = 6dx$

Integrating, we get,

$y = 2x + c$ or $y = 6x + c$

i.e. 2x - y + c = 0 or 6x - y + c = 0

∴ The G. S. of given equation is

(2x - y + c) (6x - y + c) = 0

==

Ex. Solve $p^2 - 7p + 10 = 0$

Solution: Let $p^2 - 7p + 10 = 0$

i.e. (p - 2)(p - 5)=0

be the given differential equation, which is solvable for p

∴ p - 2 = 0 or p - 5 = 0

i.e. $\frac{dy}{dx} - 2 = 0$ or $\frac{dy}{dx} - 5 = 0$

∴ dy = 2dx or dy = 5dx

Integrating, we get,

y = 2x + c or y = 5x + c

i.e. 2x - y + c = 0 or 5x - y + c = 0

∴ The G. S. of given equation is

(2x - y + c) (5x - y + c) = 0

==

Ex. Solve p(p-y) = x(x+y)

Solution: Let p(p-y) = x(x+y)

i.e. $p^2 - py - x(x+y) = 0$

i.e. (p+x)(p-x-y) = 0

be the given differential equation, which is solvable for p

∴ p + x = 0 or p - x - y = 0

i.e. $\frac{dy}{dx} + x = 0$ or $\frac{dy}{dx} - x - y = 0$

i) Consider $\frac{dy}{dx} + x = 0$

∴ dy + xdx = 0

Integrating, we get,

$y + \frac{1}{2}x + c_1 = 0$

i.e. 2y + x + c = 0......(1) where $2c_1 = c$

ii) Consider $\frac{dy}{dx} - y = x$

Which is linear differential equation with P = -1 and Q = x having G.S.

$ye^{\int P dx} = \int e^{\int P dx} Q dx + c$

i.e. $ye^{\int (-1) dx} = \int e^{\int (-1) dx} x dx + c$

∴ $ye^{-x} = \int e^{-x} x dx + c$

∴ $ye^{-x} = -xe^{-x} - \int (-e^{-x}) dx + c$

∴ $ye^{-x} = -xe^{-x} - e^{-x} + c$

∴ $y = -x - 1 + ce^{x}$

∴ $x + y + 1 - ce^{x} = 0$ ...... (2)

From equation (1), (2) the G. S. of given equation is

$(x + 2y + c)(x + y + 1 - ce^{x}) = 0$

==

Ex. Solve $\frac{1}{p}$ - p = $\frac{y}{x}$ - $\frac{x}{y}$

Solution: Let $\frac{1}{p}$ - p = $\frac{y}{x}$ - $\frac{x}{y}$

i.e. $\frac{1-p^2}{p} = \frac{y^2-x^2}{xy}$

i.e. $\frac{p^2-1}{p} = \frac{x^2-y^2}{xy}$

i.e. $xyp^2-xy-(x^2-y^2)p = 0$

i.e. $xyp^2-x^2p+y^2p-xy = 0$

i.e. (yp-x)(xp+y) = 0

be the given differential equation, which is solvable for p

∴ yp - x = 0 or xp + y = 0

i) Consider yp - x = 0 i.e.y $\frac{dy}{dx} - x = 0$

∴ ydy - xdx = 0

Integrating, we get,

$\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1$

i.e. $y^2 - x^2 - c = 0$......(1) where $2c_1 = c$

ii) Consider xp + y = 0 i.e.x $\frac{dy}{dx}$ + y = 0

∴ $\frac{dy}{y} + \frac{dx}{x} = 0$

Integrating, we get,

logy + logx = logc

∴ log(xy) = logc

∴ xy = c

∴ xy - c = 0 ...... (2)

From equation (1), (2) the G. S. of given equation is

$(y^2 - x^2 - c)(xy - c) = 0$

==

Ex. Solve $x^2(\frac{dy}{dx})^2 + xy\frac{dy}{dx} - 6y^2 = 0$

Solution: Let $x^2(\frac{dy}{dx})^2 + xy\frac{dy}{dx} - 6y^2 = 0$

i.e. $x^2p^2 + xyp - 6y^2 = 0$

i.e. (xp-2y)(xp+3y) = 0

be the given differential equation, which is solvable for p

∴ xp - 2y = 0 or xp + 3y = 0

i) Consider xp - 2y = 0 i.e.x $\frac{dy}{dx} - 2y = 0$

∴ $\frac{dy}{y} - 2\frac{dx}{x} = 0$

Integrating, we get,

logy - 2 logx = logc

$\therefore$ log $\frac{y}{x^2}$ = logc

$\therefore$ $\frac{y}{x^2}$ = c

$\therefore$ y - cx^2 = 0 ...... (1)

ii) Consider xp + 3y = 0 i.e. x $\frac{dy}{dx}$ + 3y = 0

$\therefore$ $\frac{dy}{y}$ +3 $\frac{dx}{x}$ = 0

Integrating, we get,

logy +3logx = logc

$\therefore$ log(x^3y) = logc

$\therefore$ x^3y = c

$\therefore$ x^3y - c = 0 ...... (2)

From equation (1), (2) the G. S. of given equation is

(y - cx^2) (x^3y - c) = 0

==

Equation Solvable for y: An equation F(x, y, p) = 0, where p = $\frac{dy}{dx}$ is said to be solvable for y if it can be expressed as y = f(x, p).

==

Method of finding the solution of equation solvable for y:

Let an equation F(x, y, p) = 0 ...... (1) is solvable for y.

$\therefore$ it expressed as y = f(x, p) ...... (2)

Differentiating equation (2) w.r.t. x, we get,

$$\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{dp}{dx}$$

$$\therefore p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{dp}{dx} \text{ ...... (3)}$$

Equation (3) is the differential equation of first order and first degree in p and x. Solving it, we get general solution as

ϕ(x, p, c) = 0 ...... (4)

Eliminating p from given equations (1) and (4) we get required general solution of equation (1).

If elimination of p from given equations (1) and (4) is not possible, then equations (1) and (4) represent general solution of equation (1) with p as parameter.

==

Ex. Solve px - x^4 p^2 = -y

Solution: Let px - x^4 p^2 = -y i.e. y = -px + x^4 p^2 ...... (1)

be the given differential equation, which is solvable for y.

Differentiating equation (1) w.r.t. x, we get,

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

$$\therefore p = -p + 4x^3p^2 - x\frac{dp}{dx}(1 - 2x^3p)$$

$$\therefore 2p - 4x^3p^2 + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

$\therefore 2p(1 - 2x^3p) + x\frac{dp}{dx}(1 - 2x^3p) = 0$

$\therefore (1 - 2x^3p)(2p + x\frac{dp}{dx}) = 0$

We reject the factor $(1 - 2x^3p)$ which does not contain $\frac{dp}{dx}$.

$\therefore$ Consider $(2p + x\frac{dp}{dx}) = 0$

$\therefore 2\frac{dx}{x} + \frac{dp}{p} = 0$

Integrating, we get,

$2\log x + \log p = \log c$

$\therefore x^2p = c$

$\therefore p = \frac{c}{x^2}$ ...... (2)

Eliminating p from given equations (1) and (2), we get,

$y = -x(\frac{c}{x^2}) + x^4(\frac{c^2}{x^4})$ i.e. $y = -\frac{c}{x} + c^2$

be the required general solution of equation (1).

==

Ex. Solve $y = 2px + x^2p^4$

Solution: Let $y = 2px + x^2p^4$ ...... (1)

be the given differential equation, which is solvable for y.

Differentiating equation (1) w.r.t. x, we get,

$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx}$

$\therefore 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx} = p$

$\therefore p + 2xp^4 + 2x\frac{dp}{dx} + 4x^2p^3\frac{dp}{dx} = 0$

$\therefore p(1 + 2xp^3) + 2x\frac{dp}{dx}(1 + 2xp^3) = 0$

$\therefore (1 + 2xp^3)(p + 2x\frac{dp}{dx}) = 0$

We reject the factor $(1 + 2xp^3)$ which does not contain $\frac{dp}{dx}$.

$\therefore$ Consider $(p + 2x\frac{dp}{dx}) = 0$

$\therefore \frac{dx}{x} + 2\frac{dp}{p} = 0$

Integrating, we get,

$\log x + 2\log p = \log c$

$\therefore xp^2 = c$

$\therefore p^2 = \frac{c}{x}$

$\therefore p = \sqrt{\frac{c}{x}}$ ...... (2)

Eliminating p from given equations (1) and (2), we get,

$y = 2x\sqrt{\frac{c}{x}} + x^2(\frac{c^2}{x^2})$

i.e. $y = 2\sqrt{cx} + c^2$ be the required general solution of equation (1).

==

Ex. Solve y - 2px = f(xp²)

Solution: Let $y - 2px = f(xp^2)$ i.e. $y = 2px + f(xp^2)$ …… (1)

be the given differential equation, which is solvable for y.

Differentiating equation (1) w.r.t. x, we get,

$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + f'(xp^2).[p^2 + 2xp\frac{dp}{dx}]$

$\therefore 2p + 2x\frac{dp}{dx} + f'(xp^2).[p^2 + 2xp\frac{dp}{dx}] = p$

$\therefore p + p^2 f'(xp^2) + 2x\frac{dp}{dx} + 2xp\, f'(xp^2)\frac{dp}{dx} = 0$

$\therefore p[1 + pf'(xp^2)] + 2x\frac{dp}{dx}[1 + pf'(xp^2)] = 0$

$\therefore [1 + pf'(xp^2)](p + 2x\frac{dp}{dx}) = 0$

We reject the factor $[1 + pf'(xp^2)]$ which does not contain $\frac{dp}{dx}$.

$\therefore$ Consider $(p + 2x\frac{dp}{dx}) = 0$

$\therefore \frac{dx}{x} + 2\frac{dp}{p} = 0$

Integrating, we get,

logx + 2logp = logc

$\therefore xp^2 = c$

$\therefore p^2 = \frac{c}{x}$

$\therefore p = \sqrt{\frac{c}{x}}$ …… (2)

Eliminating p from given equations (1) and (2), we get,

$y = 2x\sqrt{\frac{c}{x}} + f(x.\frac{c}{x})$

i.e. $y = 2\sqrt{cx} + f(c)$ be the required general solution of equation (1).

==

Ex. Solve $y + p^2 = 2px$

Solution: Let $y + p^2 = 2px$ i.e. $y = 2px - p^2$ …… (1)

be the given differential equation, which is solvable for y.

Differentiating equation (1) w.r.t. x, we get,

$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx}$

$\therefore 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} = p$

$\therefore p + 2\frac{dp}{dx}(x - p) = 0$

$\therefore p\frac{dx}{dp} + 2x - 2p = 0$

$\therefore p\frac{dx}{dp} + 2x = 2p$

$\therefore \quad \frac{dx}{dp} + \frac{2}{p}x = 2$

Which is linear differential equation in x and p with $P = \frac{2}{p}$ and Q = 2.

∴ It's G. S. is

$xe^{\int P dp} = \int e^{\int P dp} Q dp + c$

i.e. $xe^{\int(\frac{2}{p})dp} = \int e^{\int(\frac{2}{p})dp} 2dp + c$

$\therefore xe^{2\log p} = 2\int e^{2\log p} dp + c$

$\therefore xp^2 = 2\int p^2 dp + c$

$\therefore xp^2 = \frac{2}{3}p^3 + c$ ...... (2)

Elimination of p from given equations (1) and (2) is not possible.

∴ equations (1) and (2) represents G. S. of equation (1) with p as parameter.

==

Ex. Solve $y = (1+p)x + e^p$

Solution: Let $y = (1+p)x + e^p$ …… (1)

be the given differential equation, which is solvable for y.

Differentiating equation (1) w.r.t. x, we get,

$\frac{dy}{dx} = (1+p) + x\frac{dp}{dx} + e^p\frac{dp}{dx}$

$\therefore 1 + p + \frac{dp}{dx}(x + e^p) = p$

$\therefore 1 + \frac{dp}{dx}(x + e^p) = 0$

$\therefore \frac{dx}{dp} + (x + e^p) = 0$

$\therefore \frac{dx}{dp} + x = -e^p$

Which is linear differential equation in x and p with P = 1 and $Q = -e^p$.

∴ It's G. S. is

$xe^{\int P dp} = \int e^{\int P dp} Q dp + c$

i.e. $xe^{\int(1)dp} = \int e^{\int(1)dp}(-e^p)dp + c$

$\therefore xe^p = -\int e^p e^p dp + c$

$\therefore xe^p = -\int e^{2p} dp + c$

$\therefore xe^p = -\frac{1}{2}e^{2p} + c$ ...... (2)

Eliminating p from given equations (1) and (2) is not possible.

∴ equations (1) and (2) represents G. S. of equation (1) with p as parameter.

==

Equation Solvable for x: An equation F(x, y, p) = 0, where $p = \frac{dy}{dx}$ is said to be solvable for x if it can be expressed as x = f(y, p).

==

Method of finding the solution of equation solvable for x:

Let an equation F(x, y, p) = 0 …… (1) is solvable for x.

∴ it expressed as x = f(y, p) …… (2)

Differentiating equation (2) w.r.t. y, we get,

$\frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} . \frac{dp}{dy}$

$\therefore \frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} . \frac{dp}{dy}$ …… (3)

Equation (3) is the differential equation of first order and first degree in p and y. Solving it, we get general solution as

$\phi(y, p, c) = 0$ …… (4)

Eliminating p from given equations (1) and (4) we get required general solution of equation (1).

If elimination of p from given equations (1) and (4) is not possible, then equations (1) and (4) represent general solution of equation (1) with p as parameter.

==

Ex. Solve $y = 2px + yp^2$

Solution: Let $y = 2px + yp^2$ i.e. $2px = y - yp^2$ i.e. $2x = \frac{y}{p} - yp$ …… (1)

be the given differential equation, which is solvable for x.

Differentiating equation (1) w.r.t. y, we get,

$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - p - y\frac{dp}{dy}$

$\therefore \frac{2}{p} = \frac{1}{p} - p - y\frac{dp}{dy}(\frac{1}{p^2} + 1)$

$\therefore \frac{1}{p} + p + y\frac{dp}{dy}(\frac{1}{p^2} + 1) = 0$

$\therefore p(\frac{1}{p^2} + 1) + y\frac{dp}{dy}(\frac{1}{p^2} + 1) = 0$

$\therefore (\frac{1}{p^2} + 1)(p + y\frac{dp}{dy}) = 0$

We reject the factor $(\frac{1}{p^2} + 1)$ which does not contain $\frac{dp}{dy}$.

∴ Consider $(p + y\frac{dp}{dy}) = 0$

$\therefore \frac{dy}{y} + \frac{dp}{p} = 0$

Integrating, we get,

$\log y + \log p = \log c$

$\therefore yp = c$

$\therefore p = \frac{c}{y}$ …… (2)

Eliminating p from given equations (1) and (2), we get,

$y = 2x(\frac{c}{y}) + y(\frac{c^2}{y^2})$

i.e. $y^2 = 2cx + c^2$ is the required general solution of equation (1).

==

Ex. Solve $p^3 - 4xyp + 8y^2 = 0$

Solution: Let $p^3 - 4xyp + 8y^2 = 0$ i.e. $p^3 + 8y^2 = 4xyp$ i.e. $4x = \frac{8y}{p} + \frac{p^2}{y}$ …… (1)

be the given differential equation, which is solvable for x.

Differentiating equation (1) w.r.t. y, we get,

$$4\frac{dx}{dy} = \frac{8}{p} - \frac{8y}{p^2}\frac{dp}{dy} - \frac{p^2}{y^2} + \frac{2p}{y}\frac{dp}{dy}$$

$$\therefore \frac{4}{p} = \frac{8}{p} - \frac{p^2}{y^2} - \frac{8y}{p^2}\frac{dp}{dy} + \frac{2p}{y}\frac{dp}{dy}$$

$$\therefore \frac{4}{p} - \frac{p^2}{y^2} - \frac{8y}{p^2}\frac{dp}{dy} + \frac{2p}{y}\frac{dp}{dy} = 0$$

$$\therefore \left(\frac{4}{p} - \frac{p^2}{y^2}\right) - \frac{2y}{p}\frac{dp}{dy}\left(\frac{4}{p} - \frac{p^2}{y^2}\right) = 0$$

$$\therefore \left(\frac{4}{p} - \frac{p^2}{y^2}\right)\left(1 - \frac{2y}{p}\frac{dp}{dy}\right) = 0$$

We reject the factor $\left(\frac{4}{p} - \frac{p^2}{y^2}\right)$ which does not contain $\frac{dp}{dy}$.

$\therefore$ Consider $1 - \frac{2y}{p}\frac{dp}{dy} = 0$

$\therefore \frac{dy}{y} - 2\frac{dp}{p} = 0$

$\therefore 2\frac{dp}{p} - \frac{dy}{y} = 0$

Integrating, we get,

$2\log p - \log y = \log c$

$\therefore \frac{p^2}{y} = c$

$\therefore p^2 = cy$

$\therefore p = \sqrt{cy}$ …… (2)

Eliminating p from given equations (1) and (2), we get,

$(\sqrt{cy})^3 - 4xy\sqrt{cy} + 8y^2 = 0$

i.e. $c\sqrt{c} - 4x\sqrt{c} + 8\sqrt{y} = 0$ is the required general solution of equation (1).

==

Ex. Solve $4(xp^2 + yp) = y^4$

Solution: Let $4(xp^2 + yp) = y^4$ i.e. $4xp^2 = y^4 - 4yp$ i.e. $4x = \frac{y^4}{p^2} - \frac{4y}{p}$ …… (1)

be the given differential equation, which is solvable for x.

Differentiating equation (1) w.r.t. y, we get,

$$4\frac{dx}{dy} = \frac{4y^3}{p^2} - \frac{2y^4}{p^3}\frac{dp}{dy} - \frac{4}{p} + \frac{4y}{p^2}\frac{dp}{dy}$$

$$\therefore \frac{4}{p} = -\frac{4}{p} + \frac{4y^3}{p^2} - \frac{2y^4}{p^3}\frac{dp}{dy} + \frac{4y}{p^2}\frac{dp}{dy}$$

$$\therefore \frac{8}{p} - \frac{4y^3}{p^2} + \frac{2y^4}{p^3}\frac{dp}{dy} - \frac{4y}{p^2}\frac{dp}{dy} = 0$$

$$\therefore 2\left(\frac{4}{p} - \frac{2y^3}{p^2}\right) - \frac{y}{p}\frac{dp}{dy}\left(\frac{4}{p} - \frac{2y^3}{p^2}\right) = 0$$

$$\therefore \left(\frac{4}{p} - \frac{2y^3}{p^2}\right)\left(2 - \frac{y}{p}\frac{dp}{dy}\right) = 0$$

We reject the factor $(\frac{4}{p} - \frac{2y^3}{p^2})$ which does not contain $\frac{dp}{dy}$.

∴ Consider $2 - \frac{y}{p}\frac{dp}{dy} = 0$

∴ $2\frac{dy}{y} - \frac{dp}{p} = 0$

∴ $\frac{dp}{p} - 2\frac{dy}{y} = 0$

Integrating, we get,

logp - 2logy = logc

∴ $\frac{p}{y^2} = c$

∴ p = cy^2 ...... (2)

Eliminating p from given equations (1) and (2), we get,

$4(xc^2y^4 + cy^3) = y^4$

i.e. 4c(cxy + 1) = y is the required general solution of equation (1).

==

Ex. Solve $(\frac{dy}{dx})^2 - 2x\frac{dy}{dx} + y = 0$

Solution: Let $(\frac{dy}{dx})^2 - 2x\frac{dy}{dx} + y = 0$

i.e. $p^2 - 2xp + y = 0$

i.e. $p^2 + y = 2xp$

i.e. $2x = p + \frac{y}{p}$

be the given differential equation, which is solvable for x.

Differentiating equation (1) w.r.t. y, we get,

$2\frac{dx}{dy} = \frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$

∴ $\frac{2}{p} = \frac{1}{p} + \frac{dp}{dy}(1 - \frac{y}{p^2})$

∴ $\frac{1}{p} - \frac{dp}{dy}(1 - \frac{y}{p^2}) = 0$

∴ $\frac{dy}{dp} - p(1 - \frac{y}{p^2}) = 0$

∴ $\frac{dy}{dp} - p + \frac{y}{p} = 0$

∴ $\frac{dy}{dp} + \frac{y}{p} = p$

Which is linear differential equation in y and p with P = $\frac{1}{p}$ and Q = p

∴ It's G. S. is

$ye^{\int P dp} = \int e^{\int P dp} Q dp + c$

i.e. $ye^{\int (\frac{1}{p}) dp} = \int e^{\int (\frac{1}{p}) dp}(p) dp + c$

∴ $ye^{\log p} = \int e^{\log p} p dp + c$

∴ $yp = \int p^2 dp + c$

∴ $yp = \frac{1}{3}p^3 + c$ ...... (2)

Eliminating p from given equations (1) and (2) is not possible.

∴ equations (1) and (2) represents G. S. of equation (1) with p as parameter.

Clairaut's Equation: A differential equation of type y = px + f(p), where p = $\frac{dy}{dx}$ is said to be Clairut's equation.

==

Method of solving the Clairaut's equation:

Let y = px + f(p) …… (1) be the Clairut's equation, where p = $\frac{dy}{dx}$

Which is solvable for y.

∴ Differentiating equation (1) w.r.t. x, we get,

$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

∴ $p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

∴ $\frac{dp}{dx}[x + f'(p)] = 0$

We reject the factor [x + f'(p)] which does not contain $\frac{dp}{dx}$.

∴ Consider $\frac{dp}{dx} = 0$

∴ dp = 0

Integrating, we get,

p = c...... (2)

Eliminating p from given equations (1) and (2), we get,

y = cx + f(c) is the required general solution of Clairaut's equation.

==

Remark: The G.S. of Clairaut's equation y = px + f(p) is obtained by putting p = c.

==

Ex. Solve $y = px + p - p^2$

Solution: Let $y = px + p - p^2$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + c - c^2$

==

Ex. Solve $y = px + \sqrt{4 + p^2}$

Solution: Let $y = px + \sqrt{4 + p^2}$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + \sqrt{4 + c^2}$

==

Ex. Solve $y = px + \sqrt{a^2p^2 + b}$

Solution: Let $y = px + \sqrt{a^2p^2 + b}$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + \sqrt{a^2c^2 + b}$

==

Ex. Solve $yp = a + xp^2$

Solution: Let $yp = a + xp^2$

i.e. $y = px + \frac{a}{p}$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + \frac{a}{c}$

==

Ex. Solve $y - a\sqrt{1+p^2} - px = 0$

Solution: Let $y - a\sqrt{1+p^2} - px = 0$

i.e. $y = px + a\sqrt{1+p^2}$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + a\sqrt{1+c^2}$

==

Ex. Solve $p = \cot(px - y)$

Solution: Let $p = \cot(px - y)$ i.e. $\cot^{-1}p = px - y$ i.e. $y = px - \cot^{-1}p$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx - \cot^{-1}c$

==

Ex. Solve $p = \sin(y - px)$

Solution: Let $p = \sin(y - px)$ i.e. $y - px = \sin^{-1}p$ i.e. $y = px + \sin^{-1}p$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + \sin^{-1}c$

==

Ex. Solve $p = \log(y - px)$

Solution: Let $p = \log(y - px)$ i.e. $y - px = e^p$ i.e. $y = px + e^p$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + e^c$

==

Ex. Solve $(y - px)(p - 1) + p = 0$

Solution: Let $(y - px)(p - 1) + p = 0$ i.e. $y - px = \frac{-p}{p-1}$ i.e. $y = px + \frac{p}{1-p}$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + \frac{c}{1-c}$

==

Ex. Solve cospx.cosy = p^2 - sinpx.siny

Solution: Let cospx.cosy = p^2 - sinpx.siny

i.e. cospx.cosy+ sinpx.siny = p^2

i.e. cos(y - px) = p^2

i.e. y - px = $\cos^{-1}p^2$

i.e. y = px + $\cos^{-1}p^2$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

y = cx + $\cos^{-1}c^2$

==

Ex. Solve sinpx.cosy - cospx.siny - p = 0

Solution: Let sinpx.cosy - cospx.siny - p = 0

i.e. sin(px - y) = p

i.e. px - y = $\sin^{-1}p$

i.e. y = px - $\sin^{-1}p$ …… (1)

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

y = cx - $\sin^{-1}c$

==

Ex. Solve $y = x(\frac{dy}{dx}) + (\frac{dy}{dx})^2$

Solution: Let $y = x(\frac{dy}{dx}) + (\frac{dy}{dx})^2$ i.e. $y = px + p^2$ ...... (1) where $p = \frac{dy}{dx}$

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

$y = cx + c^2$

==

Ex. Solve $y = x\frac{dy}{dx} + a(\frac{dy}{dx})(1 - \frac{dy}{dx})$

Solution: Let $y = x\frac{dy}{dx} + a(\frac{dy}{dx})(1 - \frac{dy}{dx})$ i.e. y = px + ap(1 - p)...... (1) where $p = \frac{dy}{dx}$

be the given differential equation, which is in Clairaut's form.

∴ It's G.S. is obtained by putting p = c in equation (1) as

y = cx + ac(1 - c)

==

Equations reducible to Clairaut's form: By using some proper substitution given differential equation can be reduced to Clairut's form and it's G.S. is obtained by putting p = c and resubstituting the values.

==

Ex. Solve $e^{4x}(p - 1) + e^{2y}p^2 = 0$ by putting $e^{2x} = u$ and $e^{2y} = v$.

Solution: Let $e^{4x}(p - 1) + e^{2y}p^2 = 0$...... (1)

be the given differential equation,

Putting $e^{2x} = u$ and $e^{2y} = v$, we get,

$2e^{2x}dx = du$ and $2e^{2y}dy = dv$

$\therefore \frac{2e^{2y}dy}{2e^{2x}dx} = \frac{dv}{du}$

$\therefore \frac{dy}{dx} = \frac{e^{2x}}{e^{2y}}\frac{dv}{du} = \frac{u}{v}\frac{dv}{du}$

i.e. $p = \frac{u}{v}P$ where $P = \frac{dv}{du}$

∴ Putting this values in (1), we get,

$u^2(\frac{u}{v}P - 1) + v(\frac{u}{v}P)^2 = 0$

i.e. $\frac{u^2}{v}(Pu - v) + \frac{u^2}{v}P^2 = 0$

i.e. $Pu - v + P^2 = 0$

i.e. $v = Pu + P^2$

which is in Clairaut's form.

∴ It's G.S. is obtained by putting P = c as

$v = cu + c^2$

$\therefore e^{2y} = ce^{2x} + c^2$ is the G. S. of equation (1).

==

Ex. Solve $(x - py)(px - y) = 2p$ by putting $x^2 = u$ and $y^2 = v$.

Solution: Let $(x - py)(px - y) = 2p$...... (1)

be the given differential equation,

Putting $x^2 = u$ and $y^2 = v$, we get,

$2xdx = du$ and $2ydy = dv$

$\therefore \frac{2ydy}{2xdx} = \frac{dv}{du}$

$\therefore \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$

i.e. $p = \frac{\sqrt{u}}{\sqrt{v}}P$ where $P = \frac{dv}{du}$

∴ Putting this values in (1), we get,

$(\sqrt{u} - \frac{\sqrt{u}}{\sqrt{v}}P\sqrt{v})(\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{u} - \sqrt{v}) = 2\frac{\sqrt{u}}{\sqrt{v}}P$

i.e. $\frac{\sqrt{u}}{\sqrt{v}}(1 - P)(Pu - v) = 2\frac{\sqrt{u}}{\sqrt{v}}P$

i.e. $(P - 1)(v - Pu) = 2P$

i.e. $v = Pu + \frac{2P}{P-1}$

which is in Clairaut's form.

∴ It's G.S. is obtained by putting P = c as

$v = cu + \frac{2c}{c-1}$

$\therefore y^2 = cx^2 + \frac{2c}{c-1}$ is the G. S. of equation (1).

==

Ex. Solve (x + py) (px - y) = λ^2p by putting x^2 = u and y^2 = v.

Solution: Let (x + py) (px - y) = λ^2p...... (1)

be the given differential equation,

Putting x^2 = u and y^2 = v, we get,

2xdx= du and 2ydy = dv

$\therefore \frac{2ydy}{2xdx} = \frac{dv}{du}$

$\therefore \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$

i.e. p = $\frac{\sqrt{u}}{\sqrt{v}}$ P where P = $\frac{dv}{du}$

∴ Putting this values in (1), we get,

$(\sqrt{u} + \frac{\sqrt{u}}{\sqrt{v}}P\sqrt{v})(\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{u} - \sqrt{v}) = \lambda^2\frac{\sqrt{u}}{\sqrt{v}}P$

i.e. $\frac{\sqrt{u}}{\sqrt{v}}(1 + P)(Pu - v) = \lambda^2\frac{\sqrt{u}}{\sqrt{v}}P$

i.e. (1 + P) (Pu - v) = λ^2P

i.e. Pu - v = $\frac{\lambda^2 P}{1+P}$

i.e. v = Pu - $\frac{\lambda^2 P}{1+P}$

which is in Clairaut's form.

∴ It's G.S. is obtained by putting P = c as

v = cu - $\frac{\lambda^2 c}{1+c}$

∴ $y^2 = cx^2 - \frac{\lambda^2 c}{1+c}$ is the G. S. of equation (1).

==

Ex. Solve xy(y - px) = (x + py), using x^2 = u and y^2 = v.

Solution: Let xy(y - px) = (x + py)...... (1)

be the given differential equation,

Putting x^2 = u and y^2 = v, we get,

2xdx= du and 2ydy = dv

$\therefore \frac{2ydy}{2xdx} = \frac{dv}{du}$

$\therefore \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$

i.e. p = $\frac{\sqrt{u}}{\sqrt{v}}$ P where P = $\frac{dv}{du}$

∴ Putting this values in (1), we get,

$\sqrt{u}\sqrt{v}(\sqrt{v} - \frac{\sqrt{u}}{\sqrt{v}}P\sqrt{u}) = (\sqrt{u} + \frac{\sqrt{u}}{\sqrt{v}}P\sqrt{v})$

i.e. $\sqrt{u}$(v - Pu) = $\sqrt{u}$(1+ P)

i.e. (v -Pu) = 1+P

i.e. v = Pu + 1+P

which is in Clairaut's form.

∴ It's G.S. is obtained by putting P = c as

v = cu + 1+ c

∴ $y^2 = cx^2$ + 1+ c is the G. S. of equation (1).

==

Ex. Solve $y^2 = pxy + f(p.\frac{y}{x})$, using $x^2 = u$ and $y^2 = v$.

Solution: Let $y^2 = pxy + f(p.\frac{y}{x})$...... (1)

be the given differential equation,

Putting $x^2 = u$ and $y^2 = v$, we get,

2xdx= du and 2ydy = dv

$\therefore \frac{2ydy}{2xdx} = \frac{dv}{du}$

$\therefore \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}$

i.e. $p = \frac{\sqrt{u}}{\sqrt{v}} P$ where $P = \frac{dv}{du}$

∴ Putting this values in (1), we get,

$v = \frac{\sqrt{u}}{\sqrt{v}} P\sqrt{u}\sqrt{v} + f(\frac{\sqrt{u}}{\sqrt{v}} P.\frac{\sqrt{v}}{\sqrt{u}})$

i.e. $v = Pu + f(P)$

which is in Clairaut's form.

∴ It's G.S. is obtained by putting P = c as

$v = cu + f(c)$

$\therefore y^2 = cx^2 + f(c)$ is the G. S. of equation (1).

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MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) If $A_1, A_2, \ldots, A_{n-1}, A_n$ are functions of x and y and $p = \frac{dy}{dx}$, then an equation $F(x, y, p) = p^n + A_1p^{n-1} + A_2p^{n-2} + \ldots + A_{n-1}p + A_n = 0$ is called differential equations of ......

A) first order and first degree B) first order and higher degree

C) higher order and first degree D) None of these

2) The differential equation F(x, y, p) = 0 is factorized into linear factors then it said to be ....

A) solvable for p B) solvable for y

C) solvable for x D) None of these

3) The differential equation $p^2 - 7p + 10 = 0$ is ....

A) solvable for x B) solvable for y

C) solvable for p D) None of these

4) The differential equation $p^2 - 8p + 12 = 0$ is ....

A) solvable for x B) solvable for p

C) solvable for y D) None of these

5) The differential equation $p^2 + 6p + 8 = 0$ is ....

A) solvable for x B) solvable for y

C) solvable for p D) None of these

==

6) The differential equation p(p - y) = x(x+y) is ……..

A) solvable for p B) solvable for y
C) solvable for x D) None of these

7) The differential equation $\frac{1}{p}$ - p = $\frac{y}{x}$ - $\frac{x}{y}$ is ……..

A) solvable for p B) solvable for y
C) solvable for x D) None of these

8) The differential equation $x^2 (\frac{dy}{dx})^2 - xy\frac{dy}{dx} - 6y^2 = 0$ is ……..

A) solvable for x B) solvable for y
C) solvable for p D) None of these

9) The differential equation F(x, y, p) = 0 is solvable for y if it can be expresed as

A) y = f(x, p) B) x = f(y, p) C) p = f(x, y) D) None of these

10) If the differential equation F(x, y, p) = 0 is expressed as y = f(x, p) then it said to be …..

A) solvable for p B) solvable for y
C) solvable for x D) None of these

11) The differential equation y = 2px + f(xp^2) is ……..

A) solvable for x B) solvable for y
C) solvable for p D) None of these

12) The differential equation y = 2px + x^2p^4 is ……..

A) solvable for p B) solvable for y
C) solvable for x D) None of these

13) The differential equation 4y = $x^2 + p^2$ is ……..

A) solvable for y B) solvable for x
C) solvable for p D) None of these

14) The differential equation y + p^2 = $2px^2$ is ……..

A) solvable for x B) solvable for y
C) solvable for p D) None of these

15) The differential equation F(x, y, p) = 0 is solvable for x if it can be expresed as

A) y = f(x, p) B) x = f(y, p) C) p = f(x, y) D) None of these

16) If the differential equation F(x, y, p) = 0 is expresed as x = f(y, p), then it said to be …..

A) solvable for p B) solvable for y
C) solvable for x D) None of these

17) The differential equation y = 2px + y^3p^2 is ……..

A) solvable for p B) solvable for y
C) solvable for x D) None of these

18) The differential equation y = 2px + y^2p^3 is ……..

A) solvable for x B) solvable for y
C) solvable for p D) None of these

==

19) The differential equation $p^3 - 4xyp + 8y^2 = 0$ is ……..

A) solvable for p B) solvable for x
C) solvable for y D) None of these

20) The differential equation $4(xp^2 + yp) = y^4$ is ……..

A) solvable for x B) solvable for y
C) solvable for p D) None of these

21) The differential equation $(\frac{dy}{dx})^2y^2 - 2x\frac{dy}{dx} + y = 0$ is ……..

A) solvable for x B) solvable for y
C) solvable for p D) None of these

22) The differential equation $\frac{1}{p} = \cot(x - \frac{p}{1+p^2})$ is ……..

A) solvable for p B) solvable for x
C) solvable for y D) None of these

23) The equation of type y = px + f(p), where $p = \frac{dy}{dx}$ is called ….

A) Clairaut's equation B) Linear equation
C) Bernoulli's equation D) None of these

24) The solution of Clairaut's equation y = px + f(p) is obtained by putting p = ….

A) y B) x C) c D) None of these

25) The solution of Clairaut's equation y = px + f(p) is ….

A) y = cx + f(c) B) $y = pc + e^c$ C) y = px - f(c) D) None of these

26) The solution of differential equation p = cot(px - y) is ……..

A) $y = cx - \cot^{-1}c$ B) y = cx - cotp C) y = cx + cotp D) None of these

27) The solution of differential equation p = log(y - px) is ……..

A) y = log(y - cx) B) $y = cx + e^c$ C) $y = pc - e^c$ D) None of these

28) The solution of differential equation (y - px) (p - 1) + p = 0 is ……..

A) $x = cy - \frac{c}{1-c}$ B) $y = cx - \frac{c}{1-c}$ C) $y = cx + \frac{c}{1-c}$ D) None of these

29) The solution of differential equation (y - px) (p - 1) = p is ……..

A) $y = cx + \frac{c}{c-1}$ B) $y = cx + \frac{c}{1-c}$ C) $y = cx + \frac{c}{1-c}$ D) None of these

30) The solution of differential equation $y - a\sqrt{1+p^2} - px = 0$ is ….

A) $y = a\sqrt{1+p^2} - cx$ B) $y = a\sqrt{1+p^2} - px$
C) $y = cx + a\sqrt{1+c^2}$ D) None of these

31) The solution of differential equation $yp = a + xp^2$ is ….

A) $yp = a + c^2x$ B) $y = \frac{a}{c} + cx$ C) $cy = a + xp^2$ D) None of these

32) The solution of differential equation $y = x(\frac{dy}{dx}) + (\frac{dy}{dx})^2$ is ….

A) $y = cx - c^2$ B) $y = -cx + c^2$ C) $y = cx + c^2$ D) None of these

33) The solution of differential equation $y = x\frac{dy}{dx} + a\frac{dy}{dx}(1 - \frac{dy}{dx})$ is ….

A) y = cx + ac(1- c) B) y = cx - ac(1- c)
C) y = -cx + ac(1- c) D) None of these

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UNIT-3: LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER AND HIGHER DEGREE

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Linear Differential Equation with Constant Coefficients: A differential equation of the form $\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + p_{n-1} \frac{dy}{dx} + p_n y = X$ i.e. $f(D)y = X$, where $D \equiv \frac{d}{dx}$; $p_1, p_2, \ldots, p_n$ are constants and X is a function of x only, is called a linear differential equation with constant co-efficients.

Associated Equation of the Linear Differential Equation: If $f(D)y = X$ is the linear differential equation with constant co-efficients, then $f(D)y = 0$ is called its associated equation.

Auxiliary Equation of the Linear Differential Equation: If $f(D)y = X$ is the linear differential equation with constant co-efficients, then $f(D) = 0$ is called its auxiliary equation (A.E.).

Complementary Function (C.F.): The part of G.S. which is solution of the associated equation $f(D)y = 0$ containing arbitrary constants is called Complementary Function (C.F.).

Particular Integral (P.I.) The part of G.S. which is solution of $f(D)y = X$ not involving arbitrary constants is called Particular Integral (P.I.) and denoted by P.I. $= \frac{1}{f(D)}X$

Remark: i) If $y = u$ is the Complementary Function (C.F.) and $y = v$ is Particular Integral (P.I.) of LDE $f(D)y = X$, then $y = u + v$ is the General Solution (G.S.) of it.

ii) If operator $D \equiv \frac{d}{dx}$, then 1) $D^r y = \frac{d^r y}{dx^r}$, 2) $D^r D^k = D^{r+k}$, 3) $[f(x)\, D^r]y = f(x).D^r y$

4) $[f(x) + g(x)]\, D^n = f(x).D^n + g(x).D^n$, 5) $f(x)[D^m + D^n] = f(x).D^m + f(x).D^n$

6) $f_1(D)$ and $f_2(D)$ be operational factors, then $[f_1(D).f_2(D)]y = f_1(D)[f_2(D)y]$

7) If in LDE $f(D)y = X$, $X = 0$, then P.I. = 0 i.e. G.S. = C.F.

8) $\frac{1}{f(D)}$ is called inverse operator of $f(D)$.

Process of Finding Complementary Function (C.F.):

i) If an A.E. $f(D) = 0$ of LDE $f(D)y = X$ has n distinct roots $m_1, m_2, m_3, \ldots, m_n$, then C.F.= $C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots + C_n e^{m_n x}$

ii) If an A.E. $f(D) = 0$ of LDE $f(D)y = X$ has root m, repeated k times, then C.F.= $(C_1 + C_2 x + C_3 x^2 + \ldots + C_k x^{k-1})e^{mx}$

iii) If an A.E. $f(D) = 0$ of LDE $f(D)y = X$ has complex roots $\alpha \pm i\beta$, then C.F.= $e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

iv) If an A.E. $f(D) = 0$ of LDE $f(D)y = X$ has complex roots $\alpha \pm i\beta$ occurs twice, then C.F.= $e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]$

==

Properties of Inverse operator $\frac{1}{f(D)}$:

i) $\frac{1}{f(D)}[C_1X_1 + C_2X_2] = C_1\frac{1}{f(D)}X_1 + C_2\frac{1}{f(D)}X_2$ where C_1 and C_2 are constants.

ii) $\frac{1}{(D-\alpha)(D-\beta)} = \frac{1}{(\alpha-\beta)}\left[\frac{1}{(D-\alpha)} - \frac{1}{(D-\beta)}\right]$

==

Ex.: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 7y = 0$

Solution: Let $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 7y = 0$

i.e. $(D^2 + 6D - 7)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^2 + 6D - 7 = 0$

i.e. $(D - 1)(D + 7) = 0$

$\therefore D = 1, -7$ are the distinct roots of an A.E.

$\therefore$ C.F. $= C_1e^{x} + C_2e^{-7x}$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^{x} + C_2e^{-7x}$

be the required G.S. of given equation.

==

Ex.: Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

Solution: Let $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

i.e. $(D^2 - 7D + 12)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^2 - 7D + 12 = 0$

i.e. $(D - 3)(D - 4) = 0$

$\therefore D = 3, 4$ are the distinct roots of an A.E.

$\therefore$ C.F. $= C_1e^{3x} + C_2e^{4x}$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^{3x} + C_2e^{4x}$

be the required G.S. of given equation.

==

Ex.: Solve $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$

Solution: Let $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$

i.e. $(2D^2 + 5D - 12)y = 0$ be the given LDE with constant coefficients, Its A.E. is $2D^2 + 5D - 12 = 0$

i.e. $2D^2 + 8D - 3D - 12 = 0$

i.e. $2D(D + 4) - 3(D + 4) = 0$

i.e. $(D + 4)(2D - 3) = 0$

$\therefore$ $D = -4, \frac{3}{2}$ are the distinct roots of an A.E.

$\therefore$ C.F. $= C_1e^{-4x} + C_2e^{\frac{3}{2}x}$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^{-4x} + C_2e^{\frac{3}{2}x}$

be the required G.S. of given equation.

==

Ex.: Find the complementary function of $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0$

Solution: Let $2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 0$

i.e. $(2D^2 + 3D - 2)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $2D^2 + 3D - 2 = 0$

i.e. $2D^2 + 4D - D - 2 = 0$

i.e. $2D(D + 2) - (D + 2) = 0$

i.e. $(D + 2)(2D - 1) = 0$

$\therefore$ $D = -2, \frac{1}{2}$ are the distinct roots of an A.E.

$\therefore$ C.F. $= C_1e^{-2x} + C_2e^{\frac{1}{2}x}$

==

Ex.: Solve $(D^3 + 3D^2 - D - 3)y = 0$

Solution: Let $(D^3 + 3D^2 - D - 3)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^3 + 3D^2 - D - 3 = 0$

i.e. $D^2(D + 3) - (D + 3) = 0$

i.e. $(D + 3)(D^2 - 1) = 0$

i.e. $(D + 3)(D - 1)(D + 1) = 0$

$\therefore$ $D = -3, 1, -1$ are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{-3x} + C_2e^{x} + C_3e^{-x}$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^{-3x} + C_2e^{x} + C_3e^{-x}$

be the required G.S. of given equation.

==

Ex.: Solve $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

Solution: Let $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

i.e. $(4D^2 - 4D + 1)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $4D^2 - 4D + 1 = 0$

i.e. $(2D - 1)^2 = 0$

$\therefore D = \frac{1}{2}, \frac{1}{2}$ (repeated two times) are the roots of an A.E.

$\therefore$ C.F. $= (C_1 + C_2x)e^{\frac{1}{2}x}$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F. i.e. $y = (C_1 + C_2x)e^{\frac{1}{2}x}$

be the required G.S. of given equation.

==

Ex.: Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$

Solution: Let $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$

i.e. $(D^3 + D^2 - D - 1)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^3 + D^2 - D - 1 = 0$

i.e. $D^2 (D + 1) - (D + 1) = 0$

i.e. $(D + 1) (D^2 - 1) = 0$

i.e. $(D - 1) (D + 1)^2 = 0$

$\therefore D = 1, D = -1, -1$ (repeated two times) are the roots of an A.E.

$\therefore$ C.F. $= C_1e^x + (C_2 + C_3x)e^{-x}$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^x + (C_2 + C_3x)e^{-x}$

be the required G.S. of given equation.

==

Ex.: Find the complementary function of $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$

Solution: Let $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x$

i.e. $(D^3 + 2D^2 + D)y = e^x$ be the given LDE with constant coefficients,

Its A.E. is $D^3 + 2D^2 + D = 0$

i.e. $D(D^2 + 2D + 1) = 0$ i.e. $D (D + 1)^2 = 0$

$\therefore D = 0, D = -1, -1$ (repeated two times) are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{0x} + (C_2 + C_2x)e^{-x}$

$= C_1 + (C_2 + C_3x)e^{-x}$

==

Ex.: Solve $(D-1)^2(D^2-1)y=0$

Solution: Let $(D-1)^2(D^2-1)y=0$ be the given LDE with constant coefficients,

Its A.E. is $(D-1)^2(D^2-1)=0$

i.e. $(D+1)(D-1)^3=0$

$\therefore D=-1,\ D=1,1,1$ (repeated three times) are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{-x}+(C_2+C_3x+C_4x^2)e^x$

Here X = 0 ∴ P.I. = 0.

∴ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^{-x}+(C_2+C_3x+C_4x^2)e^x$

be the required G.S. of given equation.

==

Ex.: Solve $(D-1)^2(D^2+1)y=0$

Solution: Let $(D-1)^2(D^2+1)y=0$ be the given LDE with constant coefficients,

Its A.E. is $(D-1)^2(D^2+1)=0$

$\therefore D=1,1,\pm i$ are the roots of an A.E.

$\therefore$ C.F. $=(C_1+C_2x)e^x+e^{0x}(C_3\cos x+C_4\sin x)$

$=(C_1+C_2x)e^x+C_3\cos x+C_4\sin x$

Here X = 0 ∴ P.I. = 0.

∴ G.S. = C.F.+ P.I. = C.F.

i.e. $y=(C_1+C_2x)e^x+C_3\cos x+C_4\sin x$

be the required G.S. of given equation.

==

Ex.: Solve $(D^2-6D+13)y=0$

Solution: Let $(D^2-6D+13)y=0$ be the given LDE with constant coefficients,

Its A.E. is $D^2-6D+13=0$

$\therefore D=\frac{6\pm\sqrt{36-52}}{2}=3\pm 2i$ are the roots of an A.E.

$\therefore$ C.F. $=e^{3x}(C_1\cos 2x+C_2\sin 2x)$

Here X = 0 ∴ P.I. = 0.

∴ G.S. = C.F.+ P.I. = C.F.

i.e. $y=e^{3x}(C_1\cos 2x+C_2\sin 2x)$

be the required G.S. of given equation.

==

Ex.: Solve $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

Solution: Let $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

i.e. $(D^3 - 2D^2 + 4D - 8)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^3 - 2D^2 + 4D - 8 = 0$

i.e. $D^2(D - 2) + 4(D - 2) = 0$

i.e. $(D - 2)(D^2 + 4) = 0$

$\therefore D = 2, \pm 2i$ are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{2x} + e^{0x}(C_2\cos 2x + C_3\sin 2x)$

$= C_1e^{2x} + C_2\cos 2x + C_3\sin 2x$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1e^{2x} + C_2\cos 2x + C_3\sin 2x$

be the required G.S. of given equation.

==

Ex.: Solve $(D^4 + 18D^2 + 81)y = 0$

Solution: Let $(D^4 + 18D^2 + 81)y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^4 + 18D^2 + 81 = 0$

i.e. $(D^2 + 9)^2 = 0$

$\therefore D = \pm 3i$ (repeated two times) are the roots of an A.E.

$\therefore$ C.F. $= e^{0x}[(C_1 + C_2x)\cos 3x + (C_3 + C_4x)\sin 3x]$

$= (C_1 + C_2x)\cos 3x + (C_3 + C_4x)\sin 3x$

Here X = 0 $\therefore$ P.I. = 0.

$\therefore$ G.S. = C.F.+ P.I. = C.F.

i.e. $y = (C_1 + C_2x)\cos 3x + (C_3 + C_4x)\sin 3x$

be the required G.S. of given equation.

==

Ex.: Solve $D^2(D^2 + 3)^2y = 0$

Solution: Let $D^2(D^2 + 3)^2y = 0$ be the given LDE with constant coefficients,

Its A.E. is $D^2(D^2 + 3)^2 = 0$

$\therefore D = 0, 0, \pm\sqrt{3}i, \pm\sqrt{3}i$ (repeated two times) are the roots of an A.E.

$\therefore$ C.F. $= (C_1 + C_2x)e^{0x} + e^{0x}[(C_3 + C_4x)\cos\sqrt{3}x + (C_5 + C_6x)\sin\sqrt{3}x]$

$= C_1 + C_2x + (C_3 + C_4x)\cos\sqrt{3}x + (C_5 + C_6x)\sin\sqrt{3}x$

Here X = 0 ∴ P.I. = 0.

∴ G.S. = C.F.+ P.I. = C.F.

i.e. $y = C_1 + C_2x + (C_3 + C_4x)\cos\sqrt{3}x + (C_5 + C_6x)\sin\sqrt{3}x$

be the required G.S. of given equation.

==

General Method of Finding P.I.:

Theorem: If $D \equiv \frac{d}{dx}$ and X is function of x, then $\frac{1}{D-m} X = e^{mx}\int Xe^{-mx}dx$

Proof: Let $y = \frac{1}{D-m} X \Rightarrow (D-m)y = X \Rightarrow \frac{dy}{dx} - my = X$ ……(1)

Which is linear differential equation of first order with P = -m and Q = X.

∴ $I.F. = e^{\int P\,dx} = e^{\int(-m)dx} = e^{-mx}$

G.S. of linear differential equation is

$y(I.F.) = \int (I.F)Q\,dx + c$

∴ $y(e^{-mx}) = \int (e^{-mx})X\,dx + c$

∴ $y = ce^{mx} + e^{mx}\int (e^{-mx})X\,dx$

As G.S. = C.F.+ P.I. and C.F. of equation (1) is C.F. = ce^{mx}

∴ $P.I. = e^{mx}\int (e^{-mx})X\,dx$

∴ $\frac{1}{D-m} X = e^{mx}\int Xe^{-mx}dx$

==

P.I. of Some Standard Functions:

Type-I: When $X = e^{ax}$ where a is constant.

Theorem: If $D \equiv \frac{d}{dx}$ and f(D) is polynomial in D with $f(a) \neq 0$, then $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$

Proof: Let $f(D)y = e^{ax}$ be a LDE with $f(D) = D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots + P_{n-1}D + P_n$

As $De^{ax} = ae^{ax}$, $D^2e^{ax} = a^2e^{ax}$, …, $D^re^{ax} = a^re^{ax}$ ∀ r ∈ N

∴ $f(D)e^{ax} = [D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots + P_{n-1}D + P_n]e^{ax}$

$= D^ne^{ax} + P_1D^{n-1}e^{ax} + P_2D^{n-2}e^{ax} + \ldots + P_{n-1}D\,e^{ax} + P_ne^{ax}$

$= a^ne^{ax} + P_1a^{n-1}e^{ax} + P_2a^{n-2}e^{ax} + \ldots + P_{n-1}ae^{ax} + P_ne^{ax}$

$= [a^n + P_1a^{n-1} + P_2a^{n-2} + \ldots + P_{n-1}a + P_n]e^{ax}$

∴ $f(D)e^{ax} = f(a)e^{ax}$

∴ $e^{ax} = \frac{f(D)e^{ax}}{f(a)}$ ∵ $f(a) \neq 0$

∴ $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ Hence proved.

==

Theorem: If $D \equiv \frac{d}{dx}$, then $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$

Proof: We prove $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$ by mathematical induction.

For r = 1, $\frac{1}{D-a} e^{ax} = e^{ax} \int e^{ax} e^{-ax} dx$

$= e^{ax} \int 1 dx$

$= xe^{ax}$

$= \frac{x^1}{1!} e^{ax}$

i.e. result is true for r = 1.

Suppose result is true for r = k

i.e. $\frac{1}{(D-a)^k} e^{ax} = \frac{x^k}{k!} e^{ax}$ …… (1)

Consider $\frac{1}{(D-a)^{k+1}} e^{ax} = \frac{1}{(D-a)} [\frac{1}{(D-a)^k} e^{ax}]$

$= \frac{1}{(D-a)} [\frac{x^k}{k!} e^{ax}]$ by (1)

$= e^{ax} \int (\frac{x^k}{k!} e^{ax}) e^{-ax} dx$

$= e^{ax} \int (\frac{x^k}{k!}) dx$

$= e^{ax} \frac{x^{k+1}}{k!(k+1)}$

$= \frac{x^{k+1}}{(k+1)!} e^{ax}$

i.e. result is true for r = k ⇒ result is true for r = k+1

∴ by mathematical induction the result is true for any natural number r.

i.e. $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \ \forall\ r \in N$.

==

Theorem: If $D \equiv \frac{d}{dx}$ and f(D) is polynomial in D with $f(D) = (D-a)^r \phi(D)$ and $\phi(a) \neq 0$,

then $\frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{x^r e^{ax}}{r! \phi(a)}$

Proof: Let $f(D) = (D-a)^r \phi(D)$ and $\phi(D) \neq 0$

Consider $\frac{1}{(D-a)^r \phi(D)} e^{ax} = \frac{1}{(D-a)^r} [\frac{1}{\phi(D)} e^{ax}]$

$= \frac{1}{(D-a)^r} [\frac{e^{ax}}{\phi(a)}]$ ∵ $\phi(a) \neq 0$

$= \frac{1}{\phi(a)} [\frac{1}{(D-a)^r} e^{ax}]$

$= \frac{1}{\phi(a)} [\frac{x^r}{r!} e^{ax}]$

$= \frac{x^r e^{ax}}{r! \phi(a)}$

==

Ex.: Find the particular integral of LDE $(D^2 - 3D + 2)y = e^{5x}$

Solution: Let $(D^2 - 3D + 2)y = e^{5x}$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 3D + 2 = (D - 1)(D - 2)$ and $X = e^{5x}$

Now P.I. $= \frac{1}{f(D)} X$

$= \frac{1}{(D-1)(D-2)} e^{5x}$

$= \frac{e^{5x}}{(5-1)(5-2)}$

$= \frac{e^{5x}}{12}$

==

Ex.: Find the particular integral of LDE $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 3e^x$

Solution: Let $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 3e^x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^3 - 5D^2 + 8D - 4 = (D - 1)(D - 2)^2$

and $X = e^{2x} + 3e^x$

Now P.I. $= \frac{1}{f(D)} X = \frac{1}{(D-1)(D-2)^2}(e^{2x} + 3e^x)$

$= \frac{1}{(D-1)(D-2)^2} e^{2x} + \frac{1}{(D-1)(D-2)^2} 3e^x$

$= \frac{x^2 e^{2x}}{2!(2-1)} + \frac{3xe^x}{1!(1-2)^2}$

$= \frac{1}{2} x^2 e^{2x} + 3xe^x$

==

Ex.: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{2x}$

Solution: Let $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{2x}$

i.e. $(D^2 + 4D + 4)y = e^{2x}$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 + 4D + 4 = (D + 2)^2$

and $X = e^{2x}$

∴ It's A.E. is f(D) = 0

i.e. $(D + 2)^2 = 0$

∴ $D = -2, -2$ (repeated two times) are the roots of an A.E.

$\therefore$ C.F. = $(C_1 + C_2x)e^{-2x}$

Now P.I. = $\frac{1}{f(D)}X$

$= \frac{1}{(D+2)^2}e^{2x}$

$= \frac{e^{2x}}{(2+2)^2}$

$= \frac{1}{16}e^{2x}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_1x)e^{-2x} + \frac{1}{16}e^{2x}$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 - 3D + 2)y = \cosh x$

Solution: Let $(D^2 - 3D + 2)y = \cosh x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 3D + 2 = (D - 1)(D - 2)$

and $X = \cosh x = \frac{e^x + e^{-x}}{2}$

$\therefore$ It's A.E. is f(D) = 0

i.e. (D - 1)(D - 2) = 0

$\therefore$ D = 1, 2 are the roots of an A.E.

$\therefore$ C.F. = $C_1 e^x + C_2e^{2x}$

Now P.I. = $\frac{1}{f(D)}X$

$= \frac{1}{(D-1)(D-2)}[\frac{e^x + e^{-x}}{2}]$

$= \frac{1}{2(D-1)(D-2)}e^x + \frac{1}{2(D-1)(D-2)}e^{-x}$

$= \frac{xe^x}{2(1!)(1-2)} + \frac{e^{-x}}{2(-1-1)(-1-2)}$

$= -\frac{1}{2}xe^x + \frac{1}{12}e^{-x}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1 e^x + C_2e^{2x} - \frac{1}{2}xe^x + \frac{1}{12}e^{-x}$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$

Solution: Let $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^3 + 3D^2 + 3D + 1 = (D + 1)^3$

and $X = e^{-x}$

∴ It's A.E. is f(D) = 0

i.e. $(D + 1)^3 = 0$

∴ $D = -1, -1, -1$ (repeated three times) are the roots of an A.E.

∴ C.F. = $(C_1 + C_2x + C_3x^2)e^{-x}$

Now P.I. = $\frac{1}{f(D)} X = \frac{1}{(D+1)^3} e^{-x}$

$= \frac{x^3 e^{-x}}{3!}$

$= \frac{1}{6} x^3 e^{-x}$

∴ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_2x + C_3x^2)e^{-x} + \frac{1}{6} x^3 e^{-x}$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^3 - 1)y = (e^x + 1)^2$

Solution: Let $(D^3 - 1)y = (e^x + 1)^2$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^3 - 1 = (D - 1)(D^2 + D + 1)$

and $X = (e^x + 1)^2 = e^{2x} + 2e^x + 1$

∴ It's A.E. is f(D) = 0

i.e. $(D - 1)(D^2 + D + 1) = 0$

∴ $D = 1$ or $D = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$ are the roots of an A.E.

∴ C.F. = $C_1e^x + e^{-\frac{1}{2}x}(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x)$

Now P.I. = $\frac{1}{f(D)} X$

$= \frac{1}{(D-1)(D^2 + D + 1)} (e^{2x} + 2e^x + 1)$

$= \frac{1}{(D-1)(D^2+D+1)} e^{2x} + \frac{2}{(D-1)(D^2+D+1)} e^{x} + \frac{1}{(D-1)(D^2+D+1)} e^{0x}$

$= \frac{e^{2x}}{(2-1)(4+2+1)} + \frac{2xe^{x}}{1!(1^2+1+1)} + \frac{e^{0x}}{(0-1)(0+0+1)}$

$= \frac{1}{7} e^{2x} + \frac{2}{3} xe^{x} - 1$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1e^x + e^{-\frac{1}{2}x}(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x) + \frac{1}{7} e^{2x} + \frac{2}{3} xe^{x} - 1$

be the required G.S. of given differential equation.

==

Type-II: When $X = x^m$ or polynomial in x.

Let $f(D)y = x^m$ be the given LDE with constant coefficients,

If g(D) is the lowest degree term in f(D), then

$f(D) = g(D).[1 \pm \phi(D)]$

∴ P.I. $= \frac{1}{f(D)} x^m = \frac{1}{g(D)[1\pm\phi(D)]} x^m = = \frac{1}{g(D)} [1 \pm \phi(D)]^{-1} x^m$

and use results

i) $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$

i.e. $\frac{1}{1+\phi(D)} = [1+\phi(D)]^{-1} = 1 - \phi(D) + [\phi(D)]^2 - [\phi(D)]^3 + \ldots\ldots$

ii) $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$

i.e. $\frac{1}{1+\phi(D)} = [1-\phi(D)]^{-1} = 1 + \phi(D) + [\phi(D)]^2 + [\phi(D)]^3 + \ldots\ldots$

==

Ex.: Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = x^2$

Solution: Let $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = x^2$

i.e. $(D^2 - 2D + 5)y = x^2$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 2D + 5$ and $X = x^2$

∴ It's A.E. is f(D) = 0

i.e. $D^2 - 2D + 5 = 0$

∴ $D = \frac{2\pm\sqrt{4-20}}{2} = 1 \pm 2i$ are the roots of an A.E.

$\therefore$ C.F. = $e^x(C_1\cos 2x + C_2\sin 2x)$

Now P.I. = $\frac{1}{f(D)}X = \frac{1}{D^2 - 2D + 5}x^2$

$= \frac{1}{5[1-(\frac{2}{5}D - \frac{1}{5}D^2)]}x^2$

$= \frac{1}{5}[1-(\frac{2}{5}D - \frac{1}{5}D^2)]^{-1}x^2$

$= \frac{1}{5}[1 + \left(\frac{2}{5}D - \frac{1}{5}D^2\right) + (\frac{2}{5}D - \frac{1}{5}D^2)^2 + \ldots]x^2$

$= \frac{1}{5}[1 + \frac{2}{5}D - \frac{1}{5}D^2 + \frac{4}{25}D^2 - \frac{4}{25}D^3 + \cdots]x^2$

$= \frac{1}{5}[x^2 + \frac{2}{5}(2x) - \frac{1}{5}(2) + \frac{4}{25}(2) - 0]$

$= \frac{1}{5}[x^2 + \frac{4}{5}x - \frac{2}{25}]$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = e^x(C_1\cos 2x + C_2\sin 2x) + \frac{1}{5}(x^2 + \frac{4}{5}x - \frac{2}{25})$

be the required G.S. of given differential equation.

===

Ex.: Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

Solution: Let $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

i.e. $(D^3 + 3D^2 + 2D)y = x^2$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^3 + 3D^2 + 2D = D(D+1)(D+2)$

and $X = x^2$

$\therefore$ It's A.E. is f(D) = 0

i.e. D(D+1)(D+2) = 0

$\therefore$ $D = 0, -1, -2$ are the roots of an A.E.

$\therefore$ C.F. = $C_1e^{0x} + C_2e^{-x} + C_3e^{-2x}$

$= C_1 + C_2e^{-x} + C_3e^{-2x}$

Now P.I. = $\frac{1}{f(D)}X$

$= \frac{1}{D^3 + 3D^2 + 2D}x^2$

$= \frac{1}{2D[1+(\frac{3}{2}D + \frac{1}{2}D^2)]}x^2$

$= \frac{1}{2D}[1 + (\frac{3}{2}D + \frac{1}{2}D^2)]^{-1}x^2$

$= \frac{1}{2D}[1 - \left(\frac{3}{2}D + \frac{1}{2}D^2\right) + (\frac{3}{2}D + \frac{1}{2}D^2)^2 + \ ...]x^2$

$= \frac{1}{2D}[1 - \frac{3}{2}D - \frac{1}{2}\ D^2 + \frac{9}{4}D^2 + \frac{3}{2}\ D^3 + \cdots]x^2$

$= \frac{1}{2D}[x^2 - \frac{3}{2}(2x) - \frac{1}{2}(2) + \frac{9}{4}(2) + 0]$

$= \frac{1}{2D}[x^2 - 3x + \frac{7}{2}]$

$= \frac{1}{2}\int[x^2 - 3x + \frac{7}{2}]\ dx$

$= \frac{1}{2}[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7x}{2}]$

$= \frac{1}{6}\ x^3 - \frac{3}{4}x^2 + \frac{7}{4}x$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1 + C_2e^{-x} + C_3e^{-2x} + \frac{1}{6}\ x^3 - \frac{3}{4}x^2 + \frac{7}{4}x$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 + 2D + 3)y = x - 2x^2$

Solution: Let $(D^2 + 2D + 3)y = x - 2x^2$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 + 2D + 3$

and $X = x - 2x^2$

∴ It's A.E. is f(D) = 0

i.e. $D^2 + 2D + 3 = 0$

∴ $D = \frac{-2\pm\sqrt{4-12}}{2} = -1 \pm \sqrt{2}i$ are the roots of an A.E.

∴ C.F. $= e^{-x}\ [C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x]$

Now P.I. $= \frac{1}{f(D)}\ X$

$= \frac{1}{D^2+2D+3}(x - 2x^2)$

$= \frac{1}{3[1+(\frac{2}{3}D + \frac{1}{3}D^2)]}x^2$

$= \frac{1}{3}[1 + (\frac{2}{3}D + \frac{1}{3}D^2)]^{-1}(x - 2x^2)$

$= \frac{1}{3}[1 - \left(\frac{2}{3}D + \frac{1}{3}D^2\right) + (\frac{2}{3}D + \frac{1}{3}D^2)^2 + \cdots\ ](x - 2x^2)$

$= \frac{1}{3}[1 - \frac{2}{3}D - \frac{1}{3}D^2 + \frac{4}{9}D^2 + \frac{4}{9}D^3 + \ldots](x - 2x^2)$

$= \frac{1}{3}[(x - 2x^2) - \frac{2}{3}(1 - 4x) - \frac{1}{3}(-4) + \frac{4}{9}(-4) + 0]$

$= \frac{1}{3}[x - 2x^2 - \frac{2}{3} + \frac{8}{3}x + \frac{4}{3} - \frac{16}{9}]$

$= \frac{1}{3}[-2x^2 + \frac{11}{3}x - \frac{10}{9}]$

$= -\frac{2}{3}x^2 + \frac{11}{9}x - \frac{10}{27}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = e^{-x}[C_1\cos\sqrt{2}x + C_2\sin\sqrt{2}x] - \frac{2}{3}x^2 + \frac{11}{9}x - \frac{10}{27}$

be the required G.S. of given differential equation.

===

Ex.: Solve $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$

Solution: Let $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^3 + 2D^2 + D = D(D+1)^2$

and $X = e^{2x} + x^2 + x$

$\therefore$ It's A.E. is f(D) = 0

i.e. $D(D+1)^2 = 0$

$\therefore D = 0, -1, -1$ are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{0x} + (C_2 + C_3x)e^{-x}$

$= C_1 + (C_2 + C_3x)e^{-x}$

Now P.I. $= \frac{1}{f(D)}X$

$= \frac{1}{D^3 + 2D^2 + D}[e^{2x} + x^2 + x]$

$= \frac{1}{D^3 + 2D^2 + D}e^{2x} + \frac{1}{D^3 + 2D^2 + D}(x^2 + x)$

$= \frac{e^{2x}}{8+8+2} + \frac{1}{D[1 + (2D + D^2)]}(x^2 + x)$

$= \frac{e^{2x}}{18} + \frac{1}{D}[1 + (2D + D^2)]^{-1}(x^2 + x)$

$= \frac{e^{2x}}{18} + \frac{1}{D}[1 - (2D + D^2) + (2D + D^2)^2 - \cdots](x^2 + x)$

$= \frac{e^{2x}}{18} + \frac{1}{D}[1 - 2D - D^2 + 4D^2 + 4D^3 + \ldots](x^2 + x)$

$= \frac{e^{2x}}{18} + \frac{1}{D}[(x^2 + x) - 2(2x + 1) - (2) + 4(2) + 0]$

$= \frac{e^{2x}}{18} + \frac{1}{D}[x^2 - 3x + 4]$

$= \frac{e^{2x}}{18} + \int[x^2 - 3x + 4]\, dx$

$= \frac{e^{2x}}{18} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$

$= \frac{1}{18}e^{2x} + \frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1 + (C_2 + C_3x)e^{-x} + \frac{1}{18}e^{2x} + \frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x$

be the required G.S. of given differential equation.

===

Type-III: When X = sin(ax+b) or cos(ax+b)

Theorem: If $f(D^2)$ is polynomial in D^2 with constant coefficients and $f(-a^2) \neq 0$, then

a) $\frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}$ b) $\frac{1}{f(D^2)}\sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}$

Proof: a) By taking successive derivatives, we get,

$D\cos(ax+b) = -a\sin(ax+b)$,

$D^2\cos(ax+b) = (-a).a\cos(ax+b)$

i.e. $D^2\cos(ax+b) = (-a^2)\cos(ax+b)$

$D^3\cos(ax+b) = (-a^2).(-a)\sin(ax+b)$

$D^4\cos(ax+b) = (-a^2).(-a).a\cos(ax+b)$

i.e. $(D^2)^2\cos(ax+b) = (-a^2)^2\cos(ax+b)$

Similarly, $(D^2)^3\cos(ax+b) = (-a^2)^3\cos(ax+b)$

and so on, in general,

$(D^2)^r\cos(ax+b) = (-a^2)^r\cos(ax+b) \forall\, r \in \mathbb{N}$

∴ $f(D^2)\cos(ax+b) = f(-a^2)\cos(ax+b)$

where $f(D^2)$ is polynomial in D^2 with constant coefficients and $f(-a^2) \neq 0$

∴ $\cos(ax+b) = \frac{f(D^2)\cos(ax+b)}{f(-a^2)}$ $\because f(-a^2) \neq 0$

Operating $\frac{1}{f(D^2)}$ on both sides, we get,

∴ $\frac{1}{f(D^2)}\cos(ax+b) = \frac{\cos(ax+b)}{f(-a^2)}$ Hence proved.

b) By taking successive derivatives, we get,

$D\sin(ax+b) = a\cos(ax+b)$,

$D^2\sin(ax+b) = a.(-a)\sin(ax+b)$

i.e. $D^2\sin(ax+b) = (-a^2)\sin(ax+b)$

$D^3\sin(ax+b) = (-a^2).a\cos(ax+b)$

$D^4\sin(ax+b) = (-a^2).a.(-a)\sin(ax+b)$

i.e. $(D^2)^2\sin(ax+b)= (-a^2)^2\sin(ax+b)$

Similarly, $(D^2)^3\sin(ax+b) = (-a^2)^3\sin(ax+b)$

and so on, in general,

$(D^2)^r \sin(ax+b) = (-a^2)^r\sin(ax+b) \forall r \in \mathbb{N}$

$\therefore f(D^2) \sin(ax+b) = f(-a^2)\sin(ax+b)$

where $f(D^2)$ is polynomial in D^2 with constant coefficients and $f(-a^2) \neq 0$

$\therefore \sin(ax+b) = \frac{f(D^2)\sin(ax+b)}{f(-a^2)} \quad \because f(-a^2) \neq 0$

Operating $\frac{1}{f(D^2)}$ on both sides, we get,

$\therefore \frac{1}{f(D^2)} \sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}$ Hence proved.

==

Theorem: $a)\ \frac{1}{(D^2+a^2)^r} \cos(ax) = \frac{(-1)^r x^r \cos(ax+\frac{\pi}{2}r)}{r!(2a)^r}$ $b)\ \frac{1}{(D^2+a^2)^r} \sin(ax) = \frac{(-1)^r x^r \sin(ax+\frac{\pi}{2}r)}{r!(2a)^r}$

Proof: Consider $\frac{1}{(D^2+a^2)^r} e^{iax} = \frac{1}{(D+ai)^r(D-ai)^r} e^{iax}$

$$= \frac{x^r e^{iax}}{r!(ai+ai)^r}$$

$$= \frac{x^r e^{iax}}{r!(2ai)^r}$$

$$= \frac{(-i)^r x^r e^{iax}}{r!(2a)^r} \quad \because \frac{1}{i} = -i$$

$$= \frac{(-1)^r i^r x^r e^{iax}}{r!(2a)^r}$$

$$= \frac{(-1)^r e^{i\frac{\pi}{2}r} x^r e^{iax}}{r!(2a)^r} \because e^{i\frac{\pi}{2}} = i$$

$$= \frac{(-1)^r x^r e^{i(ax+\frac{\pi}{2}r)}}{r!(2a)^r}$$

$\therefore \frac{1}{(D^2+a^2)^r} [\cos(ax)+i\sin(ax)] = \frac{(-1)^r x^r}{r!(2a)^r} [\cos(ax+\frac{\pi}{2}r)+i\sin(ax+\frac{\pi}{2}r)]$

Equating real and imaginary parts, we get,

$a)\ \frac{1}{(D^2+a^2)^r} \cos(ax) = \frac{(-1)^r x^r \cos(ax+\frac{\pi}{2}r)}{r!(2a)^r}$

$b)\ \frac{1}{(D^2+a^2)^r} \sin(ax) = \frac{(-1)^r x^r \sin(ax+\frac{\pi}{2}r)}{r!(2a)^r}$

==

Note: 1) For r = 1, we get, $a)\ \frac{1}{D^2+a^2}$ cos(ax) = $\frac{x\sin(ax)}{2a}$, $b)\ \frac{1}{D^2+a^2}$ sin(ax) = $\frac{-x\cos(ax)}{2a}$

2) If X = cosax or sinax, then express $\frac{1}{D+\alpha}$ as $\frac{1}{D+\alpha} = (D-\alpha).\frac{1}{D^2-\alpha^2}$ and $\frac{1}{D-\alpha}$ as $\frac{1}{D-\alpha} = (D+\alpha).\frac{1}{D^2-\alpha^2}$

==

Ex.: Solve $(D^4 + 10D^2 + 9)y = \cos(2x+3)$

Solution: Let $(D^4 + 10D^2 + 9)y = \cos(2x+3)$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^4 + 10D^2 + 9 = (D^2+1)(D^2+9)$

and X = cos(2x+3)

∴ It's A.E. is f(D) = 0

i.e. $(D^2+1)(D^2+9) = 0$

∴ D = $\pm i, \pm 3i$ are the roots of an A.E.

∴ C.F. = $e^{0x}(C_1\cos x + C_2\sin x) + e^{0x}(C_3\cos 3x + C_4\sin 3x)$

i.e. C.F. = $C_1\cos x + C_2\sin x + C_3\cos 3x + C_4\sin 3x$

Now P.I. = $\frac{1}{f(D)}$ X

$= \frac{1}{(D^2+1)(D^2+9)}\cos(2x+3)$

$= \frac{\cos(2x+3)}{(-4+1)(-4+9)}$ $\quad \because D^2 = -a^2 = -2^2 = -4$

$= \frac{\cos(2x+3)}{-15}$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1\cos x + C_2\sin x + C_3\cos 3x + C_4\sin 3x - \frac{1}{15}\cos(2x+3)$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^3 + D)y = \sin 3x$

Solution: Let $(D^3 + D)y = \sin 3x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^3 + D = D(D^2+1)$

and X = sin3x

∴ It's A.E. is f(D) = 0

i.e. $D(D^2+1) = 0$

∴ D = 0, $\pm i$ are the roots of an A.E.

∴ C.F. = $C_1 e^{0x} + e^{0x}(C_2\cos x + C_3\sin x)$

i.e. C.F. = $C_1 + C_2\cos x + C_3\sin x$

Now P.I. = $\frac{1}{f(D)}$ X

$= \frac{1}{D(D^2+1)} sin3x$

$= \frac{1}{D}\frac{sin3x}{(-9+1)}$ $\quad\because D^2 = -a^2 = -3^2 = -9$

$= \frac{1}{-8}\int sin3x\, dx$

$= \frac{1}{24} cos3x$

∴ G.S.

= C.F. + P.I.

i.e. y = $C_1 + C_2\cos x + C_3\sin x + \frac{1}{24} cos3x$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 + 4)y = \sin 3x + e^x + x^2$

Solution: Let $(D^2 + 4)y = \sin 3x + e^x + x^2$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

f(D) = $D^2 + 4$ and X = $\sin 3x + e^x + x^2$

∴ It's A.E. is f(D) = 0

i.e. $D^2 + 4 = 0$

∴ D = $\pm 2i$ are the roots of an A.E.

∴ C.F. = $e^{0x}(C_1\cos 2x + C_2\sin 2x)$

i.e. C.F. = $C_1\cos 2x + C_2\sin 2x$

Now P.I. = $\frac{1}{f(D)}$ X = $\frac{1}{(D^2+4)}(\sin 3x + e^x + x^2)$

$= \frac{1}{D^2+4}\sin 3x + \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}x^2$ $\quad\because D^2 = -a^2 = -3^2 = -9$

$= \frac{sin3x}{-9+4} + \frac{e^x}{1+4} + \frac{1}{4(1+\frac{1}{4}D^2)}x^2$

$= -\frac{1}{5}sin3x + \frac{1}{5}e^x + \frac{1}{4}[1 - \frac{1}{4}D^2 + \frac{1}{16}D^4 - \ldots]x^2$

$= -\frac{1}{5}sin3x + \frac{1}{5}e^x + \frac{1}{4}[x^2 - \frac{1}{4}(2) + 0 \ldots]$

$= -\frac{1}{5}sin3x + \frac{1}{5}e^x + \frac{1}{4}x^2 - \frac{1}{8}$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{5}\sin 3x + \frac{1}{5}e^x + \frac{1}{4}x^2 - \frac{1}{8}$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 - 1)y = 10\sin^2 x$

Solution: Let $(D^2 - 1)y = 10\sin^2 x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 1 = (D - 1)(D + 1)$

and $X = 10\sin^2 x = 10\left(\frac{1-\cos 2x}{2}\right) = 5 - 5\cos 2x$

∴ It's A.E. is f(D) = 0

i.e. $(D - 1)(D + 1) = 0$

∴ D = 1, −1 are the roots of an A.E.

∴ C.F. = $C_1e^x + C_2e^{-x}$

Now P.I. = $\frac{1}{f(D)}X$

$= \frac{1}{(D^2-1)}(5 - 5\cos 2x)$

$= \frac{5}{D^2-1}e^{0x} - \frac{5}{D^2-1}\cos 2x$

$= \frac{5e^{0x}}{0-1} - \frac{5\cos 2x}{-4-1}$

$= -5 + \cos 2x$

$= \cos 2x - 5$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1e^x + C_2e^{-x} + \cos 2x - 5$

be the required G.S. of given differential equation.

==

Type-IV: When $X = e^{ax}V$, where V is a function of x.

Theorem: If $D \equiv \frac{d}{dx}$, f(D) is polynomial in D and V is a function of x, then

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.$$

Proof: Let $f(D)y = e^{ax}V$, where V is a function of x

For any function U of x, we have

$De^{ax}U = e^{ax}DU + ae^{ax}U = e^{ax}(D + a)U$

$D^2e^{ax}U = e^{ax}D(D + a)U + a\,e^{ax}(D + a)U$

$= e^{ax}(D + a)(D + a)U$

$= e^{ax}(D + a)^2U$

and so on, in general,

$D^r e^{ax} U = e^{ax} (D + a)^r U \quad \forall\, r \in N$

Let $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_{n-1} D + P_n$

$\therefore f(D)e^{ax}U = [D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_{n-1} D + P_n]e^{ax}U$

$= D^n e^{ax}U + P_1 D^{n-1} e^{ax}U + P_2 D^{n-2} e^{ax}U + \dots + P_{n-1} D e^{ax}U + P_n e^{ax}U$

$= e^{ax}(D+a)^n U + P_1 e^{ax}(D+a)^{n-1}U + P_2 e^{ax}(D+a)^{n-2}U + \dots + P_{n-1} e^{ax}(D+a)U + P_n e^{ax}U$

$= e^{ax}[(D+a)^n + P_1(D+a)^{n-1} + P_2(D+a)^{n-2} + \dots + P_{n-1}(D+a) + P_n]U$

$\therefore f(D)e^{ax}U = e^{ax} f(D+a)U$

By taking $U = \frac{1}{f(D+a)}V$, we get,

$f(D)e^{ax}\frac{1}{f(D+a)}V = e^{ax} f(D+a)\frac{1}{f(D+a)}V = e^{ax}V$

i.e. $e^{ax}V = f(D)e^{ax}\frac{1}{f(D+a)}V$

Operating $\frac{1}{f(D)}$ on both sides, we get,

$\frac{1}{f(D)} e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$ Hence proved.

==

Ex.: Solve $(D^4 - 1)y = e^x \cos x$

Solution: Let $(D^4 - 1)y = e^x \cos x$

be the given LDE with constant coefficients,

comparing it with $f(D)y = X$, we get,

$f(D) = D^4 - 1 = (D^2 - 1)(D^2 + 1)$

and $X = e^x \cos x$

$\therefore$ It's A.E. is $f(D) = 0$

i.e. $(D^2 - 1)(D^2 + 1) = 0$

$\therefore D = \pm 1, \pm i$ are the roots of an A.E.

$\therefore$ C.F. $= C_1 e^x + C_2 e^{-x} + e^{0x}(C_3 \cos x + C_4 \sin x)$

i.e. C.F. $= C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$

Now P.I. $= \frac{1}{f(D)} X$

$= \frac{1}{(D^2-1)(D^2+1)} e^x \cos x$

$= e^x \frac{1}{[(D+1)^2 - 1][(D+1)^2 + 1]} \cos x$

$= e^x \frac{1}{(D^2 + 2D)(D^2 + 2D + 2)} \cos x$

$= e^x \frac{1}{(-1 + 2D)(-1 + 2D + 2)} \cos x$

$= e^x \frac{1}{(2D-1)(2D+1)} \cos x$

$= e^x \frac{1}{(4D^2-1)} \cos x$

$= e^x \frac{\cos x}{(-4-1)}$

$= -\frac{1}{5} e^x \cos x$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x - \frac{1}{5} e^x \cos x$

be the required G.S. of given differential equation.

===

Ex.: Solve $(D^2 - 6D + 13)y = e^{3x}\sin 2x$

Solution: Let $(D^2 - 6D + 13)y = e^{3x}\sin 2x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 6D + 13$

and $X = e^{3x}\sin 2x$

$\therefore$ It's A.E. is f(D) = 0

i.e. $D^2 - 6D + 13 = 0$

$\therefore D = \frac{6\pm\sqrt{36-52}}{2} = \frac{6\pm 4i}{2} = 3\pm 2i$ are the roots of an A.E.

$\therefore$ C.F. $= e^{3x}(C_1\cos 2x + C_2 \sin 2x)$

Now P.I. $= \frac{1}{f(D)} X$

$= \frac{1}{(D^2 - 6D + 13)} e^{3x}\sin 2x$

$= e^{3x} \frac{1}{[(D+3)^2 - 6(D+3) + 13]} \sin 2x$

$= e^{3x} \frac{1}{(D^2 + 6D + 9 - 6D - 18 + 13)} \sin 2x$

$= e^{3x} \frac{1}{(D^2+4)} \sin 2x$

$= e^{3x} \frac{-x\cos 2x}{(2\times 2)}$

$= -\frac{1}{4} xe^{3x} \sin 2x$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = e^{3x}(C_1\cos 2x + C_2 \sin 2x) - \frac{1}{4} xe^{3x} \sin 2x$

be the required G.S. of given differential equation.

===

Ex.: Solve $(D^2 - 4D + 3)y = e^x\cos2x$

Solution: Let $(D^2 - 4D + 3)y = e^x\cos2x$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 4D + 3 = (D - 1)(D - 3)$

and $X = e^x\cos2x$

∴ It's A.E. is f(D) = 0

i.e. (D – 1)(D – 3) = 0

∴ D = 1, 3 are the roots of an A.E.

∴ C.F. = $C_1e^{x} + C_2e^{3x}$

Now P.I. = $\frac{1}{f(D)}X$

$= \frac{1}{(D^2 - 4D + 3)} e^x\cos2x$

$= e^x \frac{1}{[(D+1)^2 - 4(D+1) + 3]} \cos2x$

$= e^x \frac{1}{(D^2 + 2D + 1 - 4D - 4 + 3)} \cos2x$

$= e^x \frac{1}{(D^2 - 2D)} \cos2x$

$= e^x \frac{1}{(-4 - 2D)} \cos2x$

$= -\frac{1}{2} e^x \frac{1}{(D+2)} \cos2x$

$= -\frac{1}{2} e^x \frac{(D-2)}{(D^2-4)} \cos2x$

$= -\frac{1}{2} e^x \frac{(-2\sin2x - 2\cos2x)}{(-4-4)}$

$= -\frac{1}{8} e^x(\sin2x + \cos2x)$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1e^{x} + C_2e^{3x} - \frac{1}{8} e^x(\sin2x + \cos2x)$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 - 2D + 1)y = x^2e^{3x}$

Solution: Let $(D^2 - 2D + 1)y = x^2e^{3x}$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 - 2D + 1 = (D-1)^2$ and $X = x^2e^{3x}$

$\therefore$ It's A.E. is $f(D) = 0$

i.e. $(D-1)^2 = 0$

$\therefore$ D = 1, 1 are the roots of an A.E.

$\therefore$ C.F. $= (C_1 + C_2x)e^x$

Now P.I. $= \frac{1}{f(D)} X$

$= \frac{1}{(D^2-2D+1)} x^2e^{3x}$

$= e^{3x} \frac{1}{[(D+3)^2 - 2(D+3) + 1]} x^2$

$= e^{3x} \frac{1}{(D^2+6D+9-2D-6+1)} x^2$

$= e^{3x} \frac{1}{(D^2+4D+4)} x^2$

$= e^{3x} \frac{1}{4(1+D+\frac{D^2}{4})} x^2$

$= \frac{1}{4} e^{3x}[1-(D+\frac{D^2}{4}) + (D+\frac{D^2}{4})^2 - \ldots] x^2$

$= \frac{1}{4} e^{3x}[1- D - \frac{D^2}{4} + D^2 + \frac{D^3}{2} + \ldots] x^2$

$= \frac{1}{4} e^{3x}[x^2 - 2x - \frac{1}{2} + 2 + 0]$

$= \frac{1}{8} e^{3x}(2x^2 - 4x + 3)$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_2x)e^x + \frac{1}{8} e^{3x}(2x^2 - 4x + 3)$

be the required G.S. of given differential equation.

==

Type-V: When X = xV, where V is a function of x only.

Theorem: If $D \equiv \frac{d}{dx}$, f(D) is polynomial in D and V is a function of x only, then

$$\frac{1}{f(D)} xV = \left[x - \frac{1}{f(D)} f'(D)\right] \frac{1}{f(D)} V.$$

Proof: Let f(D)y = xV, where V is a function of x only.

For any function U of x, By using Leibnitz's theorem, we get,

$D^r xU = xD^rU + rD^{r-1}U \quad \forall\, r \in N$ … (i)

Let $f(D) = D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots + P_{n-1}D + P_n$ … (ii)

$\therefore f'(D) = nD^{n-1} + P_1(n-1)D^{n-2} + P_2(n-2)D^{n-3} + \ldots + P_{n-1}$ … (iii)

$\therefore f(D)(xU) = [D^n + P_1D^{n-1} + P_2D^{n-2} + \dots + P_{n-1}D + P_n](xU)$

$= D^n(xU) + P_1D^{n-1}(xU) + P_2D^{n-2}(xU) + \dots + P_{n-1}D(xU) + P_n(xU)$

$= xD^nU + nD^{n-1}U + P_1[xD^{n-1}U + (n-1)D^{n-2}U] + P_2[xD^{n-2}U + (n-2)D^{n-3}U) + \dots$

$+ P_{n-1}(xDU + U) + P_n(xU)$

$= x[D^n + P_1D^{n-1} + P_2D^{n-2} + \dots + P_{n-1}D + P_n]U$

$+ [nD^{n-1} + P_1(n-1)D^{n-2} + P_2(n-2)D^{n-3} + \dots + P_{n-1}]U$

$\therefore f(D)(xU) = xf(D)U + f'(D)U$

By taking $U = \frac{1}{f(D)}V$, we get,

$f(D)(x\frac{1}{f(D)}V) = xf(D)\frac{1}{f(D)}V + f'(D)\frac{1}{f(D)}V$

i.e. $f(D)(x\frac{1}{f(D)}V) = xV + f'(D)\frac{1}{f(D)}V$

$\therefore xV = f(D)(x\frac{1}{f(D)}V) - f'(D)\frac{1}{f(D)}V$

Operating $\frac{1}{f(D)}$ on both sides, we get,

$\therefore \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V$

$\therefore \frac{1}{f(D)}(xV) = [x - \frac{1}{f(D)}f'(D)]\frac{1}{f(D)}V$

Hence proved.

==

Ex.: Solve $(D^2 + 1)y = x\cos 2x$

Solution: Let $(D^2 + 1)y = x\cos 2x$

be the given LDE with constant coefficients,

comparing it with $f(D)y = X$, we get,

$f(D) = D^2 + 1$ and $X = x\cos 2x$

$\therefore$ It's A.E. is $f(D) = 0$

i.e. $D^2 + 1 = 0$

$\therefore D = \pm i$ are the roots of an A.E.

$\therefore$ C.F. $= e^{0x}(C_1\cos x + C_2\sin x)$

i.e. C.F. $= C_1\cos x + C_2\sin x$

Now P.I. $= \frac{1}{f(D)}X$

$= \frac{1}{(D^2 + 1)}x\cos 2x$

$= [x- \frac{1}{(D^2 + 1)}(2D)] \frac{1}{(D^2 + 1)} cos2x$

$= [x- \frac{1}{(D^2 + 1)}(2D)] \frac{cos2x}{(-4 + 1)}$

$= -\frac{1}{3}[xcos2x - \frac{1}{(D^2 + 1)}(2Dcos2x)]$

$= -\frac{1}{3}[xcos2x - \frac{1}{(D^2 + 1)}(-4sin2x)]$

$= -\frac{1}{3}[xcos2x + \frac{4sin2x}{(-4 + 1)}]$

$= -\frac{1}{3} xcos2x + \frac{4}{9} sin2x$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1cosx + C_2sinx - \frac{1}{3} xcos2x + \frac{4}{9} sin2x$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 + 2D + 1)y = xcosx$

Solution: Let $(D^2 + 2D + 1)y = xcosx$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 + 2D + 1 = (D + 1)^2$

and X = xcosx

∴ It's A.E. is f(D) = 0

i.e. $(D + 1)^2 = 0$

∴ D = -1, -1 are the roots of an A.E.

∴ C.F. = $(C_1 + C_2x)e^{-x}$

Now P.I. = $\frac{1}{f(D)} X$

$= \frac{1}{(D^2 + 2D + 1)} xcosx$

$= [x- \frac{1}{(D^2 + 2D + 1)}(2D + 2)] \frac{1}{(D^2 + 2D + 1)} cosx$

$= [x- \frac{2}{(D^2 + 2D + 1)}(D + 1)] \frac{1}{(-1+2D+ 1)} cosx$

$= \frac{1}{2}[x- \frac{2}{(D+1)^2}(D + 1)] \int cosx dx$

$= \frac{1}{2}[x- \frac{2}{(D + 1)}] sinx$

$= \frac{1}{2}[xsinx - \frac{2}{(D^2 - 1)}(D - 1)]sinx$

$= \frac{1}{2}[xsinx - \frac{2}{(D^2-1)}(cosx - sinx)]$

$= \frac{1}{2}[xsinx - \frac{2(cosx-sinx)}{(-1-1)}]$

$= \frac{1}{2}(xsinx + cosx - sinx)$

∴ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_2x)e^{-x} + \frac{1}{2}(xsinx + cosx - sinx)$

be the required G.S. of given differential equation.

==

Ex.: Solve $(D^2 + 4)y = xsinx$

Solution: Let $(D^2 + 4)y = xsinx$

be the given LDE with constant coefficients,

comparing it with f(D)y = X, we get,

$f(D) = D^2 + 4$ and X = xsinx

∴ It's A.E. is f(D) = 0

i.e. $D^2 + 4 = 0$

∴ $D = \pm 2i$ are the roots of an A.E.

∴ C.F. = $e^{0x}(C_1cos2x + C_2sin2x)$

i.e. C.F. = $C_1cos2x + C_2sin2x$

Now P.I. = $\frac{1}{f(D)}X$

$= \frac{1}{(D^2+4)}xsinx$

$= [x - \frac{1}{(D^2+4)}(2D)]\frac{1}{(D^2+4)}sinx$

$= [x - \frac{2}{(D^2+4)}D]\frac{1}{(-1+4)}sinx$

$= \frac{1}{3}[xsinx - \frac{2}{(D^2+4)}cosx]$

$= \frac{1}{3}[xsinx - \frac{2cosx}{(-1+4)}]$

$= \frac{1}{3}[xsinx - \frac{2}{3}cosx]$

$= \frac{1}{9}(3xsinx - 2cosx)$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1cos2x + C_2sin2x + \frac{1}{9}(3xsinx - 2cosx)$

be the required G.S. of given differential equation.

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Ex.: Solve $(D^2 - 1)y = x\sin x$

Solution: Let $(D^2 - 1)y = x\sin x$

be the given LDE with constant coefficients,

comparing it with $f(D)y = X$, we get,

$f(D) = D^2 - 1 = (D - 1)(D + 1)$

and $X = x\sin x$

∴ It's A.E. is $f(D) = 0$

i.e. $(D - 1)(D + 1) = 0$

∴ $D = 1, -1$ are the roots of an A.E.

∴ C.F. $= C_1e^{x} + C_2e^{-x}$

Now P.I. $= \frac{1}{f(D)} X$

$= \frac{1}{(D^2 - 1)} x\sin x$

$= [x - \frac{1}{(D^2 - 1)}(2D)] \frac{1}{(D^2 - 1)} \sin x$

$= [x - \frac{2}{(D^2 - 1)} D] \frac{1}{(-1-1)} \sin x$

$= -\frac{1}{2}[x\sin x - \frac{2}{(D^2 - 1)} \cos x]$

$= -\frac{1}{2}[x\sin x - \frac{2\cos x}{(-1 - 1)}]$

$= -\frac{1}{2}(x\sin x + \cos x)$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1e^{x} + C_2e^{-x} - \frac{1}{2}(x\sin x + \cos x)$

be the required G.S. of given differential equation.

===

Ex.: Solve $(D^2 - 2D + 1)y = x\sin x$

Solution: Let $(D^2 - 2D + 1)y = x\sin x$

be the given LDE with constant coefficients,

comparing it with $f(D)y = X$, we get,

$f(D) = D^2 - 2D + 1 = (D - 1)^2$

and $X = x\sin x$

∴ It's A.E. is $f(D) = 0$

i.e. $(D - 1)^2 = 0$

∴ $D = 1, 1$ are the roots of an A.E.

$\therefore$ C.F. = $(C_1 + C_2x)e^x$

Now P.I. = $\frac{1}{f(D)}$ X

$= \frac{1}{(D^2 - 2D + 1)}$ xsinx

$= [x - \frac{1}{(D^2 - 2D + 1)}(2D - 2)]\frac{1}{(D^2 - 2D + 1)} sinx$

$= [x - \frac{2}{(D^2 - 2D + 1)}(D - 1)]\frac{1}{(-1 - 2D + 1)} sinx$

$= \frac{1}{2}[x - \frac{2}{(D-1)^2}(D - 1)]\int(-sinx)dx$

$= \frac{1}{2}[x - \frac{2}{(D - 1)}] cosx$

$= \frac{1}{2}[xcosx - \frac{2}{(D^2 - 1)}(D + 1)]cosx$

$= \frac{1}{2}[xcosx - \frac{2}{(D^2 - 1)}(-sinx + cosx)]$

$= \frac{1}{2}[xcosx - \frac{2(-sinx + cosx)}{(-1-1)}]$

$= \frac{1}{2}(xcosx - sinx + cosx)$

$\therefore$ G.S. = C.F. + P.I.

i.e. y = $(C_1 + C_2x)e^x + \frac{1}{2}(xcosx - sinx + cosx)$

be the required G.S. of given differential equation.

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MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) A differential equation of the form $\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_{n-1}\frac{dy}{dx} + p_ny = X$ i.e. f(D)y = X, where D $\equiv \frac{d}{dx}$; $p_1, p_2, \ldots, p_n$ are constants and X is a function of x only, is called a …… differential equation with constant co-efficients.

A) linear B) homogeneous C) quadratic D) non-homogeneous

2) An associated equation of linear differential equation with constant coefficient's f(D)y = X is ……

A) f(D) = 0 B) f(D) = X C) f(D)y = 0 D) None of these

3) An auxiliary equation (A.E.) of linear differential equation with constant coefficient's f(D)y = X is ……

A) f(D) = 0 B) f(D) = X C) f(D)y = 0 D) None of these

4) If f(D)y = X is a LDE with constant coefficient's, then f(D)y = 0 is called …… equation.

A) complementary B) auxiliary C) associated D) None of these

5) If f(D)y = X is a LDE with constant coefficient's, then f(D) = 0 is called …… equation

A) complementary B) auxiliary C) associated D) None of these

6) If LDE f(D)y = X has C.F. = u and P.I. = v, then its G.S. is y = ……

A) uv B) u - v C) u + v D) None of these

7) If LDE is f(D)y = 0, then P.I. = ……

A) -1 B) 0 C) 1 D) None of these

8) If an A.E. f(D) = 0 of LDE f(D)y = X has n distinct roots $m_1, m_2, m_3, \ldots, m_n$ then C.F.= ……

A) $C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + \ldots + C_ne^{m_nx}$

B) $C_1e^{-m_1x} + C_2e^{-m_2x} + C_3e^{-m_3x} + \ldots + C_ne^{-m_nx}$

C) $C_1e^{x} + C_2e^{m_2x} + C_3e^{m_3x} + \ldots + C_ne^{m_nx}$

D) None of these

9) If an A.E. f(D) = 0 of LDE f(D)y = 0 has n distinct roots $m_1, m_2, m_3, \ldots, m_n$ then its G.S. is y = ……

A) $C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x} + \ldots + C_ne^{m_nx}$

B) $C_1e^{-m_1x} + C_2e^{-m_2x} + C_3e^{-m_3x} + \ldots + C_ne^{-m_nx}$

C) $C_1e^{x} + C_2e^{m_2x} + C_3e^{m_3x} + \ldots + C_ne^{m_nx}$

D) None of these

10) If an A.E. f(D) = 0 of LDE f(D)y = X has root m, repeated k times, then C.F. = ……

A) $C_1e^{mx}+C_2e^{mx}+C_3e^{mx}+\ldots+C_ke^{mx}$ B) Ce^{mx}

C) $(C_1+ C_2x+ C_3x^2 + \ldots + C_kx^{k-1})e^{mx}$ D) None of these

11) If an A.E. f(D) = 0 of LDE f(D)y = 0 has root m, repeated k times, then its G.S. is y = ……

A) $C_1e^{mx}+C_2e^{mx}+C_3e^{mx}+\ldots+C_ke^{mx}$ B) Ce^{mx}

C) $(C_1+ C_2x+ C_3x^2 + \ldots + C_kx^{k-1})e^{mx}$ D) None of these

12) If an A.E. f(D) = 0 of LDE f(D)y = X has complex roots $\alpha \pm i\beta$, then C.F.= ……

A) $C_1e^{\alpha x} + C_2e^{\beta x}$ B) $e^{\beta x}(C_1\cos\alpha x + C_2\sin\alpha x)$

C) $e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$ D) None of these

13) If an A.E. f(D) = 0 of LDE f(D)y = 0 has complex roots $\alpha \pm i\beta$, then its G.S. is y = ……

A) $C_1e^{\alpha x} + C_2e^{\beta x}$ B) $e^{\beta x}(C_1\cos\alpha x + C_2\sin\alpha x)$

C) $e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$ D) None of these

14) If an A.E. f(D) = 0 of LDE f(D)y = X has complex roots $\alpha \pm i\beta$ occurs twice, then C.F.= ……

A) $(C_1+ C_2x)e^{\alpha x} + (C_3+ C_4x)e^{\beta x}$ B) $e^{\alpha x}[(C_1+ C_2x)\cos\beta x + (C_3+ C_4x)\sin\beta x]$

C) $e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$ D) None of these

15) If an A.E. f(D) = 0 of LDE f(D)y = 0 has complex roots $\alpha \pm i\beta$ occurs twice, then its G.S. is y = ……

A) $(C_1+ C_2x)e^{\alpha x} + (C_3+ C_4x)e^{\beta x}$ B) $e^{\alpha x}[(C_1+ C_2x)\cos\beta x + (C_3+ C_4x)\sin\beta x]$

C) $(C_1\cos\beta x + C_2\sin\beta x)$ D) None of these

16) If a and b are real roots of LDE with constant coefficient's f(D)y = 0, then its G.S. is y = ……

A) $C_1e^{ax} + C_2e^{bx}$ B) $C_1e^{ax} + C_2e^{-bx}$

C) $C_1e^{-ax} + C_2e^{-bx}$ D) None of these

17) The solution of LDE with constant coefficient's $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 7y = 0$ is y = ……

A) $C_1e^{x} + C_2e^{-7x}$ B) $C_1e^{-x} + C_2e^{-7x}$ C) $C_1e^{-x} + C_2e^{7x}$ D) None of these

18) The G.S of LDE $(D^2 + 6D - 7)y = 0$ is y = ……

A) $C_1e^{x} + C_2e^{-7x}$ B) $C_1e^{-x} + C_2e^{-7x}$ C) $C_1e^{-x} + C_2e^{7x}$ D) None of these

19) The solution of LDE with constant coefficient's $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ is y = ……

A) $C_1e^{2x} + C_2e^{-3x}$ B) $C_1e^{-2x} + C_2e^{-3x}$ C) $C_1e^{2x} + C_2e^{3x}$ D) None of these

20) The G.S of LDE $(D^2 - 5D + 6)y = 0$ is y = ……

A) $C_1e^{2x} + C_2e^{-3x}$ B) $C_1e^{-2x} + C_2e^{-3x}$ C) $C_1e^{2x} + C_2e^{3x}$ D) None of these

21) The solution of LDE with constant coefficient's $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$ is y = ……

A) $C_1e^{3x} + C_2e^{4x}$ B) $C_1e^{-3x} + C_2e^{-4x}$ C) $C_1e^{-3x} + C_2e^{4x}$ D) None of these

22) The G.S of $(D^2 - 7D + 12)y = 0$ is y = ……

A) $C_1e^{3x} + C_2e^{4x}$ B) $C_1e^{-3x} + C_2e^{-4x}$ C) $C_1e^{-3x} + C_2e^{4x}$ D) None of these

23) The G.S of $(2D^2 + 5D - 12)y = 0$ is y = ……

A) $C_1e^{-\frac{3}{2}x} + C_2e^{-4x}$ B) $C_1e^{\frac{3}{2}x} + C_2e^{-4x}$ C) $C_1e^{-3x} + C_2e^{4x}$ D) None of these

24) The C.F. of $(2D^2 + 3D - 2)y = 0$ is ……

A) $C_1e^{\frac{1}{2}x} + C_2e^{-2x}$ B) $C_1e^{\frac{3}{2}x} + C_2e^{-4x}$ C) $C_1e^{-3x} + C_2e^{4x}$ D) None of these

25) The G.S of $(D - 1)^2(D^2 - 1)y = 0$ is y = ……

A) $C_1e^x + C_2e^x + C_3e^{-x}$ B) $(C_1+ C_2x)e^x + C_3e^{-x}$

C) $(C_1+ C_2x+ C_3x^2)e^x+ C_4e^{-x}$ D) None of these

26) The solution of LDE with constant coefficient's $(D^2 +16)y = 0$ is y = ……..

A) $e^{-x}(C_1\cos 4x + C_2\sin 4x)$ B) $C_1\cos 4x + C_2\sin 4x$

C) $e^x(C_1\cos 4x + C_2\sin 4x)$ D) None of these

27) The solution of LDE with constant coefficient's $(D^2 - 6D + 13)y = 0$ is y = ……..

A) $e^{3x}(C_1\cos x + C_2\sin x)$ B) $e^{2x}(C_1\cos 3x + C_2\sin 3x)$

C) $e^{3x}(C_1\cos 2x + C_2\sin 2x)$ D) None of these

28) The solution of LDE with constant coefficient's $(D^2 - 6D + 9)y = 0$ is y = ……..

A) $(C_1 + C_2x)e^{-3x}$ B) $(C_1 + C_2x)e^{3x}$

C) $(C_1e^{3x} + C_2e^{3x})$ D) None of these

29) If f(D)y = X is a linear differential equation with constant coefficient's then P.I.= ……

A) $\frac{1}{f(D)}X$ B) f(D)y C) f(D)X D) $\frac{1}{f(D)}y$

30) If $D \equiv \frac{d}{dx}$ and X is a function of x then $\frac{1}{D}$ X = ……

A) $e^{\int X dx}$ B) $\int X$ dx C) $\frac{dX}{dx}$ D) None of these

31) If $D \equiv \frac{d}{dx}$ and $f(0) \neq 0$, then $\frac{1}{f(D)}$ k = ……

A) $\frac{1}{f(0)}$ B) $\frac{k}{f(0)}$ C) $\frac{1}{f(a)}$ D) None of these

32) If $D \equiv \frac{d}{dx}$ then $\frac{1}{D}$ sinx = ……

A) cosx B) $-cosx$ C) $sinx$ D) None of these

33) If $D \equiv \frac{d}{dx}$ then $\frac{1}{D}$ cosx = ……

A) cosx B) –sinx C) $sinx$ D) None of these

34) If X is a function of x then $\frac{1}{D-m}$ X = ……

A) $e^{-mx}\int Xe^{-mx}$ dx B) $e^{mx}\int Xe^{mx}$ dx

C) $e^{mx}\int Xe^{-mx}$ dx D) None of these

35) If $f(D)y = e^{ax}$ is linear differential equation with constant coefficient's with $f(a) \neq 0$ then $\frac{1}{f(D)}$ e^{ax} = ……

A) $\frac{x^re^{ax}}{r!}$ B) $\frac{e^{ax}}{f(a)}$ C) $\frac{x^re^{ax}}{f(a)}$ D) None of these

36) $\frac{1}{(D-a)^r}$ e^{ax} = ……

A) $\frac{x^r e^{ax}}{r!}$ B) $\frac{e^{ax}}{f(a)}$ C) $\frac{x^r e^{ax}}{f(a)}$ D) None of these

37) If $\phi(a) \neq 0$, then $\frac{1}{(D-a)^r \phi(D)}$ e^{ax} = ……

A) $\frac{x^r e^{ax}}{r!}$ B) $\frac{x^r e^{ax}}{r!\phi(a)}$ C) $\frac{x^r e^{ax}}{f(a)}$ D) None of these

38) P.I. of LDE $(D^2 - 3D + 2)y = e^{5x}$ is ……

A) $\frac{xe^{5x}}{12}$ B) $\frac{e^{5x}}{2}$ C) $\frac{e^{5x}}{12}$ D) None of these

39) P.I. of LDE $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$ is ……

A) $\frac{x^3e^{-x}}{6}$ B) $\frac{e^{-x}}{8}$ C) $\frac{e^{-x}}{3}$ D) None of these

40) P.I. of LDE $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{2x}$ is ……

A) $\frac{e^{2x}}{4}$ B) $\frac{e^{2x}}{16}$ C) $\frac{e^{2x}}{32}$ D) None of these

41) $1 + x + x^2 + x^3 +$ ……. is an expansion of…

A) $(1 - x)^{-1}$ B) $(1 + x)^{-1}$ C) $(1 - x)^{-n}$ D) None of these

42) $1 - x + x^2 - x^3 +$ ……. is an expansion of…

A) $\frac{1}{1-x}$ B) $\frac{1}{1+x}$ C) $(1 - x)^n$ D) None of these

43) If $f(-a^2) \neq 0$, then $\frac{1}{f(D^2)}$ sin(ax+b) = …….

A) $\frac{tan(ax+b)}{f(-a^2)}$ B) $\frac{sin(ax+b)}{f(-a^2)}$ C) $\frac{cos(ax+b)}{f(-a^2)}$ D) None of these

44) If $f(-9) \neq 0$ then $\frac{1}{f(D^2)}$ sin(3x+5) = …….

A) $\frac{tan(3x+5)}{f(-25)}$ B) $\frac{sin(3x+5)}{f(-9)}$ C) $\frac{cos(3x+5)}{f(-25)}$ D) None of these

45) $\frac{1}{D^2+a^2}$ sinax = …….

A) $\frac{-x}{2a}$ cosax B) $\frac{-x}{2a}$ sinax C) $\frac{x}{2a}$ cosax D) $\frac{x}{2a}$ sinax

46) $\frac{1}{D^2+36}$ sin(6x) = …….

A) $\frac{-x}{12}$ cos6x B) $\frac{-x}{12}$ sin6x C) $\frac{x}{12}$ cos6x D) $\frac{x}{12}$ sin6x

47) $\frac{1}{D^2+16}$ sin(3x-5) = …….

A) $\frac{\sin(3x-5)}{16}$ B) $\frac{\sin(3x-5)}{7}$ C) $\frac{\cos(3x-5)}{-7}$ D) None of these

48) If $f(D^2)$ is polynomial in D^2 with constant coefficient's and and $f(-a^2) \neq 0$ then $\frac{1}{f(D^2)}$ cos(ax+b) = …….

A) $\frac{\tan(ax+b)}{f(-a^2)}$ B) $\frac{\sin(ax+b)}{f(-a^2)}$ C) $\frac{\cos(ax+b)}{f(-a^2)}$ D) None of these

49) If $f(D^2)$ is polynomial in D^2 with constant coefficient's and and $f(-4) \neq 0$ then $\frac{1}{f(D^2)}\cos(2x+3)$ = …….

A) $\frac{\tan(2x+3)}{f(-4)}$ B) $\frac{\sin(2x+3)}{f(-4)}$ C) $\frac{\cos(2x+3)}{f(-4)}$ D) None of these

50) $\frac{1}{D^2+a^2}\cos ax$ = …….

A) $\frac{-x}{2a}\cos ax$ B) $\frac{-x}{2a}\sin ax$ C) $\frac{x}{2a}\cos ax$ D) $\frac{x}{2a}\sin ax$

51) $\frac{1}{D^2+16}\cos 4x$ = …….

A) $\frac{-x}{8}\cos 4x$ B) $\frac{-x}{8}\sin 4x$ C) $\frac{x}{8}\cos 4x$ D) $\frac{x}{8}\sin 4x$

52) If $f(D)y = e^{ax}V$ where V is function of x then $\frac{1}{f(D)}e^{ax}V$ = …….

A) $e^{ax}\frac{1}{f(D-a)}V$ B) $e^{ax}\frac{1}{f(D+a)}V$ C) $V\frac{1}{f(D+a)}e^{ax}$ D) None of these

53) If $f(D)y = e^{4x}V$ where V is function of x then $\frac{1}{f(D)}e^{4x}V$ = …….

A) $e^{4x}\frac{1}{f(D-4)}V$ B) $e^{4x}\frac{1}{f(D+4)}V$ C) $V\frac{1}{f(D+4)}e^{4x}$ D) None of these

54) If $f(D)y = e^{-3x}V$ where V is function of x then $\frac{1}{f(D)}e^{-3x}V$ = …….

A) $e^{-3x}\frac{1}{f(D-3)}V$ B) $e^{-3x}\frac{1}{f(D+3)}V$ C) $V\frac{1}{f(D-3)}e^{-3x}$ D) None of these

55) If $f(D)y = xV$ where V is function of x then $\frac{1}{f(D)}(xV)$ = …….

A) $[x - \frac{1}{f(D)}f'(D)]\frac{1}{f(D)}V$ B) $[x + \frac{1}{f(D)}f'(D)]\frac{1}{f(D)}V$

C) $[x - \frac{1}{f'(D)}f(D)]\frac{1}{f(D)}V$ D) None of these

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UNIT-4: HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Homogeneous Linear Differential Equation: A differential equation of the form

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_{n-1}\, x\frac{dy}{dx} + p_n y = X$$

where $p_1, p_2, \ldots, p_n$ are constants and X is a function of x only, is called a homogeneous linear differential equation of order n.

Remark: A homogeneous linear differential is also called **Cauchy's** linear equation.

e.g. i) $x^4 \frac{d^4y}{dx^4} + 7x^3 \frac{d^3y}{dx^3} - 12x \frac{dy}{dx} + 5y = \log x$

is a homogeneous linear differential equation of order 4.

ii) $x^8 \frac{d^7y}{dx^7} + xy = x^4$ is a homogeneous linear differential equation of order 7.

iii) $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$

is a homogeneous linear differential equation of order 2.

Method of Solving Homogeneous Linear Differential Equation:

Consider a homogeneous linear differential equation

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_{n-1}\, x\frac{dy}{dx} + p_n y = X \ \ldots \text{(i)}$$

To solve it we change variable x to z by putting

$x = e^z$ i.e. $z = \log x$ and $D = \frac{d}{dz}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}.\frac{1}{x}$ $\qquad \because \frac{dz}{dx} = \frac{1}{x}$

$\therefore x\frac{dy}{dx} = \frac{dy}{dz} = Dy$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{1}{x}.\frac{dy}{dz}\right]$

$= \frac{1}{x}.\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}.\frac{dy}{dz}$

$= \frac{1}{x}.\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}.\frac{dy}{dz}$

$= \frac{1}{x^2}.\frac{d^2y}{dz^2} - \frac{1}{x^2}.\frac{dy}{dz}$

$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (D^2 - D)y = D(D-1)y$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

and so on, in general $x^r\frac{d^ry}{dx^r} = D(D-1)(D-2)(D-3)\ldots(D-r+1)y$

$\therefore$ Equation (i) becomes,

$[D(D-1)(D-2)\ldots(D-n+1) + p_1 D(D-1)(D-2)\ldots(D-n+2)$

$+ p_2D(D-1)(D-2)\ldots(D-n+3) + \ldots\ldots + p_{n-1}D + p_n]y = Z$

Which is linear differential equation with constant coefficients and Z is function of z. Using usual method, G.S. in y and z is obtained. In this solution putting z = logx, we get required G.S. of given equation.

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Ex. Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$

Solution: Let $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$ … (i)

be the given homogeneous linear differential equation.

To solve it we put $x = e^z$ i.e. z = logx and $D = \frac{d}{dz}$, we get,

$x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) + D - 4]y = 0$

i.e. $(D^2 - 4)y = 0$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 4 = 0$

i.e. $(D-2)(D+2) = 0$

$\therefore$ D = 2, -2 are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{2z} + C_2e^{-2z}$

and P.I.= 0 $\because$ Z = 0.

$\therefore$ G.S. = C.F. + P.I. = C.F.

i.e. $y = C_1e^{2z} + C_2e^{-2z}$

Using z = logx i.e. $e^z = x$, we get,

$y = C_1x^2 + \frac{C_2}{x^2}$

be the G.S. of given homogeneous LDE.

==

Ex. Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$

Solution: Let $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$ … (i)

be the given homogeneous linear differential equation.

To solve it we put $x = e^z$ i.e. z = logx and $D = \frac{d}{dz}$, we get,

$x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) - 4D + 6]y = e^{5z}$

i.e. $(D^2 - 5D + 6)y = e^{5z}$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 5D + 6 = 0$

i.e. $(D - 2)(D - 3) = 0$

$\therefore$ D = 2, 3 are the roots of an A.E.

$\therefore$ C.F. = $C_1e^{2z} + C_2e^{3z}$

and P.I.= $\frac{1}{(D-2)(D-3)} e^{5z}$

$= \frac{e^{5z}}{(5-2)(5-3)}$

$= \frac{1}{6} e^{5z}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1e^{2z} + C_2e^{3z} + \frac{1}{6} e^{5z}$

Using z = logx i.e. $e^z = x$, we get,

$y = C_1x^2 + C_2x^3 + \frac{1}{6}x^5$

be the G.S. of given homogeneous LDE.

==

Ex. Solve $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} - \frac{4}{x^2}y = x^2$

Solution: Let $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} - \frac{4}{x^2}y = x^2$ i.e. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$ … (i)

be the given homogeneous linear differential equation.

To solve it we put $x = e^z$ i.e. z = logx and $D = \frac{d}{dz}$, we get,

$x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) - 2D - 4]y = e^{4z}$

i.e. $(D^2 - 3D - 4)y = e^{4z}$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 3D - 4 = 0$

i.e. $(D - 4)(D + 1) = 0$

$\therefore$ D = 4, -1 are the roots of an A.E.

$\therefore$ C.F. = $C_1e^{4z} + C_2e^{-z}$

and P.I.= $\frac{1}{(D-4)(D+1)} e^{4z}$

$= \frac{ze^{4z}}{1!(4+1)}$

$= \frac{1}{5} ze^{4z}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1e^{4z} + C_2e^{-z} + \frac{1}{5}ze^{4z}$

Using z = logx i.e. $e^z = x$, we get,

$y = C_1x^4 + \frac{C_2}{x} + \frac{1}{5}x^4 \log x$

be the G.S. of given homogeneous LDE.

==

Ex. Solve $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = \frac{2}{x^2}\log x$

Solution: Let $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = \frac{2}{x^2}\log x$ i.e. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$… (i)

be the given homogeneous linear differential equation.

To solve it we put $x = e^z$ i.e. z = logx and $D = \frac{d}{dz}$, we get,

$x\frac{dy}{dx} = Dy$ and $x^2\frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) - D + 1]y = 2z$

i.e. $(D^2 - 2D + 1)y = 2z$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 2D + 1 = 0$

i.e. $(D-1)^2 = 0$

$\therefore$ D = 1, 1 are the roots of an A.E.

$\therefore$ C.F. $= (C_1 + C_2z)e^z$

and P.I.$= \frac{1}{D^2 - 2D + 1} 2z$

$= \frac{2}{[1-(2D-D^2)]} z$

$= 2[1+(2D - D^2)+(2D - D^2)^2 + \ldots]z$

$= 2[z+2(1)+0]$

$= 2z + 4$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_2z)e^z + 2z + 4$

Using z = logx i.e. $e^z = x$, we get,

$y = (C_1 + C_2\log x)x + 2\log x + 4$

be the G.S. of given homogeneous LDE.

Ex. Solve $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2 \log x$

Solution: Let $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2 \log x$ … (i)

be the given homogeneous linear differential equation.

To solve it we put $x = e^z$ i.e. $z = \log x$ and $D = \frac{d}{dz}$, we get,

$x\frac{dy}{dx} = Dy$ and $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) - 4D + 6]y = e^{2z}z$

i.e. $(D^2 - 5D + 6)y = e^{2z}z$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 5D + 6 = 0$

i.e. $(D-2)(D-3) = 0$

$\therefore$ D = 2, 3 are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{2z} + C_2e^{3z}$

and P.I.$= \frac{1}{(D-2)(D-3)} e^{2z}z$

$= e^{2z} \frac{1}{(D+2-2)(D+2-3)} z$

$= e^{2z} \frac{1}{D(D-1)} z$

$= -e^{2z} \frac{1}{D(1-D)} z$

$= -e^{2z} \frac{1}{D}[1 + D + D^2 + \ldots]z$

$= -e^{2z} \frac{1}{D}[z + 1 + 0]$

$= -e^{2z} \int (z+1)dz$

$= -e^{2z}(\frac{1}{2}z^2 + z)$

$= -\frac{1}{2}e^{2z}(z+1)z$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1e^{2z} + C_2e^{3z} - \frac{1}{2}e^{2z}(z+1)z$

Using $z = \log x$ i.e. $e^z = x$, we get,

$y = C_1x^2 + C_2x^3 - \frac{1}{2}x^2(\log x + 1)\log x$

be the G.S. of given homogeneous LDE.

===

Ex. Solve $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$

Solution: Let $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$ i.e. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x}$… (i)

be the given homogeneous linear differential equation.

To solve it we put $x = e^z$ i.e. $z = \log x$ and $D = \frac{d}{dz}$, we get,

$x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$ and $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$

Equation (i) becomes,

$[D(D-1)(D-2) + 2D(D-1) - D + 1]y = \frac{1}{e^z}$

i.e. $(D^3 - 3D^2 + 2D + 2D^2 - 2D - D + 1)y = e^{-z}$

i.e. $(D^3 - D^2 - D + 1)y = e^{-z}$

Which is LDE with constant coefficients.

It's A.E. is $D^3 - D^2 - D + 1 = 0$

i.e. $D^2(D-1) - (D-1) = 0$

i.e. $(D-1)(D^2-1) = 0$

i.e. $(D-1)^2(D+1) = 0$

$\therefore$ D = 1, 1, -1 are the roots of an A.E.

$\therefore$ C.F. $= (C_1 + C_2z)e^z + C_3e^{-z}$

and P.I. $= \frac{1}{(D-1)^2(D+1)} e^{-z}$

$= \frac{ze^{-z}}{1!(-1-1)^2}$

$= \frac{z}{4e^z}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_2z)e^z + C_3e^{-z} + \frac{z}{4e^z}$

Using $z = \log x$ i.e. $e^z = x$, we get,

$y = (C_1 + C_2\log x)x + \frac{C_3}{x} + \frac{\log x}{4x}$

be the G.S. of given homogeneous LDE.

==

Legendre's Linear Equation: A differential equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + p_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + p_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_n y = X$$

where $p_1, p_2, \ldots, p_n$ are constants and X is a function of x only, is called a Legendre's linear equation of order n.

Remark: i) To convert Legendre's linear equation to a homogeneous linear differential equation form put ax+b = u.

ii) To convert Legendre's linear equation to a linear differential equation form put $ax+b = e^z$ i.e. z = log(ax+b).

Method of Solving Legendre's Linear Equation:

Consider the Legendre's linear equation

$$(ax+b)^n \frac{d^n y}{dx^n} + p_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + p_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_n y = X$$

To solve it we change variable x to z by putting

$ax+b = e^z$ i.e. z = log(ax+b) and $D = \frac{d}{dz}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}.\frac{a}{ax+b}$ $\qquad \because \frac{dz}{dx} = \frac{a}{ax+b}$

$\therefore (ax+b)\frac{dy}{dx} = a\frac{dy}{dz} = aDy$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{a}{ax+b}.\frac{dy}{dz}\right]$

$= \frac{a}{ax+b}.\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{a^2}{(ax+b)^2}.\frac{dy}{dz}$

$= \frac{a}{ax+b}.\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{a^2}{(ax+b)^2}.\frac{dy}{dz}$

$= \frac{a^2}{(ax+b)^2}.\frac{d^2y}{dz^2} - \frac{a^2}{(ax+b)^2}.\frac{dy}{dz}$

$\therefore (ax+b)^2\frac{d^2y}{dx^2} = a^2\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right] = a^2(D^2 - D)y = a^2D(D-1)y$

Similarly $(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$

and so on, in general $(ax+b)^r\frac{d^ry}{dx^r} = a^rD(D-1)(D-2)(D-3)\ldots(D-r+1)y$

$\therefore$ Equation (i) becomes,

$$[a^nD(D-1)(D-2)\ldots(D-n+1) + p_1a^{n-1}D(D-1)(D-2)\ldots(D-n+2) + p_2a^{n-2}D(D-1)(D-2)\ldots(D-n+3) + \ldots\ldots + p_{n-1}aD + p_n]y = Z$$

Which is linear differential equation with constant coefficients and Z is function of z. Using usual method, G.S. in y and z is obtained. In this solution putting z = log(ax+b), we get required G.S. of given equation.

===

Ex. Solve $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4$

Solution: Let $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4$ … (i)

be the given Legendre's linear equation.

To solve it we put $x+2 = e^z$ i.e. $z = \log(x+2)$ and $D = \frac{d}{dz}$, we get,

$(x+2)\frac{dy}{dx} = Dy$ and $(x+2)^2 \frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) - D + 1]y = 3(e^z-2) + 4$

i.e. $(D^2 - 2D + 1)y = 3e^z-2$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 2D + 1 = 0$

i.e. $(D-1)^2 = 0$

∴ D = 1, 1 are the roots of an A.E.

∴ C.F. = $(C_1 + C_2z)e^z$

and P.I.= $\frac{1}{(D-1)^2}(3e^z-2)$

$= \frac{1}{(D-1)^2}3e^z - \frac{1}{(D-1)^2}2e^{0z}$

$= \frac{3z^2e^z}{2!} - \frac{2e^{0z}}{(0-1)^2}$

$= \frac{3}{2}z^2e^z - 2$

∴ G.S. = C.F. + P.I.

i.e. $y = (C_1 + C_2z)e^z + \frac{3}{2}z^2e^z - 2$

Using $z = \log(x+2)$ i.e. $e^z = x+2$, we get,

$y = [C_1 + C_2\log(x+2)](x+2) + \frac{3}{2}(x+2)^2[\log(x+2)]^2 - 2$

be the G.S. of given Legendre's equation.

==

Ex. Solve $(x+3)^2 \frac{d^2y}{dx^2} - 4(x+3)\frac{dy}{dx} + 6y = \log(x+3)$

Solution: Let $(x+3)^2 \frac{d^2y}{dx^2} - 4(x+3)\frac{dy}{dx} + 6y = \log(x+3)$… (i)

be the given Legendre's linear equation.

To solve it we put $x+3 = e^z$ i.e. $z = \log(x+3)$ and $D = \frac{d}{dz}$, we get,

$(x+3)\frac{dy}{dx} = Dy$ and $(x+3)^2 \frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1)-4D+6]y = z$

i.e. $(D^2 - 5D + 6)y = z$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 5D + 6 = 0$

i.e. $(D-2)(D-3) = 0$

$\therefore$ D = 2, 3 are the roots of an A.E.

$\therefore$ C.F. = $C_1e^{2z} + C_2e^{3z}$

and P.I.= $\frac{1}{D^2-5D+6}z$

$= \frac{1}{6[1-(\frac{5}{6}D-\frac{1}{6}D^2)]}z$

$= \frac{1}{6}[1+(\frac{5}{6}D-\frac{1}{6}D^2)+(\frac{5}{6}D-\frac{1}{6}D^2)^2+\ldots]z$

$= \frac{1}{6}[z+\frac{5}{6}(1)+0]$

$= \frac{1}{6}z+\frac{5}{36}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1e^{2z} + C_2e^{3z} + \frac{1}{6}z + \frac{5}{36}$

Using $z = \log(x+3)$ i.e. $e^z = x+3$, we get,

$y = C_1(x+3)^2 + C_2(x+3)^3 + \frac{1}{6}\log(x+3) + \frac{5}{36}$

be the G.S. of given Legendre's equation.

==

Ex. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Solution: Let $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$ … (i)

be the given Legendre's linear equation.

To solve it we put $1+x = e^z$ i.e. $z = \log(1+x)$ and $D = \frac{d}{dz}$, we get,

$(1+x)\frac{dy}{dx} = Dy$ and $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1)+D+1]y = 2\sin z$

i.e. $(D^2 + 1)y = 2\sin z$

Which is LDE with constant coefficients.

It's A.E. is $D^2 + 1 = 0$

$\therefore$ $D = \pm i$ are the roots of an A.E.

$\therefore$ C.F. = $e^{0z}(C_1\cos z + C_2\sin z) = C_1\cos z + C_2\sin z$

and P.I.= $\frac{1}{D^2+1}(2\sin z)$

$= \frac{-2z\cos z}{(2\times 1)}$

$= -z\cos z$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1\cos z + C_2\sin z - z\cos z = [C_1 - z]\cos z + C_2\sin z$

Using $z = \log(1+x)$ i.e. $e^z = 1+x$, we get,

$y = [C_1 - \log(1+x)]\cos\log(1+x) + C_2\sin\log(1+x)$

be the G.S. of given Legendre's equation.

==

Ex. Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

Solution: Let $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$ … (i)

be the given Legendre's linear equation.

To solve it we put $1+x = e^z$ i.e. $z = \log(1+x)$ and $D = \frac{d}{dz}$, we get,

$(1+x)\frac{dy}{dx} = Dy$ and $(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

Equation (i) becomes,

$[D(D-1) + D + 1]y = 4\cos z$

i.e. $(D^2+1)y = 4\cos z$

Which is LDE with constant coefficients.

It's A.E. is $D^2 + 1 = 0$

∴ $D = \pm i$ are the roots of an A.E.

∴ C.F. = $e^{0z}(C_1\cos z + C_2\sin z) = C_1\cos z + C_2\sin z$

and P.I.= $\frac{1}{D^2+1}(4\cos z)$

$= \frac{4z\sin z}{(2\times 1)}$

$= 2z\sin z$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1\cos z + C_2\sin z + 2z\sin z = C_1\cos z + (C_2 + 2z)\sin z$

Using $z = \log(1+x)$ i.e. $e^z = 1+x$, we get,

$y = C_1\cos\log(1+x) + [C_2 + 2\log(1+x)]\sin\log(1+x)$

be the G.S. of given Legendre's equation.

==

Ex. Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Solution: Let $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1$ … (i)

be the given Legendre's linear equation.

To solve it we put $3x+2 = e^z$ i.e. $z = \log(3x+2)$ and $D = \frac{d}{dz}$, we get,

$(3x+2)\frac{dy}{dx} = 3Dy$ and $(3x+2)^2 \frac{d^2y}{dx^2} = 9D(D-1)y$

Equation (i) becomes,

$[9D(D-1) + 3(3D) - 36]y = 3(\frac{e^z-2}{3})^2 + 4(\frac{e^z-2}{3}) + 1$

i.e. $(9D^2 - 9D + 9D - 36)y = \frac{1}{3}(e^{2z} - 4e^z + 4) + \frac{4}{3}(e^z - 2) + 1$

i.e. $(9D^2 - 36)y = \frac{1}{3}e^{2z} - \frac{4}{3}e^z + \frac{4}{3} + \frac{4}{3}e^z - \frac{8}{3} + 1$

i.e. $9(D^2 - 4)y = \frac{1}{3}e^{2z} - \frac{1}{3}$

i.e. $(D^2 - 4)y = \frac{1}{27}e^{2z} - \frac{1}{27}$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 4 = 0$

i.e. $(D-2)(D+2) = 0$

∴ $D = 2, -2$ are the roots of an A.E.

∴ C.F. $= C_1e^{2z} + C_2e^{-2z}$

and P.I. $= \frac{1}{(D-2)(D+2)}(\frac{1}{27}e^{2z} - \frac{1}{27})$

$= \frac{1}{27}[\frac{1}{(D-2)(D+2)}e^{2z} - \frac{1}{(D-2)(D+2)}e^{0z}]$

$= \frac{1}{27}[\frac{ze^{2z}}{(1!)(2+2)} - \frac{e^{0z}}{(0-2)(0+2)}]$

$= \frac{1}{108}(ze^{2z} + 1)$

∴ G.S. = C.F. + P.I.

i.e. $y = C_1e^{2z} + C_2e^{-2z} + \frac{1}{108}(ze^{2z} + 1)$

Using $z = \log(3x+2)$ i.e. $e^z = 3x+2$, we get,

$y = C_1(3x+2)^2 + \frac{C_2}{(3x+2)^2} + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$

be the G.S. of given Legendre's equation.

==

Ex. Solve $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

Solution: Let $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$ … (i)

be the given Legendre's linear equation.

To solve it we put $2x+1 = e^z$ i.e. $z = \log(2x+1)$ and $D = \frac{d}{dz}$, we get,

$(2x+1)\frac{dy}{dx} = 2Dy$ and $(2x+1)^2\frac{d^2y}{dx^2} = 4D(D-1)y$

Equation (i) becomes,

$[4D(D-1) - 2(2D) - 12]y = 6(\frac{e^z-1}{2})$

i.e. $(4D^2 - 4D - 4D - 12)y = 3(e^z - 1)$

i.e. $(4D^2 - 8D - 12)y = 3(e^z - 1)$

i.e. $4(D^2 - 2D - 3)y = 3(e^z - 1)$

i.e. $(D^2 - 2D - 3)y = \frac{3}{4}(e^z - 1)$

Which is LDE with constant coefficients.

It's A.E. is $D^2 - 2D - 3 = 0$

i.e. $(D-3)(D+1) = 0$

$\therefore D = 3, -1$ are the roots of an A.E.

$\therefore$ C.F. $= C_1e^{3z} + C_2e^{-z}$

and P.I.$= \frac{1}{(D-3)(D+1)}\frac{3}{4}(e^z - 1)$

$= \frac{3}{4}[\frac{1}{(D-3)(D+1)}e^z - \frac{1}{(D-3)(D+1)}e^{0z}]$

$= \frac{3}{4}[\frac{e^z}{(1-3)(1+1)} - \frac{e^{0z}}{(0-3)(0+1)}]$

$= \frac{3}{4}(-\frac{1}{4}e^z + \frac{1}{3})$

$= -\frac{3}{16}e^z + \frac{1}{4}$

$\therefore$ G.S. = C.F. + P.I.

i.e. $y = C_1e^{3z} + C_2e^{-z} - \frac{3}{16}e^z + \frac{1}{4}$

Using $z = \log(2x+1)$ i.e. $e^z = 2x+1$, we get,

$y = C_1(2x+1)^3 + C_2(2x+1)^{-1} - \frac{3}{16}(2x+1) + \frac{1}{4}$

i.e. $y = C_1(2x+1)^3 + \frac{C_2}{(2x+1)} - \frac{3}{8}x + \frac{1}{16}$

be the G.S. of given Legendre's equation.

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MULTIPLE CHOICE QUESTIONS (MCQ'S)

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1) A differential equation $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2$ is ……

A) L.D.E. with constant coefficient's B) Homogeneous L.D.E.

C) Non-Homogeneous L.D.E. D) None of these

2) A differential equation $x^3 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} + 8y = 7x^2$ is ……

A) L.D.E. with constant coefficient's B) Homogeneous L.D.E.

C) Non-Homogeneous L.D.E. D) None of these

3) A differential equation $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} - \frac{4}{x^2}y = x^2$ is ……

A) L.D.E. with constant coefficient's B) Homogeneous L.D.E.

C) Non-Homogeneous L.D.E. D) None of these

4) A differential equation of the form

$x^n \frac{d^ny}{dx^n} + P_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + P_2x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_{n-1}x \frac{dy}{dx} + P_ny = X$

Where $P_1, P_2, P_3, \ldots\ldots\ldots.P_n$ are constants and X is function of x only is called ……

A) L.D.E. with constant coefficient's B) Homogeneous L.D.E.

C) Non-Homogeneous L.D.E. D) None of these

5) A homogeneous linear differential equation is also called ……

A) Cauchy's linear equation B) Legendre's linear equation

C) Non-Homogeneous L.D.E. D) None of these

6) A differential equation of the form

$x^n \frac{d^ny}{dx^n} + P_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + P_2x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots.+ P_{n-1}x \frac{dy}{dx} + P_ny = X$

Where $P_1, P_2, P_3, \ldots\ldots P_n$ are constants and X is function of x only can be reduced to L.D.E. with constant coefficient form by substitution ……

A) z = logx B) x = logz C) $z = e^x$ D) None of these

7) A differential equation $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2$ can be reduced to L.D.E. with constant coefficient form by substitution ……

A) x = logz B) z = logx C) $z = e^x$ D) None of these

8) A differential equation $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2\log x$ can be reduced to L.D.E. with constant coefficient form by substitution ……

A) x = logz B) z = logx C) $z = e^x$ D) None of these

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9) If $D \equiv \frac{d}{dz}$ and z = logx then $x\frac{dy}{dx}$ = ……

A) Dy B) D(D-1)y C) D(D-1)(D-2)y D) None of these

10) If $D \equiv \frac{d}{dz}$ and z = logx then $x^2\frac{d^2y}{dx^2}$ = ……

A) Dy B) D(D-1)y C) D(D-1)(D-2)y D) None of these

11) If $D \equiv \frac{d}{dz}$ and z = logx then $x^3\frac{d^3y}{dx^3}$ = ……

A) Dy B) D(D-1)y C) D(D-1)(D-2)y D) None of these

12) If $D \equiv \frac{d}{dz}$ and z = logx then $x^r\frac{d^ry}{dx^r}$ = ……

A) D(D-1)(D-2) …… (D-r-1)y B) D(D-1)(D-2) …… (D-r)y

C) D(D-1)(D-2) …… (D-r+1)y D) None of these

13) To reduce the Legendre's Linear Equation

$(ax+b)^n\frac{d^ny}{dx^n} + P_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_{n-1}(ax+b)\frac{dy}{dx} + P_ny = X$

in homogeneous linear differential equation form we substitute ……

A) ax+b = u B) z = log(ax+b) C) x = log(az+b) D) None of these

14) To reduce the Legendre's Linear Equation

$(ax+b)^n\frac{d^ny}{dx^n} + P_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_{n-1}(ax+b)\frac{dy}{dx} + P_ny = X$

into linear differential equation with constant coefficient form we substitute ……

A) ax+b = logz B) z = log(ax+b) C) x = log(az+b) D) None of these

15) The Legendre's Linear Equation $(2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

can be reduced to L.D.E. with constant coefficient form by substitution ……

A) 2x+1 = logz B) z = log(2x+1) C) x = log(2z+1) D) None of these

16) The Legendre's Linear Equation $(2x-1)^3\frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = 0$

can be reduced to L.D.E. with constant coefficient form by substitution ……

A) 2x-1 = logz B) z = log(2x-1) C) x = log(2z-1) D) None of these

17) The Legendre's Linear Equation $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

can be reduced to L.D.E. with constant coefficient form by substitution ……

A) 1+x = logz B) z = log(1+x) C) x = log(z+1) D) None of these

18) If $D \equiv \frac{d}{dz}$ and z = log(ax+b) then $(ax+b)\frac{dy}{dx}$ = ……

A) aDy B) $a^2D(D-1)y$ C) $a^3D(D-1)(D-2)y$ D) None of these

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19) If $D \equiv \frac{d}{dz}$ and z = log(ax+b) then $(ax+b)^2 \frac{d^2y}{dx^2}$ = ……

A) aDy B) $a^2D(D-1)y$ C) $a^3D(D-1)(D-2)y$ D) None of these

20) If $D \equiv \frac{d}{dz}$ and z = log(ax+b) then $(ax+b)^3 \frac{d^3y}{dx^3}$ = ……

A) aDy B) $a^2D(D-1)y$ C) $a^3D(D-1)(D-2)y$ D) None of these

21) If $D \equiv \frac{d}{dz}$ and z = logx then $(ax+b)^r \frac{d^ry}{dx^r}$ = ……

A) $a^rD(D-1)(D-2)$ …… $(D-r-1)y$ B) $a^rD(D-1)(D-2)$ …… $(D-r)y$

C) $a^rD(D-1)(D-2)$ …… $(D-r+1)y$ D) None of these

22) If $D \equiv \frac{d}{dz}$ and z = log(x+2) then $(x+2)\frac{dy}{dx}$ = ……

A) Dy B) 2Dy C) 2D(D-1)y D) None of these

23) If $D \equiv \frac{d}{dz}$ and z = log(x+2) then $(x+2)^2 \frac{d^2y}{dx^2}$ = ……

A) 2Dy B) D(D-1)y C) 4D(D-1)y D) None of these

24) If $D \equiv \frac{d}{dz}$ and z = log(x+2) then $(x+2)^3 \frac{d^3y}{dx^3}$ = ……

A) (D-1)y B) 8D(D-1)y C) D(D-1)(D-2)y D) None of these

25) If $D \equiv \frac{d}{dz}$ and z = log(x+2) then $(x+2)^n \frac{d^ny}{dx^n}$ = ……

A) D(D-1)(D-2)…(D-n-1)y B) D(D-1)(D-2)…(D-n)y

C) D(D-1)(D-2)……(D-n+1)y D) None of these

26) If $D \equiv \frac{d}{dz}$ and z = log(3x+2) then $(3x+2)\frac{dy}{dx}$ = ……

A) 3Dy B) 2Dy C) Dy D) None of these

27) If $D \equiv \frac{d}{dz}$ and z = log(3x+2) then $(3x+2)^2 \frac{d^2y}{dx^2}$ = ……

A) 3Dy B) 9D(D-1)y C) 27D(D-1)(D-2)y D) None of these

28) If $D \equiv \frac{d}{dz}$ and z = log(3x+2) then $(3x+2)^3 \frac{d^3y}{dx^3}$ = ……

A) 3Dy B) 9D(D-1)y C) 27D(D-1)(D-2)y D) None of these

29) If $D \equiv \frac{d}{dz}$ and z = log(3x+2) then $(3x+2)^r \frac{d^ry}{dx^r}$ = ……

A) $3^rD(D-1)(D-2)…(D-r+1)y$ B) $3^rD(D-1)(D-2)…(D-r)y$

C) $3^rD(D-1)(D-2)…(D-r-1)y$ D) None of these

30) If $D \equiv \frac{d}{dz}$ and z = log(4x+1) then $(4x+1)\frac{dy}{dx}$ = ……

A) 4Dy B) 16D(D-1)y C) 64D(D-1)(D-2)y D) None of these

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31) If $D \equiv \frac{d}{dz}$ and z = log(4x+1) then $(4x+1)^2 \frac{d^2y}{dx^2}$ = ……

A) 4Dy B) 16D(D-1)y C) 64D(D-1)(D-2)y D) None of these

32) If $D \equiv \frac{d}{dz}$ and z = log(4x+1) then $(4x+1)^3 \frac{d^3y}{dx^3}$ = ……

A) 4Dy B) 16D(D-1)y C) 64D(D-1)(D-2)y D) None of these

33) If $D \equiv \frac{d}{dz}$ and z = log(4x+1) then $(4x+1)^r \frac{d^ry}{dx^r}$ = ……

A) 4^rD(D-1)(D-2)…(D-r+1)y B) D(D-1)(D-2)…(D-r+1)y

C) 4^rD(D-1)(D-2)…(D-r-1)y D) None of these

34) If $D \equiv \frac{d}{dz}$ and z = log(2x+5) then $(2x+5)\frac{dy}{dx}$ = ……

A) 2Dy B) Dy C) 5D(D-1)y D) None of these

35) If $D \equiv \frac{d}{dz}$ and z = log(2x+5) then $(2x+5)^2 \frac{d^2y}{dx^2}$ = ……

A) 2Dy B) 4D(D-1)y C) 25D(D-1)y D) None of these

36) If $D \equiv \frac{d}{dz}$ and z = log(2x+5) then $(2x+5)^3 \frac{d^3y}{dx^3}$ = ……

A) 2Dy B) 4D(D-1)y C) 8D(D-1)(D-2)y D) None of these

37) If $D \equiv \frac{d}{dz}$ and z = log(2x+5) then $(2x+5)^r \frac{d^ry}{dx^r}$ = ……

A) 2^rD(D-1)(D-2)…(D-r+1)y B) 2^rD(D-1)(D-2)…(D-r)y

C) 2^rD(D-1)(D-2)…(D-r-1)y D) None of these

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."