Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: F.Y.B.SC SEM.-I

SUBJECT: MTH-103(B): GRAPH THEORY

PREPARED BY: PROF. K. D. KADAM

MTH 103(B): GRAPH THEORY

Unit-I. Graphs **No. of Periods – 12**

Graph, Simple graph, Multigraph, Hand shaking lemma, Types of Graphs, Operations on graphs, Subgraphs, Isomorphism of graphs, Walk, path, cycles (circuits).

Unit-2. Connected Graphs **No. of Periods – 12**

Connected and disconnected Graphs, bridges, Cut vertices, edge connectivity and vertex connectivity, Eulerian graph, Hamiltonian Graph, Planer Graph, Euler's Formula for planer graphs, Kuratowski's two graph, Geometrical dual

Unit-3. Trees and Directed Graphs **No. of Periods – 11**

Definition and some properties of trees, Distance and Centre in a tree, Definitions of Rooted and Binary trees, Spanning trees, Minimal Spanning trees, Directed graphs, some types of digraphs.

Unit-4. Applications of the Graphs **No. of Periods – 10**

Existence of Graphs for given number of Vertices and Edges, Coloring of the graphs, Konigsberg Seven Bridge Problem, Travelling salesman Problem, Dijkstra's algorithm, Warshall's algorithm, formation of flowchart using rooted trees.

Reference books:

1. Graph Theory with Applications to Engineering and Computer science. by Narsingh Deo, Prentice Hall of India Pvt. Ltd. 1979.

(Unit I: 1.1, 1.4, 1.5, 2.1, 2.2, 2.4, 2.7; Unit II: 2.5, 2.6, 2.9, 5.2, 5.3, 5.4, 5.5, 5.6; Unit III: 3.1, 3.2, 3.4, 3.5, 3.7, 9.1, 9.2; Unit IV: 1.4, 8.1, 1.2, 2.10, 3.4,11.4, 3.5).

2. Theory and Problems of Discreate Mathematics by Seymour Lipschitz and Marc Lars Lipson, Schaum's outline series, McGraw-Hill Ltd., New York, 2007.

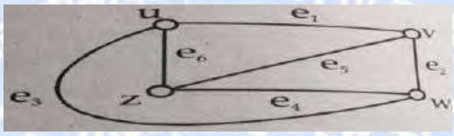
UNIT-1. GRAPHS

Graph: An ordered pair of sets (V(G), E(G)) is called a Graph, where V(G) is non empty set of elements called as vertex set of G and E(G) is family of unordered pairs of elements of V(G) called edge set of G.

Remark: 1) A vertex is also called point or junction or 0-simplex.

2) An edge is also called as line or arc or 1-simplex.

Ex. Describe the graph given below in the form of vertices and edges.



Solution. Given graph is described as G = (V(G), E(G))

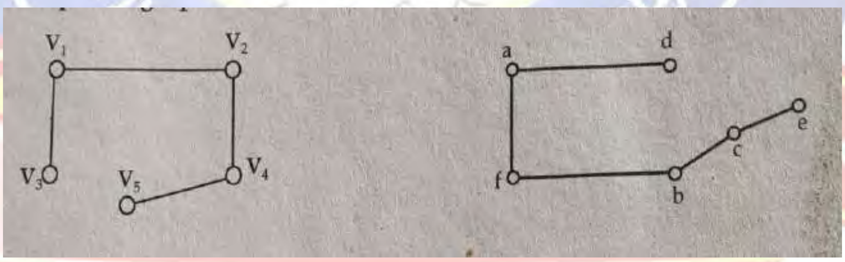
Where V(G) = {u, v, w, z} and E(G) = {e_1, e_2, e_3, e_4, e_5, e_6}

Ex. Draw the graph G = (V(G), E(G)) if

a) V(G) = {v_1, v_2, v_3, v_4, v_5} and E(G) = {(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_4, v_5)}

b) V(G) = {a, b, c, d, e, f} and E(G) = {(a, d), (a, f), (b, c), (b, f), (c, e)}

Solution. Given graphs are drawn as follows



End Vertices: Let e = (u, v) be an edge of graph G, then u and v are called an end vertices of an edge e.

Adjacent Vertices: Two vertices are said to be adjacent vertices if there is an edge between them.

Self-loop (or loop): An edge is said to be self-loop (or loop) of graph G if end vertices of an edge are same.

Multiple Edges or Parallel edges: Two or more edges of a graph are said to be multiple edges or parallel edges if they have same end vertices.

Simple Graph: A graph without self-loops and parallel edges is called simple graph.

Multiple Graph or Multi-graph: A graph with parallel edges is called as multiple graph or multi-graph.

Pseudo-graph: A graph with loop is called as pseudo-graph.

Null Graph: A graph containing no edge is called as null graph. Denoted by N_n.

Finite Graph: A graph having finite number of vertices is called as finite graph.

Infinite Graph: A graph having infinite number of vertices is called as infinite graph.

Order of a Graph: The number of vertices of graph G = (V, E) i.e.$|V|$ is called as order of a graph G.

Size of a Graph: The number of edges of graph G = (V, E) i.e.$|E|$ is called as size of a graph G.

Degree of a Vertex: The number of edges incident at any vertex v of a graph G is called the degree of vertex v. Denoted by d(v).

Note: Degree of a vertex is also called valancy of v.

Pendent Vertex: A vertex of a graph of degree one is called as pendent vertex.

Isolated Vertex: A vertex of a graph of degree zero is called an isolated vertex.

Odd Vertex: A vertex of a graph whose degree is odd is called an odd vertex.

Even Vertex: A vertex of a graph whose degree is even is called an even vertex.

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Hand Shaking Lemma: If G = (V, E) be a graph, then sum of the degrees of all vertices of G is equal to twice the number of edges of G.

i. e. if V = $\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ then $\sum_{i=1}^{n} d(v_i) = 2|E|$

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Ex. In any graph, the number of odd vertices is always even.

Proof: Let G = (V, E) be any graph with n vertices $v_1, v_2, \ldots\ldots, v_k, v_{k+1}, \ldots\ldots, v_n$ and number of edges is $|E| = q$ edges.

Without loss of generality, suppose degrees of $v_1, v_2, \ldots\ldots, v_k$ are even and degrees of $v_{k+1}, \ldots\ldots, v_n$ are odd. By hand shaking lemma

$\sum_{i=1}^{n} d(v_i) = 2|E|$ = an even number.

$\sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{n} d(v_i) = 2|E|$ = an even number.

As sum of even numbers is even.

$\therefore \sum_{i=1}^{k} d(v_i)$ is even.

$\therefore \sum_{i=k+1}^{n} d(v_i)$ is also even.

As $d(v_i)$ is odd for $k+1 \leq i \leq n$,

$\therefore \sum_{i=k+1}^{n} d(v_i)$ must contain an even number of terms.

Hence the number of vertices of odd degree is even.

Hence the number of odd vertices in any graph is always even.

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Ex. Let G = (V(G), E(G)) be a graph.

Where V = {v_1, v_2, v_3, v_4, v_5} and

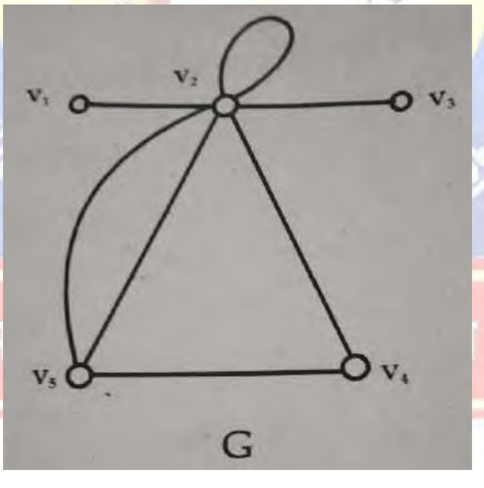
E = {(v_1, v_2), (v_2, v_2), (v_2, v_3), (v_2, v_5), (v_2, v_5), (v_2, v_4), (v_4, v_5)}

i) Draw the graph, ii) Find the degree of each vertex in G,

iii) Find the vertices of odd degree, iv) Verify the hand shaking lemma,

v) Find the order and size of G.

Solution. i) Given graph is drawn as follows



ii) $d(v_1) = 1$, $d(v_2) = 7$, $d(v_3) = 1$, $d(v_4) = 2$, $d(v_5) = 3$

iii) v_1, v_2, v_3 and v_5 are the vertices of odd degree.

iv) Given graph contain 5 vertices and 7 edges.

$\therefore 2|E| = 2 \times 7 = 14$ ......(1)

& $\sum_{i=1}^{5} d(v_i) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$

$= 1 + 7 + 1 + 2 + 3$

= 14

= 2|E| by (1)

Hence the hand shaking lemma is verified.

v) Order of graph G = |V| = 5 and Size of graph G = |E| = 7.

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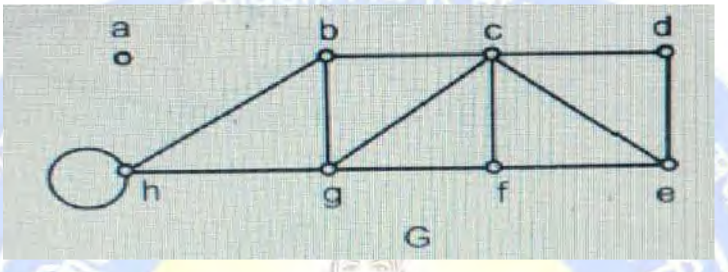
Ex. Verify the hand shaking lemma for the following graph. (Mar.2019)



Proof: Given graph contain 8 vertices and 12 edges.

$\therefore 2|E| = 2 \times 12 = 24$ ……(1)

& $\sum d(a) = d(a) + d(b) + d(c) + d(d) + d(e) + d(f) + d(g) + d(h)$

$= 0 + 3 + 5 + 2 + 3 + 3 + 4 + 4$

= 24

= 2|E| by (1)

Hence the hand shaking lemma is verified.

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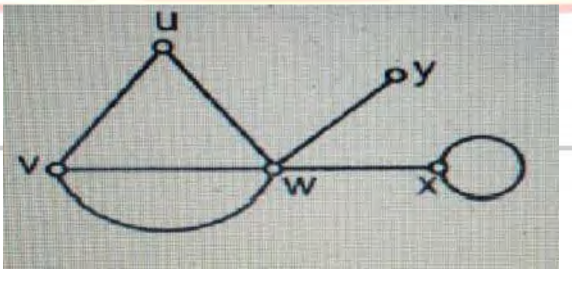
Ex. Verify the hand shaking lemma for the following graph. (Oct.2018)



Proof: Given graph contain 5 vertices and 7 edges.

$\therefore 2|E| = 2 \times 7 = 14$ ……(1)

& $\sum d(u) = d(u) + d(v) + d(w) + d(x) + d(y)$

$= 2 + 3 + 5 + 3 + 1$

= 14

= 2|E| by (1)

Hence the hand shaking lemma is verified.

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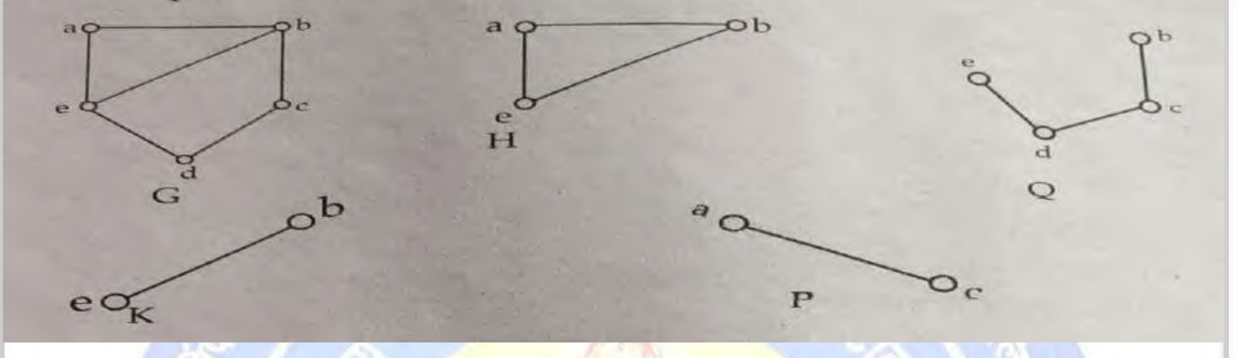
Subgraph: A graph $G_1 = (V_1, E_1)$ is said to be subgraph of graph G = (V, E) if $V_1 \subseteq V$ and $E_1 \subseteq E$ such that end vertices of every edge in E_1 lies in V_1.

e. g.



Here H, Q, K are subgraphs of graph G but P is not subgraph of G because an edge (a, c) of P is not an edge of G.

Note: 1) Every graph is its own subgraph.

2) If H is subgraph of G and K is subgraph of H then K is a subgraph of G.

3) A single vertex in a graph G is also a subgraph of G.

4) A single edge in a G, together with its end vertices is also a subgraph of G.

Vertex Disjoint Subgraphs : Two subgraphs of a graph are said to be vertex disjoint subgraphs if they have no common vertices.

Edge Disjoint Subgraphs : Two subgraphs of a graph are said to be edge disjoint subgraphs if they have no common edges.

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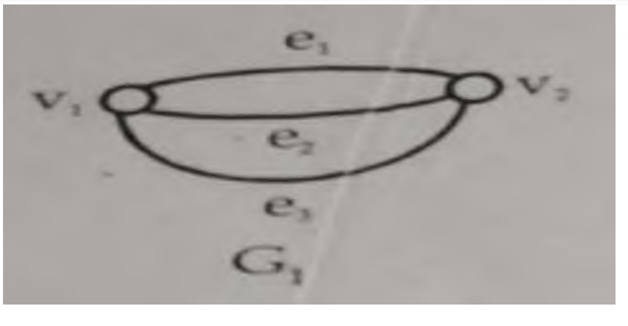
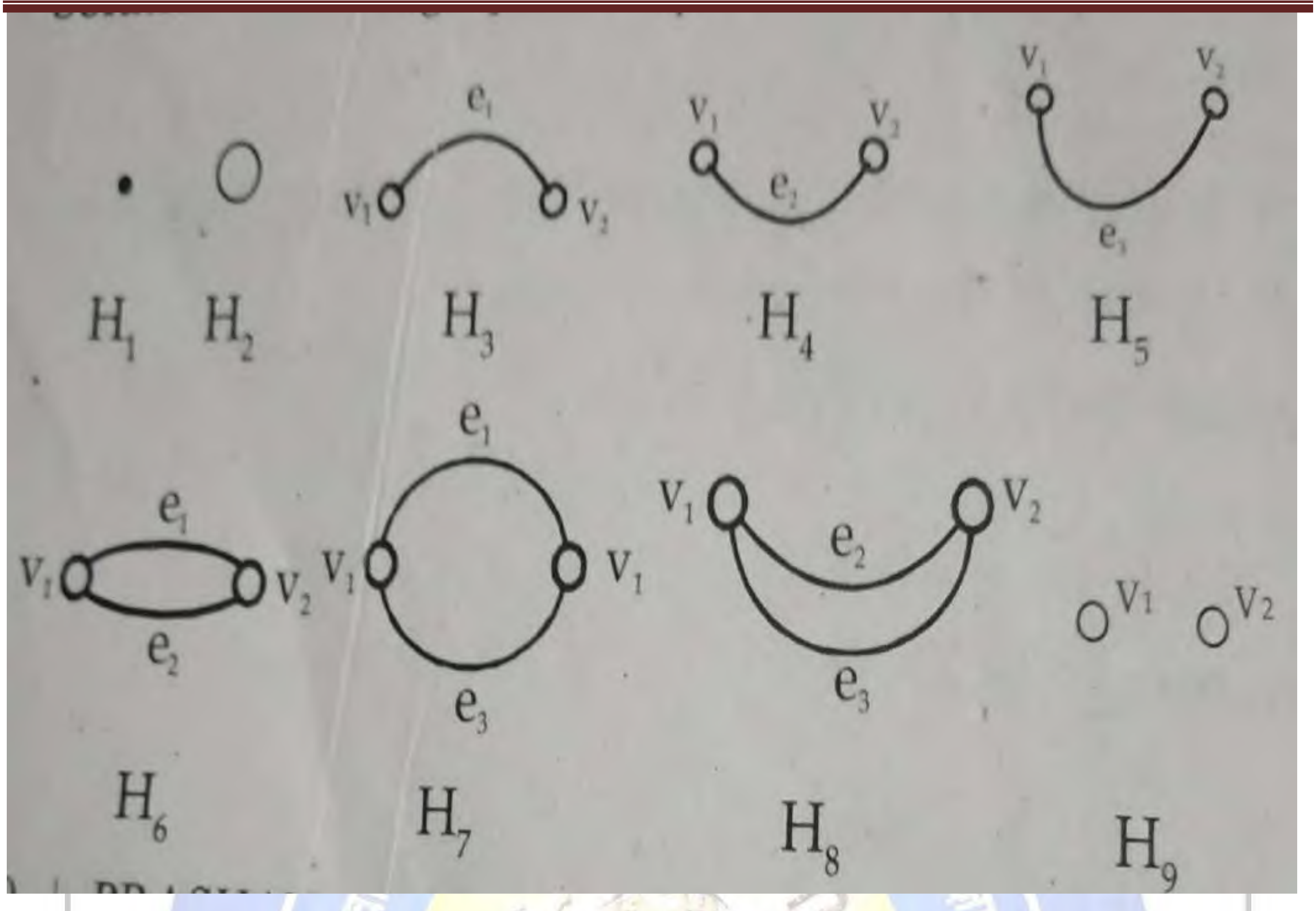
Ex. Find all subgraphs of the following graph.



Solution. All subgraphs of the given graph G_1 are as follows.



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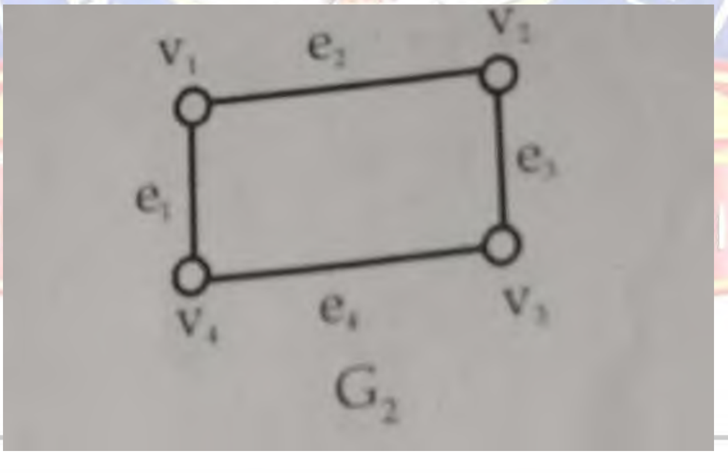
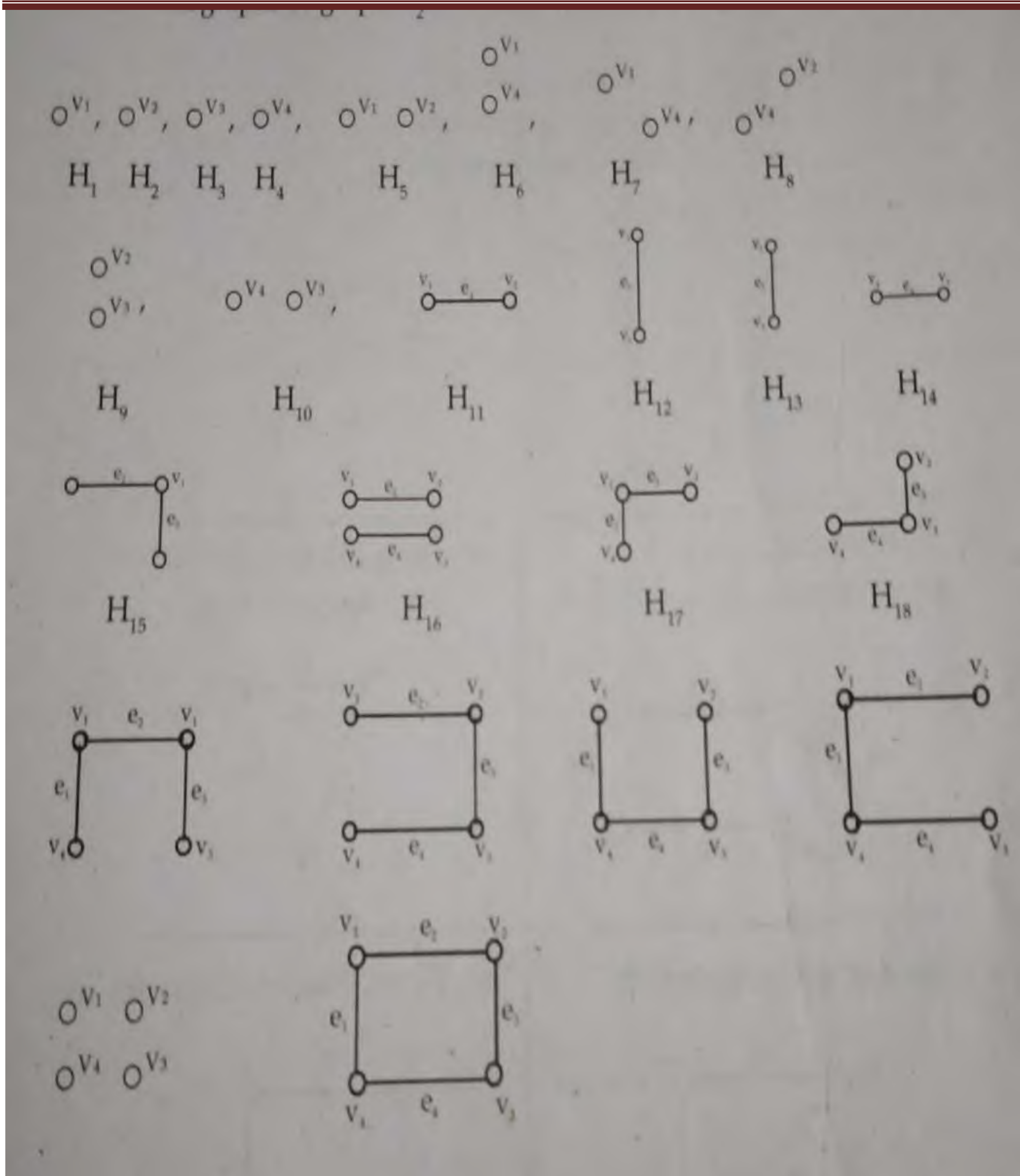
Ex. Find all subgraphs of the following graph.

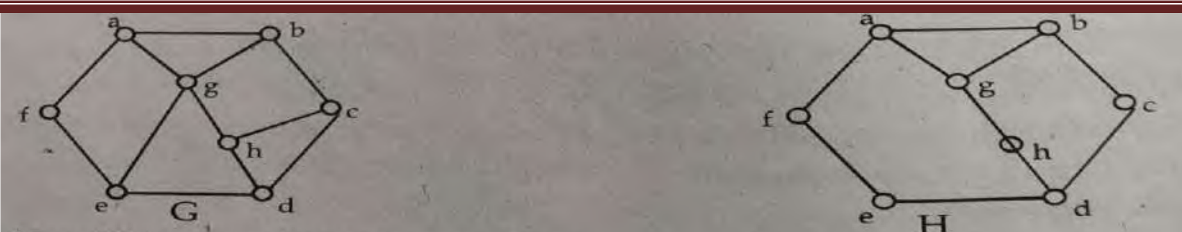


Solution. All subgraphs of the given graph G_2 are as follows.



Spannig Subgraphs: A subgraph H of a graph G is said to be spanning subgraph of G if it contain all the vertices of G.

e.g.



Here H is spanning subgraph of G.

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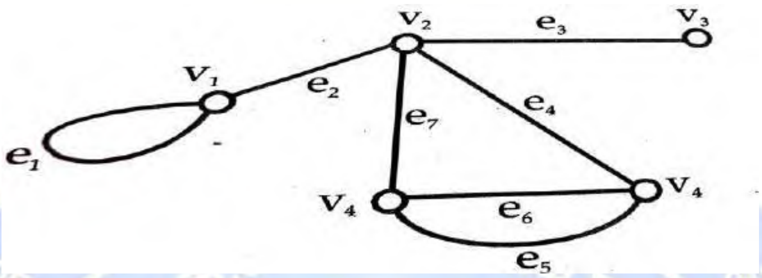
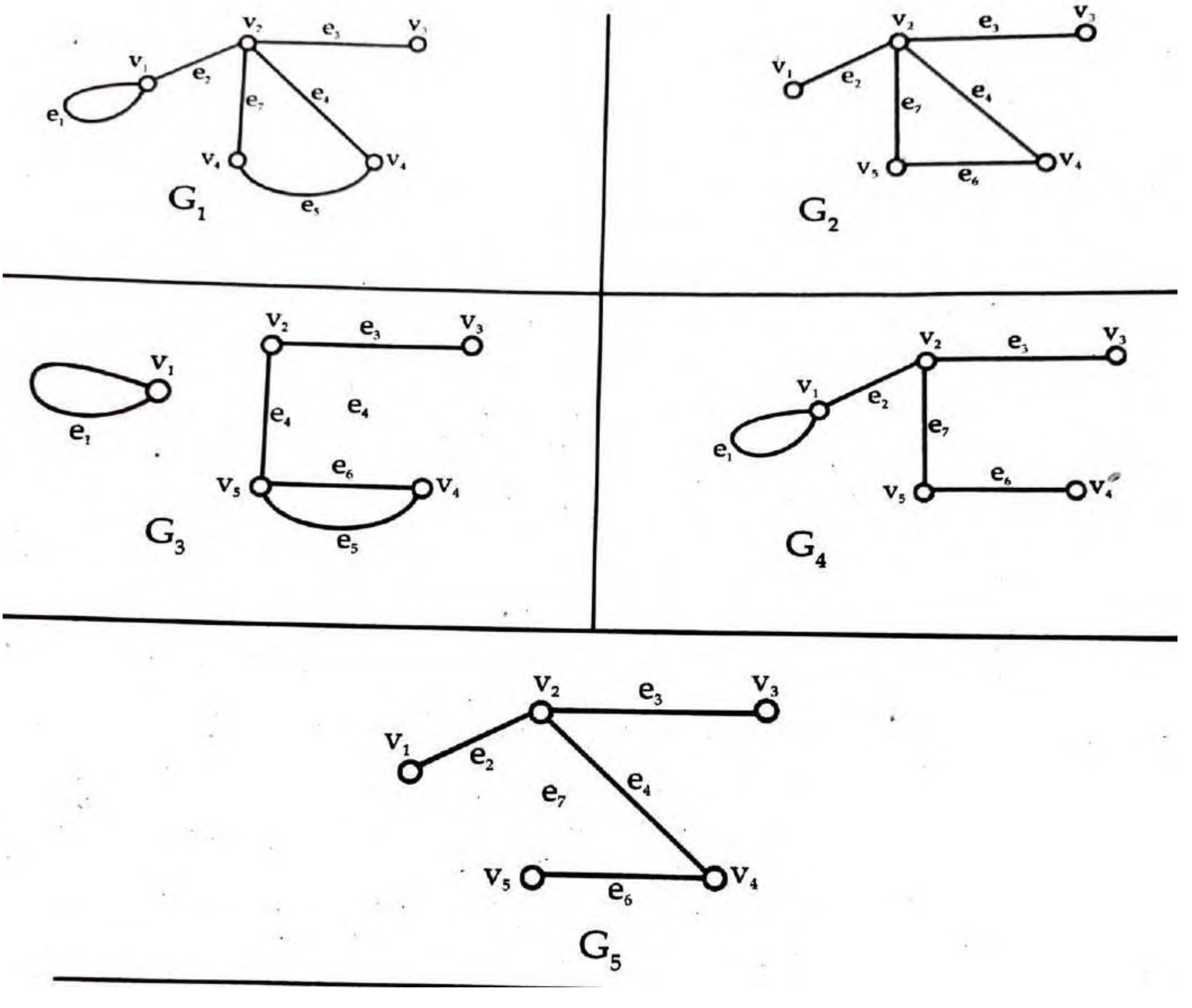
Ex. Find any five spanning subgraphs of a graph G, given below.



Solution. Five spanning subgraphs of a graph G are as follows.



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Induced Subgraph: Let U be the nonempty subset of the vertex set of graph G. Then the subgraph G<U> of G, whose vertex set is U and the edge set consists of those edges of G which are pair of elements of U is called vertex induced or induced subgraph of G.

Edge Induced Subgraph: Let S be the nonempty set of edges of graph G. Then the subgraph G<S> of G, whose vertex set consists of those vertices of G which are incident with at least one edge of S and whose edges are in S is edge induced subgraph of G.

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Isomorphism of graphs: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. A one-one and onto mapping $f : V_1 \longrightarrow V_2$ is said to be an isomorphism from G_1 to G_2, if for all $u, v \in V_1$, the number of edges joining u to v in G_1 is same as number of edges joining f(u) to f(v) in G_2. Denoted by $G_1 \cong G_2$.

Properties of Isomorphism:

1) Every group is isomorphic to itself. i.e. $G \cong G$
2) If G is isomorphic to G' then G' is isomorphic to G.
3) If G is isomorphic to G' and G' is isomorphic to G'' then G is isomorphic to G''
4) Two graphs G_1 and G_2 are isomorphic to each other then

   i) $|V(G_1)| = |V(G_2)|$, ii) $|E(G_1)| = |E(G_2)|$,

   iii) Both graphs must have equal number of vertices of a same degrees.

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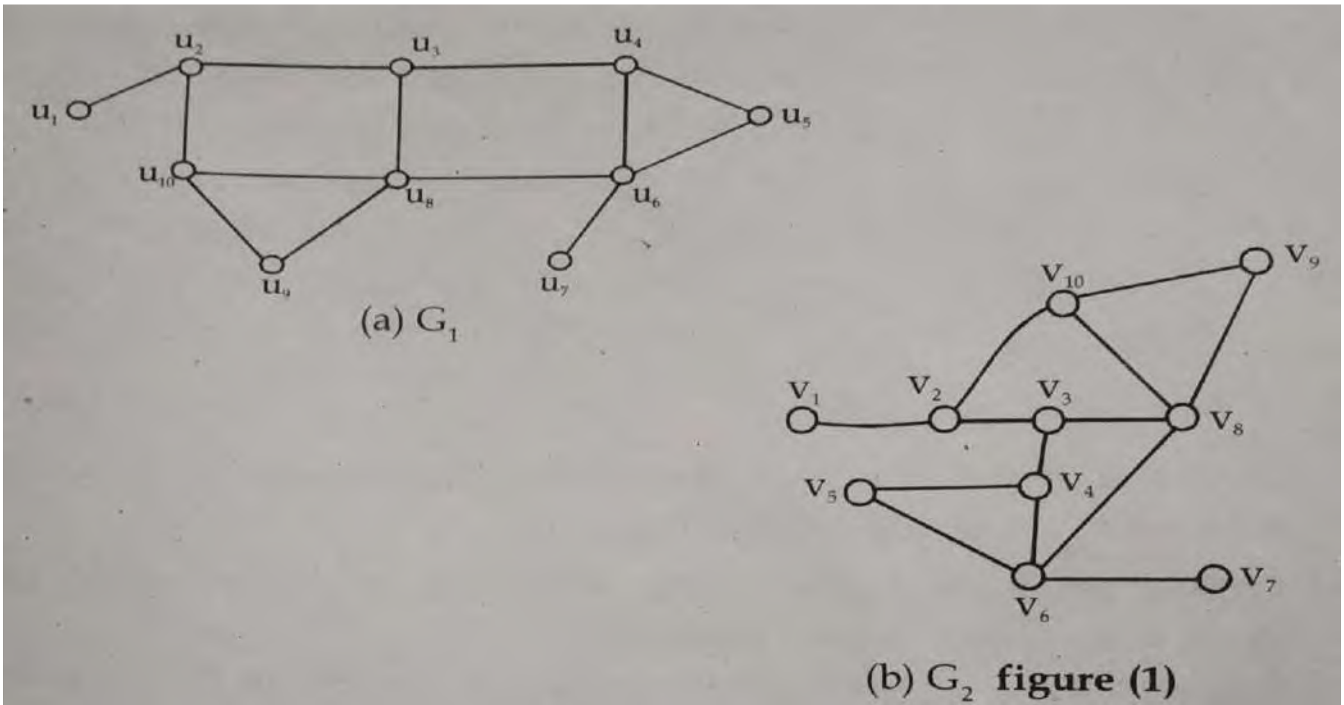
Ex. Are the following graphs isomorphic?



(a) G_1 (b) G_2 **figure (1)**

Solution. We observe that i) $|V(G_1)| = |V(G_2)| = 10$, ii) $|E(G_1)| = |E(G_2)| = 13$,

iii) Both graphs must have equal number of vertices of a same degrees.

Define mapping $f : V(G_1) \longrightarrow V(G_2)$ by $f(u_i) = v_i$ for i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Which is clearly one-one and onto. Hence f is an isomorphism from G_1 to G_2.

$\therefore G_1 \cong G_2$

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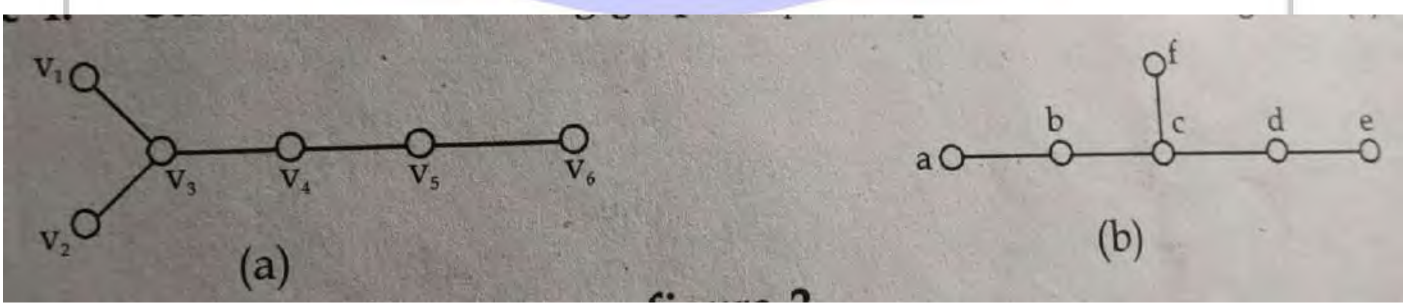
Ex. Are the following graphs T_1 and T_2 isomorphic?



(a) (b)

Solution. We observe that i) $|V(T_1)| = |V(T_2)| = 6$, ii) $|E(T_1)| = |E(T_2)| = 5$,

iii) We observe that in both graphs one vertex of degree 3 i.e. v_3 and c respectively. But in graph T_1, vertex v_3 adjacent to 2 vertices of degree 1 and 1 vertex of degree 2 while in graph T_2, vertex c adjacent to 1 vertex of degree 1 and 2 vertices of degree 2. Hence adjacency is not preserved in T_1 and T_2.

$\therefore$ T_1 is not isomorphic to T_2.

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Ex. Are the following graphs isomorphic to each other? Justify.

Solution. We observe that i) $|V(G_1)| = |V(G_2)| = 6$, ii) $|E(G_1)| = |E(G_2)| = 12$,

iii) We observe that in both graphs all vertices of degree 4.

Define mapping $f : V(G_1) \longrightarrow V(G_2)$ by $f(u_i) = v_i$ for i = 1, 2, 3, 4, 5, 6.

Which is clearly one-one and onto mapping.

Hence f is an isomorphism from G_1 to G_2.

$\therefore G_1 \cong G_2$

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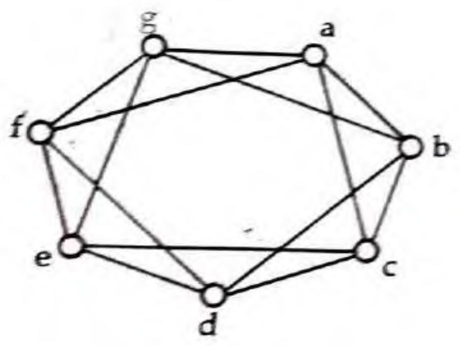
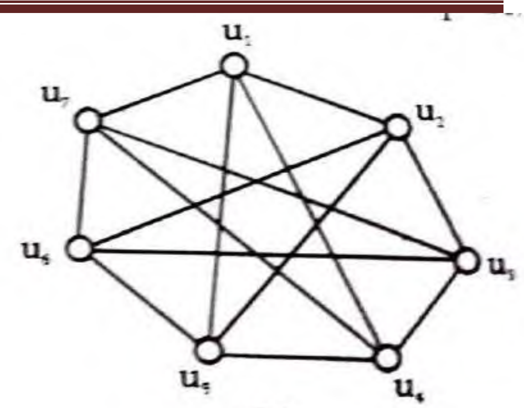
Ex. Show that the graphs G and G' given below are isomorphic.



G



G¹

Solution. We observe that i) $|V(G)| = |V(G')| = 7$, ii) $|E(G)| = |E(G')| = 14$,

iii) We observe that in both graphs all vertices of degree 4.

Define mapping $f : V(G_1) \to V(G_2)$ by

$f(a) = u_1$, $f(b) = u_2$, $f(c) = u_3$, $f(d) = u_4$, $f(e) = u_5$, $f(f) = u_6$, $f(g) = u_7$.

Which is clearly one-one and onto mapping.

Hence f is an isomorphism from G_1 to G_2.

$\therefore G_1 \cong G_2$

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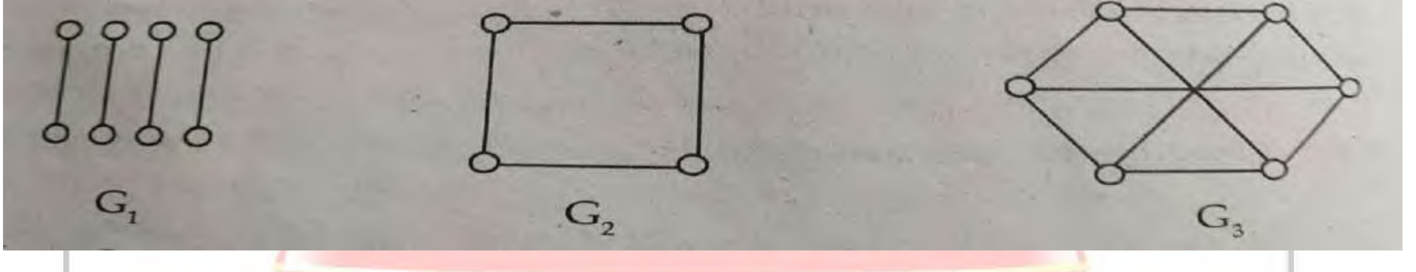
Types of Graphs:

Regular Graph: A graph having same degree of every vertex is called as regular graph.



Here G_1 is regular graph of degree 1.

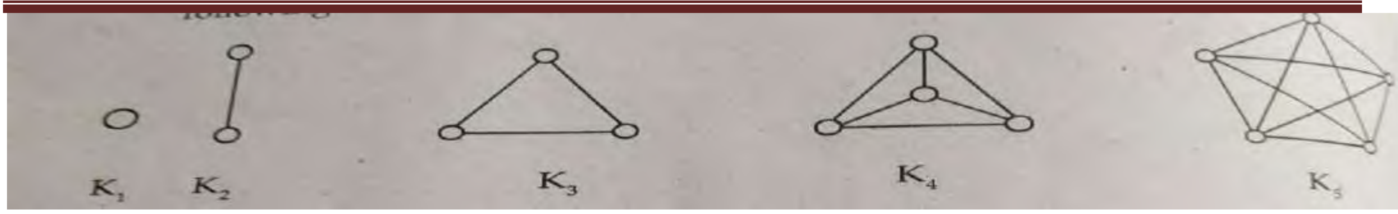
G_2 is regular graph of degree 2.

G_3 is regular graph of degree 3.

Note: 1) Any regular graph of degree 3 is called **cubic graph**.

2) A regular graph of degree 3 of 6 vertices is called **Peterson's graph**.

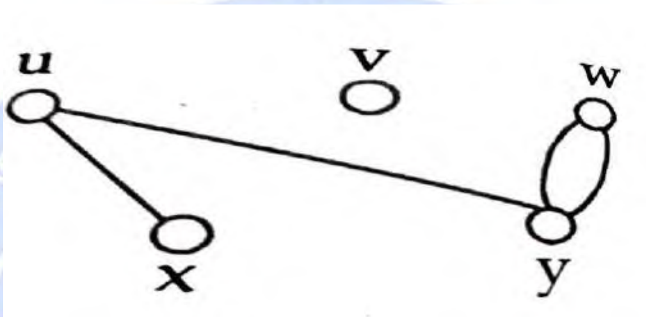
Complete Graph: A simple graph of n vertices having degree of every vertex (n-1) is called as complete graph. Denoted by K_n.



are the complete graphs of 1, 2, 3, 4 and 5 vertices respectively.

Bipartite Graph: A graph G = (V, E) is said to be bipartite graph if a vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that every edge of G has one end vertex in V_1 and another is in V_2.

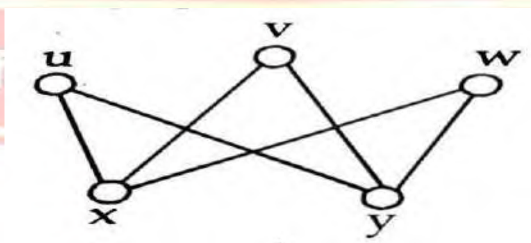
e. g.



is a bipartite graph with partition $V_1 = \{u, v, w\}$ and $V_2 = \{x, y\}$ such that every edge of G has one end vertex in V_1 and another is in V_2.

Complete Bipartite Graph: A graph G = (V, E) is said to be complete bipartite graph if a vertex set V can be partitioned into two nonempty disjoint subsets V_1 and V_2 such that every vertex in V_1 is adjacent to all vertices in V_2. If $n(V_1) = m$ and $n(V_1) = n$ then complete bipartite graph is denoted by $K_{m, n.}$

e. g.



is a complete bipartite graph with partition $V_1 = \{u, v, w\}$ and $V_2 = \{x, y\}$ such that every vertex in V_1 is adjacent to all vertices in V_2.

Remark:1) A complete bipartite graph $K_{m, n.}$ has m+n vertices and mn edges.

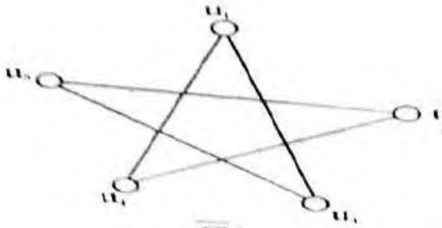
i. e. $|V(Km, n)| = m+n$ and $|E(Km, n)| = mn$

2) A complete bipartite graph $K_{1, n.}$ or $K_{n, 1}$ is **Star graph**

Complement of a Graph: A simple graph $\bar{G}$ is said to be complement of a simple graph G = (V, E) if a vertex sets of G and $\bar{G}$ are same and two vertices are adjacent in $\bar{G}$ if they are not adjacent in G.

e.g.

G



$\bar{G}$

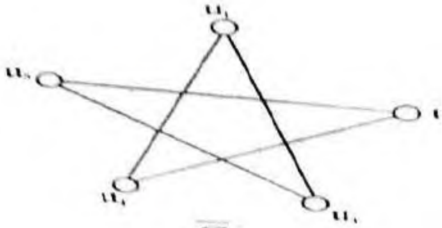
Here graph $\bar{G}$ is complement of graph G.

Note: 1) $\bar{\bar{G}}$ = G, 2) $\overline{K_n}$ = N_n and $\overline{N_n}$ = K_n, 3) Complement of regular graph of n vertices and of r degree is a regular graph of n vertices and of n-r-1 degree

Self-Complementary Graph: A simple graph G is said to be self-complementary graph if G is isomorphic to its complement $\bar{G}$.

e.g.

G

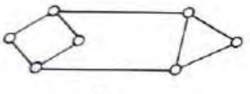


$\bar{G}$

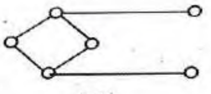
Here $\bar{G}$ = G $\therefore$ G is self-complementary graph.

Complement of a subgraph: Let H be a subgraph of graph G, then the subgraph $\bar{H}$ of graph G is said to be complement of H in G if it obtained by deleting the edges of H in G.

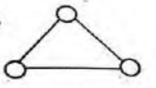
e.g. Here $\bar{H}$ is the complement of subgrah H in G.



G



H



$\bar{H}$

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Operations on graphs:

Removal of Vertex: Let v ∈ V(G), then removal of vertex v from G is a subgraph G - v of G obtained by deleting the vertex v and edges incident at v.

Note: $|V(G - v)| = |V(G)| - 1$ and $|E(G - v)| = |E(G)| - d(v)$

Removal of an edge: Let $e \in E(G)$, then removal of an edge e from G is a subgraph G-e of G obtained by deleting an edge e.

Note: $|V(G - e)| = |V(G)|$ and $|E(G - e)| = |E(G)| - 1$

Union of two graphs: A graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ is said to be union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Intersection of two graphs: A graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ is said to be intersection of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Ring sum of two graphs: A graph $G_1 \oplus G_2 = (V_1 \cup V_2, E_1 \cup E_2 - E_1 \cap E_2)$ is said to be ring sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Addition of Vertex: Let G = (V, E) be any graph and v the vertex not in V, then G + v is a graph whose vertex set is $V \cup \{v\}$ and edge set is $E \cup \{$edges joining v to all vertices of G$\}$

Note: $|V(G + v)| = |V(G)| + 1$ and $|E(G + v)| = |E(G)| + |V(G)|$

Addition of an edge: Let G = (V, E) be any graph. If u, v are in V, then G + e is the graph obtained from graph G by joining u and v by an edge e.

Note: $|V(G + e)| = |V(G)|$ and $|E(G + e)| = |E(G)| + 1$

==

Ex. Prove that the maximum number of edges in a simple graph on n vertices is $\frac{n(n-1)}{2}$

Proof: Let G be a simple graph on n vertices $v_1, v_2, \ldots\ldots, v_n$ and q edges.

In G, any vertex $v_i \in V(G)$ is adjacent to atmost (n-1) vertices of G.

$\therefore d(v_i) \leq (n-1)$ for each i = 1, 2, ……, n.

$\therefore \sum_{i=1}^{n} d(v_i) \leq \sum_{i=1}^{n} (n-1)$

$\Rightarrow 2q \leq n(n-1)$ by Hand Shaking Lemma

$\Rightarrow q \leq \frac{n(n-1)}{2}$

Thus the maximum number of edges in a G is $\frac{n(n-1)}{2}$. Hence Proved.

==

Ex. Show that the total number of edges in a complete graph on n vertices is $\frac{n(n-1)}{2}$

Proof: Let K_n be a complete graph on n vertices $v_1, v_2, \ldots\ldots, v_n$ and q edges.

In complete graph every pair of vertices is adjacent to each other.

$\therefore$ each vertex in K_n is adjacent to all remaining (n-1) vertices of G.

$\therefore d(v_i) = (n-1)$ for each i = 1, 2, ……, n.

$\therefore \sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{n}(n-1)$

$\Longrightarrow 2q = n(n-1)$ by Hand Shaking Lemma

$\Longrightarrow q = \frac{n(n-1)}{2}$

Thus the total number of edges in a K_n is $\frac{n(n-1)}{2}$. Hence Proved.

===

Ex. Does there exist a regular graph of degree 5 on 7 vertices? Justify.

Solution: Let G be a regular graph of degree 5 on 7 vertices and say q edges.

$\therefore$ every vertex of regular graph G is of degree 5.

By Hand Shaking Lemma $\sum_{i=1}^{7} d(v_i) = 2q$

$\therefore 2q = \sum_{i=1}^{7} 5$

$\Longrightarrow 2q = 5 \times 7$

$\Longrightarrow q = \frac{35}{2}$ which is impossible

Hence there does not exist a regular graph of degree 5 on 7 vertices

===

Ex. Does there exist a graph on 5 vertices whose degrees are 1, 2, 3, 4 and 5? Justify.

Solution: Let G be a graph on 5 vertices v_1, v_2, v_3, v_4 and v_5 whose degrees are 1, 2, 3, 4 and 5 respectively and say of q edges.

By Hand Shaking Lemma $\sum_{i=1}^{5} d(v_i) = 2q$

$\therefore 2q = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$

$= 1 + 2 + 3 + 4 + 5$

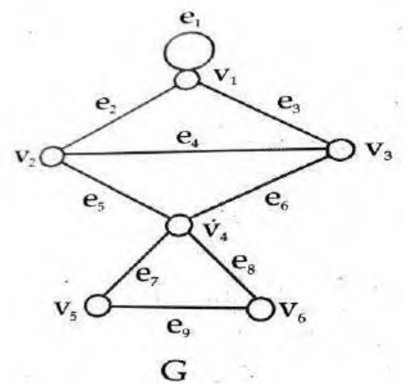
$\Longrightarrow 2q = 15$

$\Longrightarrow q = \frac{15}{2}$ which is impossible

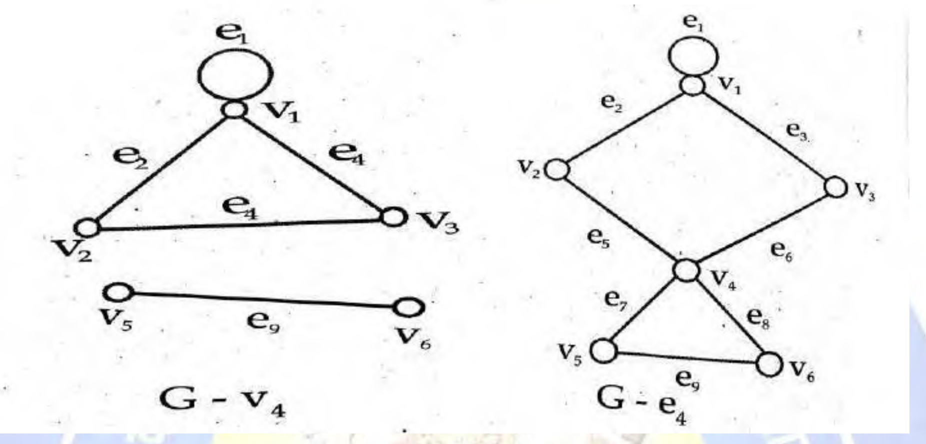
Hence there does not exist a graph on 5 vertices whose degrees are 1, 2, 3, 4 and 5.

===

Ex. Find the subgraphs G - v_4 and G - e_4 from following graph G



Solution: Subgraphs G - v_4 and G - v_4 of graph G are as follows



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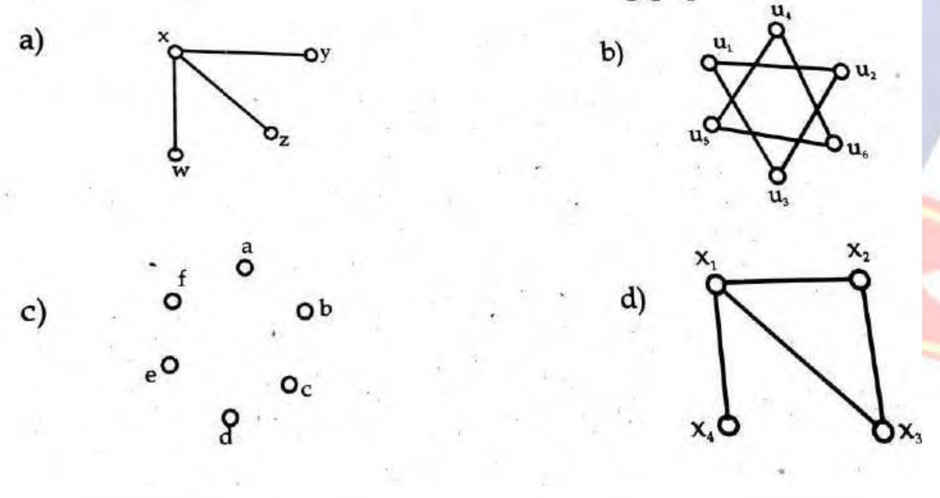
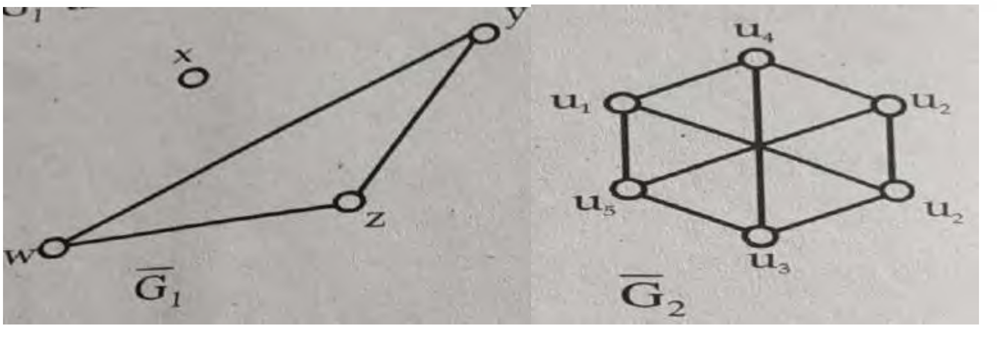
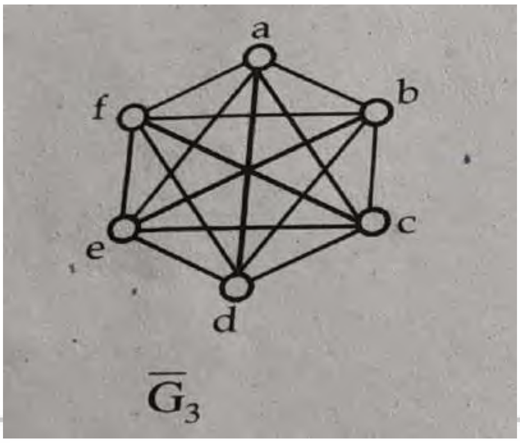
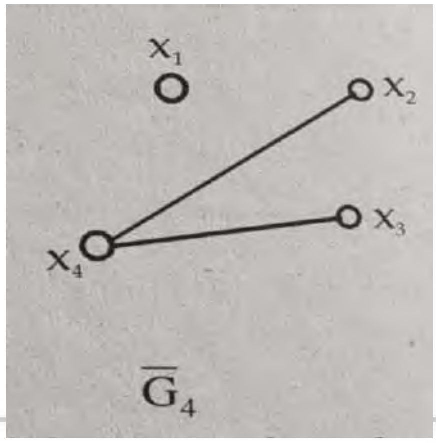
Ex. Find the complement of the following graphs.



Solution: The complements of given graphs are as follows







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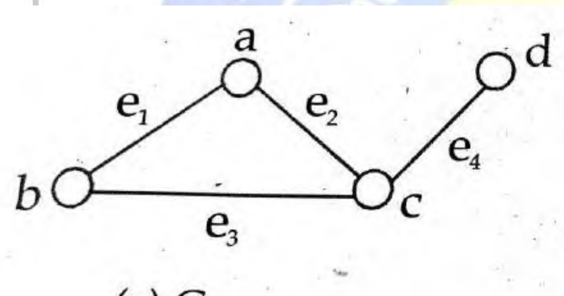
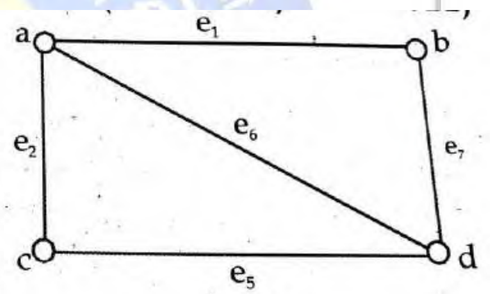
Ex. For the following graphs G_1 and G_2



(a) G_1



(b) G_2

Find the graphs a) $G_1 \cup G_2$, b) $G_1 \cap G_2$, c) $G_1 \oplus G_2$, d) $G_2 + f$, e) $G_2 + e$, $e = (a, b)$

Solution: The graphs a) $G_1 \cup G_2$, b) $G_1 \cap G_2$, c) $G_1 \oplus G_2$, d) $G_2 + f$, e) $G_2 + e$ are as follows

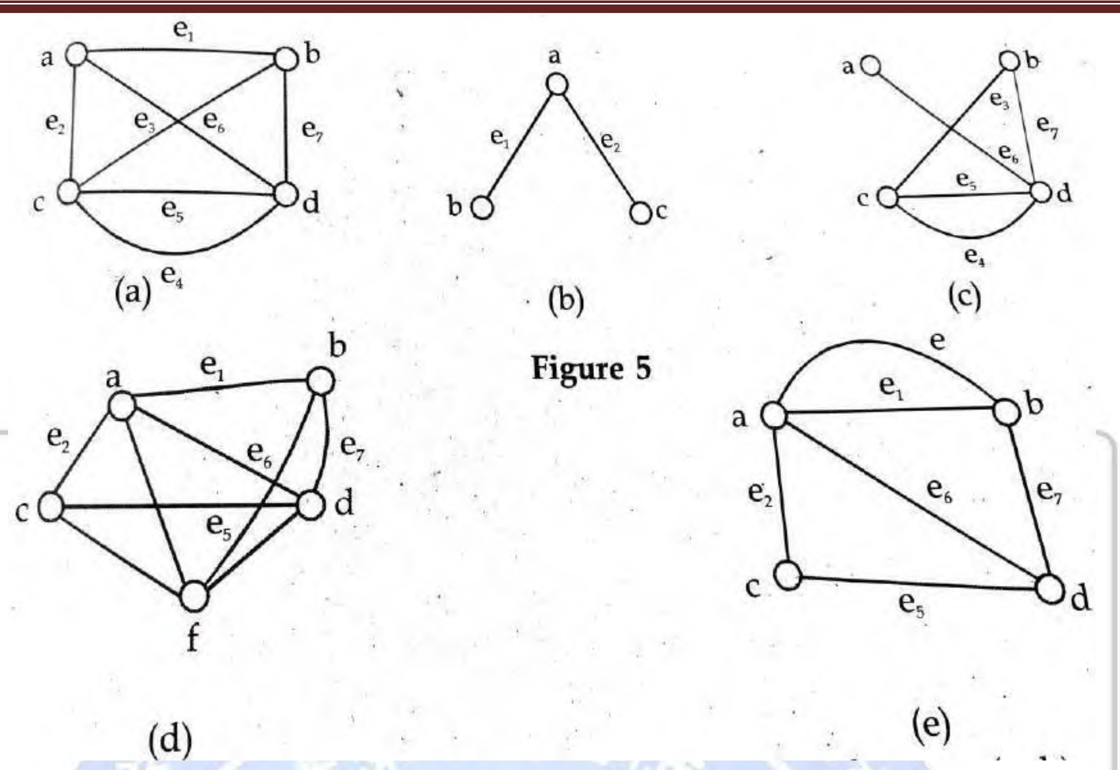


Figure 5

Walk: A finite alternating sequence of vertices and edges, starting and ending with vertices, such that each edge in the sequence is incident at the vertices preceding and following it, is called a walk or an edge sequence in the graph G.

Closed Walk: A walk is said to be closed walk if terminal vertices of walk are same.

Open Walk: A walk is said to be an open walk if terminal vertices of walk are not same.

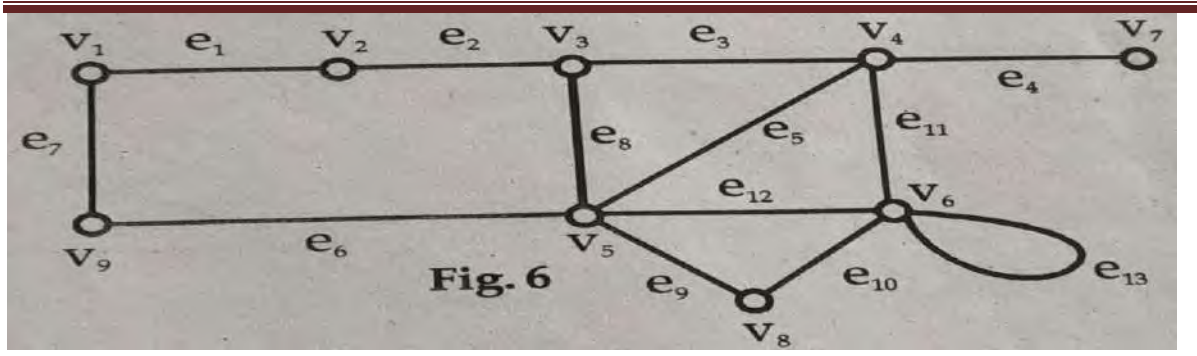
Trail: A walk is said to be trail if no edge is repeated in it.

Path: A walk is said to be path if no vertex is repeated in it.

Circuit (or Cycle): A closed walk is said to be circuit (or cycle) if no vertex is repeated in it except terminal vertices.

Length of a Walk: Total number of edges occur in a walk is called as length of a walk.

e.g. In the following graph



P: $v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_{11}\, v_6\, e_{13}\, v_6\, e_{10}\, v_8$

Q: $v_1\, e_7\, v_9\, e_6\, v_5\, e_5\, v_4\, e_3\, v_3\, e_8\, v_5\, e_{12}\, v_6$

R: $v_1\, e_1\, v_2\, e_2\, v_3\, e_8\, v_5\, e_9\, v_8$

S: $v_3\, e_3\, v_4\, e_{11}\, v_6\, e_{10}\, v_8\, e_9\, v_5\, e_8\, v_3$

T: $v_3\, e_8\, v_5\, e_{12}\, v_6\, e_{13}\, v_6\, e_{11}\, v_4\, e_3\, v_3$

Here P, Q, R, S and T all are walks. S and T are trails. S is a circuit. T is trail but not cycle. P, Q and R are open walks. Lengths of P, Q, R, S and T are 6, 6, 4, 5, 5 resp.

Note: 1) Every path is open trail but not conversely.

2) Every circuit is closed trail but not conversely.

3) Every circuit is a path is open path need not be circuit.

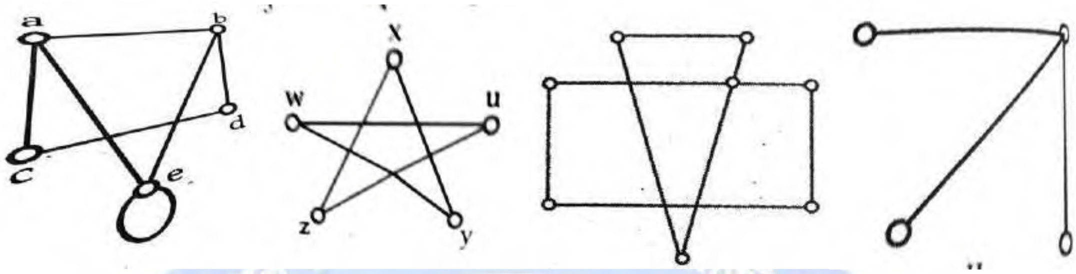
4) In graph G, walk, paths and circuits are subgraphs of G.

===

UNIT-2. CONNECTED GRAPHS

Connected Graph: A graph G = (V, E) is called as connected graph if for every u, v ∈ V there exists at least one u-v path in G.

e.g.



G_1 G_2 G_3 G_4

All above graphs G_1, G_2, G_3 and G_4 are connected, since there is path between every pair of vertices.

Disconnected Graph: A graph G = (V, E) is called as disconnected graph if for some u, v ∈ V there does not exists u-v path in G.

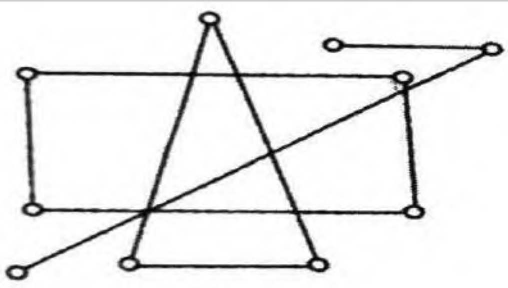
e.g.

G_1 G_2

Both above graphs G_1 and G_2 are disconnected, since there is no path between every pair of vertices.

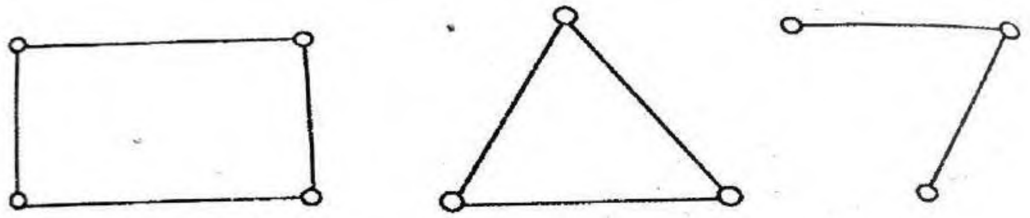
Component of a Graph: A maximal connected subgraph of a graph is called as component of a graph.

e.g.



H

Given graph H is disconnected, since there is no path between every pair of vertices. Its components are as follows.



Distance between two vertices: Minimum number of edges on u-v path is called the distance between two vertices u and v. Denoted by d(u, v).

Diameter of a graph: The maximum distance between two vertices of a graph is called diameter of a graph.

Note: 1) A disconnected graph has connected subgraphs.

2) If G is connected graph, then G is itself a component of G.

3) A graph G is connected iff it has exactly one component.

4) A graph G is disconnected iff it has at least two components.

5) A null graph N_n has n components i.e. each vertex of null graph is a component.

6) A graph G is disconnected iff its vertex set V can be partitioned into two non-empty sub-sets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and other is in V_2.

7) If a graph has exactly two vertices of odd degree then there will be a path joining these two vertices.

8) Let G = (V, E) be a simple graph with k components and |V| = n, |E| = m, then $m \geq n-k$.

==

Cut Vertex: If a graph G has k components and G – v contain more than k components then vertex v is said to be cut vertex of G.

Bridge: If a graph G has k components and G – e contain k+1 components then an edge e is said to be bridge or isthmus of G.

Note:1) An edge of a graph G is a bridge iff e does not lie in any cycle of G.

2) If G is connected graph and v is cut vertex of G, then G – v contain two or more components.

3) If G is connected graph and e is isthmus or bridge of G, then G – e contain exactly

two components.

4) Every non-trivial connected simple graph has at least two vertices which are not cut vertices.

Disconnecting Set: A subset S = {e_1, e_2, e_3,………. e_k} of E(G) is called disconnecting set of connected graph G if G – S is disconnected graph.

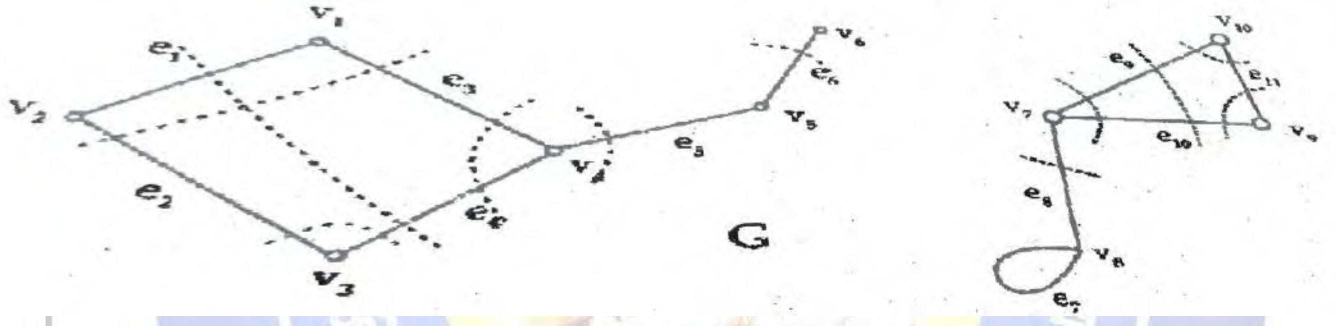
Cut Set: A minimal disconnecting set of graph G is called cut set of G.

Vertex Connectivity: The number of vertices removed to disconnect a graph or increase number of components at least by one. Denoted by K(G)

Edge Connectivity: The number of edges in a smallest cut set is called edge connectivity of G. Denoted by $\lambda(G)$

Note: $\delta(G)$ = Minimum degree of a vertex of G

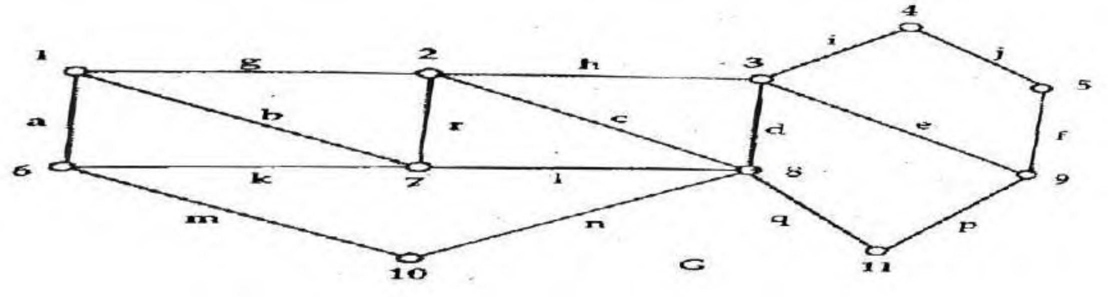
e.g.



1) In this graph the vertices v_4, v_5 and v_7 are cut vertices but other vertices are not cut vertices.
2) In this graph the edges e_5, e_6 and e_8 are bridges but other edges are not bridges.
3) In this graph the sets {e_5}, {e_6}, {e_8}, {e_1, e_2}, {e_1, e_3}, {e_1, e_4}, {e_2, e_3}, {e_2, e_4}, {e_3, e_4}, {e_9, e_{10}}, {e_9, e_{11}} and {e_{10}, e_{11}} are cut sets.
4) Here K(G) = 1, $\lambda(G)$= 1 and $\delta(G)$ = 1

==

Ex. From given graph



Find 1) A closed walk of length 8.

2) A path of length 8.

3) All edge disjoint paths between the vertices 1 and 8.

4) Distance between the vertices 6 and 9.

5) An open trail of length 8.

6) Five different cut sets of G.

7) $\lambda(G)$, $K(G)$ and $\delta(G)$.

Solution.: 1) A closed walk of length 8 is 1 b 7 r 2 c 8 d 3 h 2 r 7 k 6 a 1

2) A path of length 8 is 1 g 2 h 3 e 9 p 11 q 8 l 7 k 6 m 10

3) All edge disjoint paths between the vertices 1 and 8 are

i) 1 g 2 c 8, ii) 1 b 7 l 8, iii) 1 a 6 m 10 n 8

4) Distance between the vertices 6 and 9 is 4.

5) An open trail of length 8 is 1 g 2 h 3 i 4 j 5 f 9 p 11 q 8 n 10

6) Five different cut sets of G are

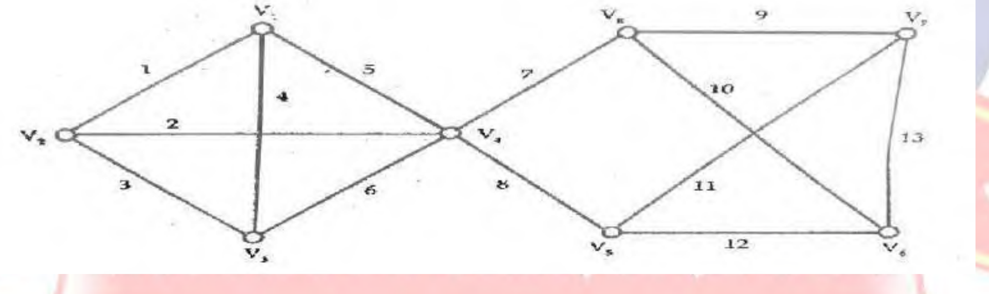
i) {a, b, g}, ii) {a, k, m}, iii) {m, n}, iv) {p, q}, v) {i, j}.

7) $\lambda(G) = 2$, $K(G) = 2$ and $\delta(G) = 2$.

===

Ex. Construct the graph in which $K(G) < \lambda(G) < \delta(G)$

Solution.: Consider the graph



In which $K(G)$ = Vertex connectivity = 1

$\lambda(G)$ = Edge connectivity = 2

& $\delta(G)$ = Minimum degree of a vertex in G = 3

Thus in above graph $K(G) < \lambda(G) < \delta(G)$.

===

Weighted Graph: A graph in which every edge is assigned by non-negative real number called as weight of an edge and graph is called as weighted graph.

Weight of the subgraph: If H is a subgraph weighted graph G, then the sum of weights of all edges of H is called as weight of the subgraph and denoted by w(H).

Eulerian Trail: A trail in a connected graph G is said to be an Eulerian trail if it passes from every edge of the graph exactly once.

Eulerian Circuit: A circuit in a connected graph G is said to be an Eulerian circuit if it passes from every edge of the graph exactly once.

Eulerian Graph: A connected graph G having at least one Eulerian circuit is called an Eulerian graph.

Note: 1) A graph has an Eulerian trail iff it is connected and it has either 0 or 2 vertices of odd degree.

2) A graph has an Eulerian circuit iff it is connected and all vertices of even degree.

3) A graph is an Eulerian graph iff it is connected and all vertices of even degree.

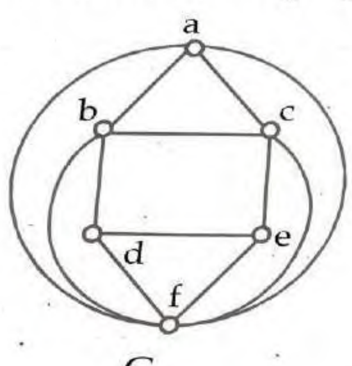
4) A complete graph K_n for $n > 1$ is Eulerian iff n is odd.

5) A complete bipartite graph $K_{m,n}$ is Eulerian iff m and n both are even.

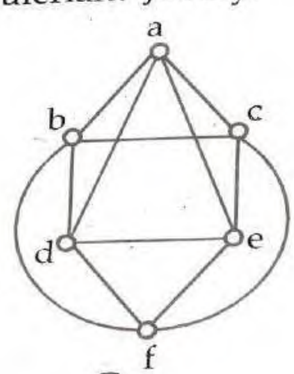
6) No edge of an Eulerian graph is an isthmus.

==

Example 7 : Are the followgin graphs Eulerian? Justify.



G_1



G_2

Solution : In graph G_1 there are exatly two vertices d and e of odd degree and G_1 connected. There fore G_1 has an Eulerian path but not eulrian circuit

$\therefore$ G_1 is not an Eulerian graph.

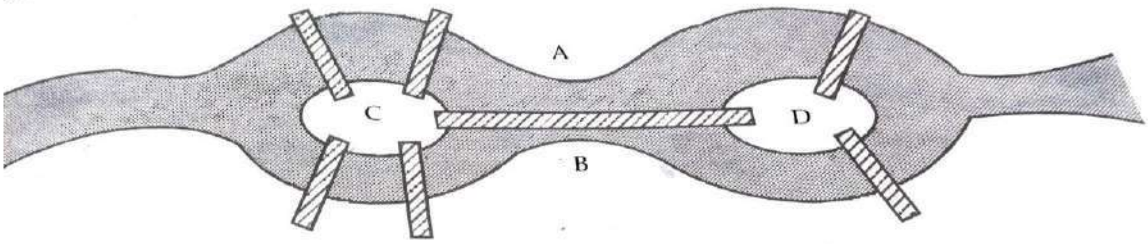
Also the graph G_2 is conneced and all the vertices of G_1 are of even degree Hen

G_2 is an Eulerian graph.

==

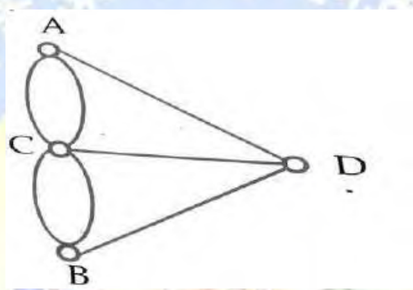
Konisgberg's Seven Bridge Problem: Konisgberg was the capital of old East Prusia, and was founded by the tetonic knigts in 1254.

The river Pregel flowed through the city forming two islands say C and D and there were seven bridges connecting the islands and two banks A and B of the river as shown in figure



The problem was start at any of the four land areas of the city say A, B, C and D, walk over each of the seven bridges exactly once and return to the starting point.

Euler presented a paper which described the solution of problem. Euler observed that such a continuous walk over the seven bridges was impossible. This paper of Euler is considered as the origin of graph theory. Euler represented the arrangement of the river and its bridges by means of a graph. In the graph land areas were shown by vertices and bridges by edges as follows.



Hamiltonian Path: A path in a connected graph G is said to be Hamiltonian Path if it passes from every vertex of the graph.

Semi-Hamiltonian Graph: A connected graph G having at least one Hamiltonian path is called semi-Hamiltonian graph.

Hamiltonian Circuit: A circuit in a connected graph G is said to be Hamiltonian circuit if it passes from every vertex of the graph.

Hamiltonian Graph: A connected graph G having at least one Hamiltonian circuit is called Hamiltonian graph.

Note: 1) Every Hamiltonian circuit is Hamiltonian path but not conversely.

2) Every Hamiltonian graph is semi-Hamiltonian graph but not conversely.

3) Deletion of any edge from Hamiltonian circuit results Hamiltonian path.

4) A simple Hamiltonian graph has at least 3 vertices.

5) For $n \geq 3$, K_n is Hamiltonian graph.

6) If $n > 3$ is even, K_n is Hamiltonian graph but not Eulerian graph.

7) If n is odd, K_n is both Hamiltonian and Eulerian.

8) A complete bipartite graph $K_{m,n}$ is Hamiltonian iff m = n.

9) If a bipartite graph G is Hamiltonian then number of vertices in G is even.

Ex. Give an example of a connected graph that has

a) Neither an Euler circuit nor a Hamiltonian circuit.

b) An Euler circuit but no Hamilton cycle.

c) A Hamiltonian circuit but no Euler circuit.

d) Both Hamiltonian circuit and an Euler circuit.

or

Ex. Give an example of a connected graph that has

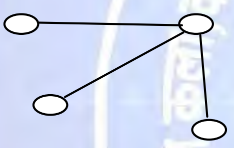
a) Neither an Euler graph nor a Hamiltonian graph.

b) An Euler graph but no Hamilton graph.

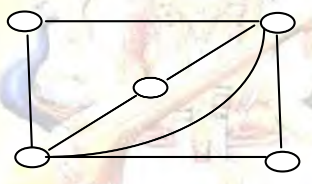
c) A Hamiltonian graph but no Euler graph.

d) Both Hamiltonian graph and an Euler graph.

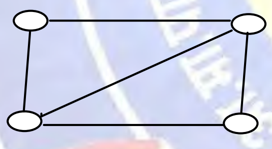
Solution: a)



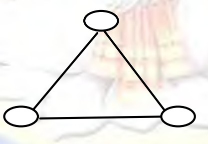
b)



c)



d)



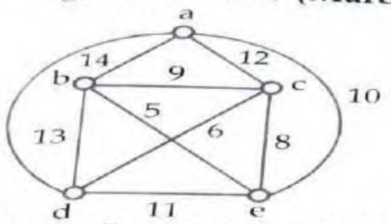
Traveling Salesman Problem: The problem is stated as “A salesman is required to visit a number of cities during his trip. Given the distance between the cities, in what order should he travel so as to visit every city precisely once and return home, with the minimum mileage traveled?”

In this problem represent the cities by vertices, roads between them by edges and associate every edge e_i by a real number $w(e_i)$ called weight which is distance between cities. If there is road between every pair of cities. Then we find Hamiltonian circuit having minimum weight. Which gives solution of problem.

Nearest Neighbour Method: Start from any vertex and find the nearest vertex to it to form an initial path of one edge and augment this path in a vertex by vertex manner such that each next vertex is nearest to previous one and continue the process until all vertices in G are included.

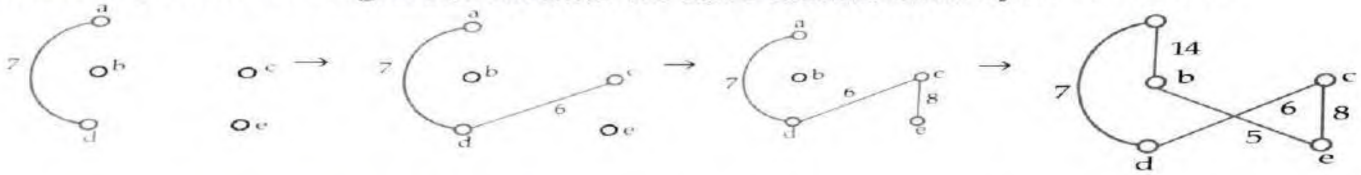
==

Example 8 : Using nearest neighbour method find the Hamiltonian cycle for the graph given below. **(March 2011)**

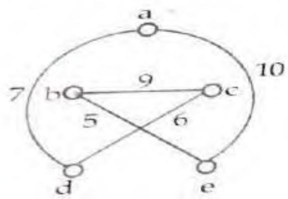


Also find the minimum Hamiltionian circuit for this graph G.

Solution : By nearest neighbour method we have Hamiltonian cycle as follows.



which is the required Hamiltonan cycle and the tatal distance of this cycle is $7 + 6 + 8 + 5 + 14 = 40$, whereas the minimum Hamiltonian circuit is as follows & its total distance is $7 + 5 + 9 + 6 + 10 = 37$



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Planar Graph: A graph which can be drawn on a plane without intersecting of edges is called planar graph.

Plane Graph: A representation of a planar graph in which no two edges intersects is called plane graph or embedding.

Faces or Regions or Windows: In a plane graph regions bounded by cycles are called faces or regions or windows.

Tree: A connected graph without a cycle is called a tree.

Note: A tree on n vertices has n-1 edges.

==

Euler's Formula For Planar Graph: If G is a connected plane graph with p vertices and q edges then $p – q + r = 2$, where r = number of faces of G.

Proof: We prove the result by induction on number of faces r of G.

If $r = 1$, then G has no cycle i. e. G is a tree.

$\therefore q = p - 1$

$\therefore p - q + r = p - (p - 1) + 1 = 2$

$\therefore$ result is true for $r = 1$.

Suppose result is true for all graphs with number of faces $< r$ & $r > 1$.

Let G be connected plane graph with p vertices q edges and r faces.

Let e be an common edge of any two faces of G.

Then graph $G_1 = G - e$ has p vertices $q - 1$ edges and $r - 1$ faces.

$\therefore$ By induction result is true for G_1.

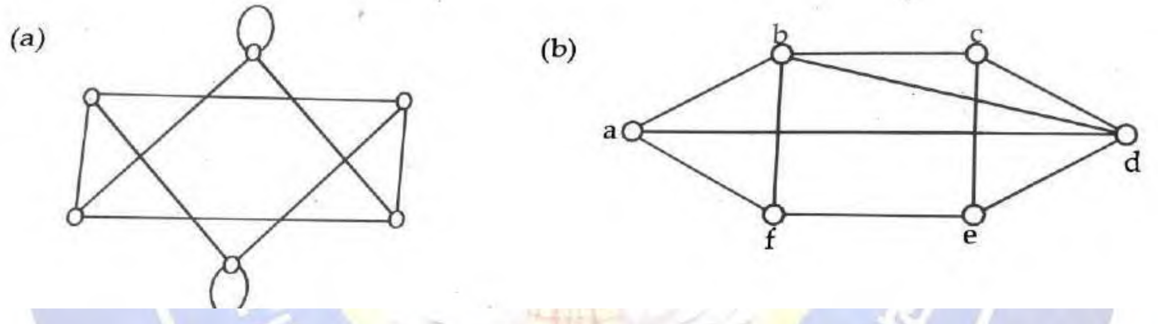
i.e. $p - (q - 1) + (r - 1) = 2$

Thus $p - q + r = 2$ i.e. result is true for G.

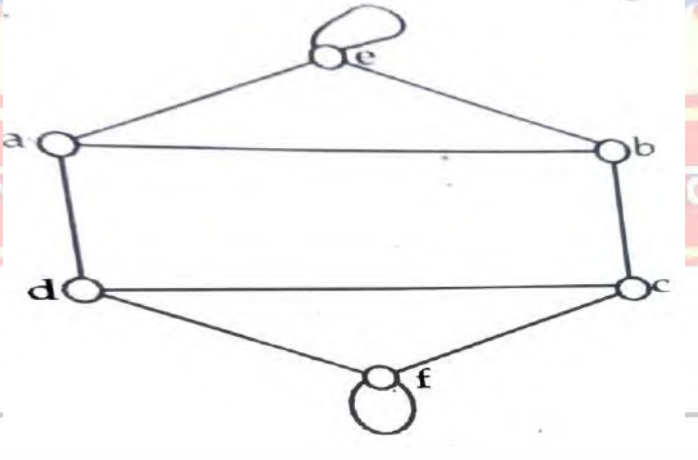
Hence by induction result is true for all connected plane graphs.

==

Ex. Verify Euler's formula for the planar graphs.



Proof: a) Given graph is drawn in plane as without intersecting of edges as follows



In this planar graph p = number of vertices = 6

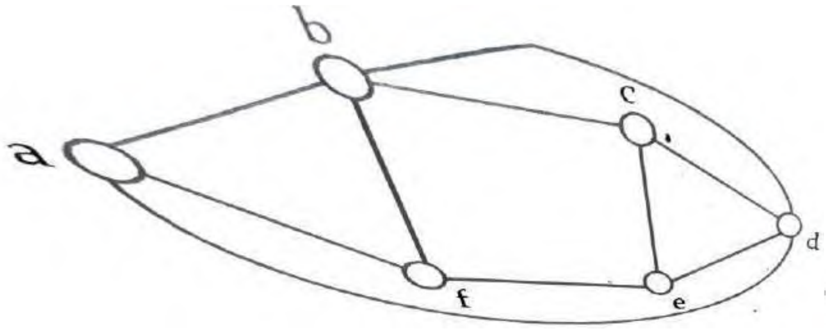
q = number of edges = 10

r = number of faces = 6

$\therefore p - q + r = 6 - 10 + 6 = 2$.

Hence Euler's formula is verified.

b) Given graph is drawn in plane as without intersecting of edges as follows



In this planar graph p = number of vertices = 6

q = number of edges = 10

r = number of regions = 6

∴ p - q + r = 6 – 10 + 6 = 2.

Hence Euler's formula is verified.

==

Ex. Find the number of edges in a planar graph with 16 vertices and 20 faces.

Solution: Let G be a planar graph with 16 vertices, 20 faces q edges.

Here p = 16, r = 20 and q = ?

By Euler's formula p - q + r = 2, we get,

16 – q + 20 = 2

∴ 36 – 2 = q

∴ q = 34.

==

Ex. Find the number of faces in a simple planar graph with 7 vertices and 7 edges.

Solution: Let G be a simple planar graph with 7 vertices, 7 edges and r faces.

Here p = 7, q = 7 and r = ?

By Euler's formula p - q + r = 2, we get,

7 – 7 + r = 2

∴ r = 2

==

Ex. Find the number of vertices in a simple planar graph with edges 16 and faces 9.

Solution: Let G be a simple planar graph with 16 edges, 9 faces and p vertices,.

Here p =? q = 16 and r = 9

By Euler's formula p - q + r = 2, we get,

$p - 16 + 9 = 2$

$\therefore p = 2 + 7 = 9$

Kuratowski's Two Graphs: A complete graph of 5 vertices i.e. K_5 is called Kuratowski's first graph and a complete bipartite graph $K_{3,3}$ is called Kuratowski's second graph.

Note: 1) Kuratowski's both graphs i.e. K_5 and $K_{3,3}$ both are not planar graphs.

2) Kuratowski's first graph K_5 is a non-planar graph with minimum number of vertices

3) If we delete any edge from K_5, it becomes a planar graph.

4) Kuratowski's second graph $K_{3,3}$ is a non-planar graph with minimum no. of edges

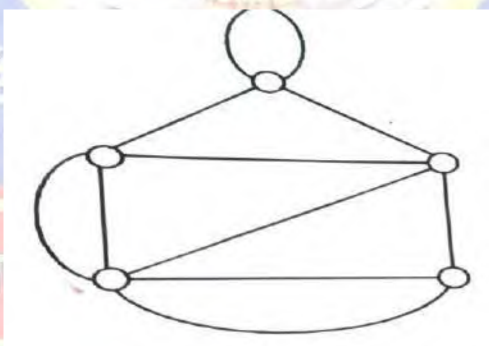
5) If we delete any vertex from $K_{3,3}$ it becomes a planar graph.

Geometrical dual: A graph G^* is called a geometrical dual of a graph G if it obtained by placing vertices in each face of its plane graph and joining these new vertices by such that each new edge crosses an old edge of G.

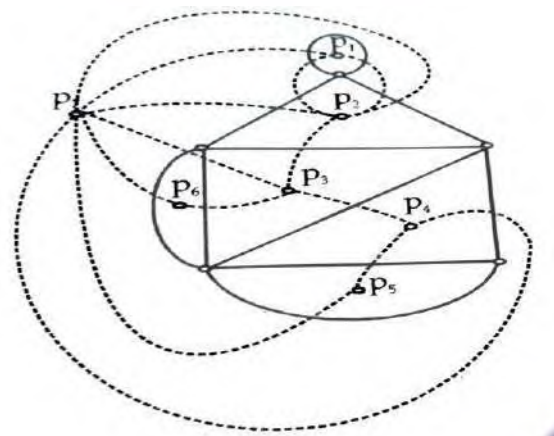
Self dual graph: If geometrical dual of G is G then G is called a self dual graph.

Note: A geometrical dual G^* is always a plane connected graph.

Ex. Find the geometrical dual of graph

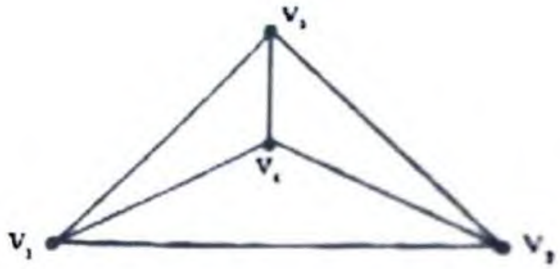


Proof: The geometrical dual of given graph is as follows



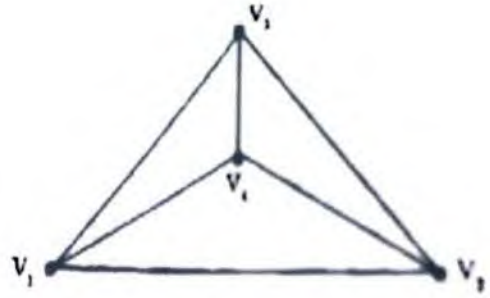
Ex. Show that geometrical dual of K_4 is K_4.

Proof: Let graph K_4 is drawn in plane as



Geometrical dual of it is

=



Geometrical dual of K_4 is K_4. Hence K_4 is self-dual graph is proved.

Colouring of Graph: An assignment of colours to the vertices of a graph such that no two adjacent vertices have a same colour.

Colour Class: A set of all vertices having the same colours is called colour class.

Chromatic Number: The minimum number of colours required to colour a graph G is called the chromatic number. Denoted by $\Psi(G)$.

Note: 1) Graph is said to be n-colouring graph if it required n colours to colouring it.

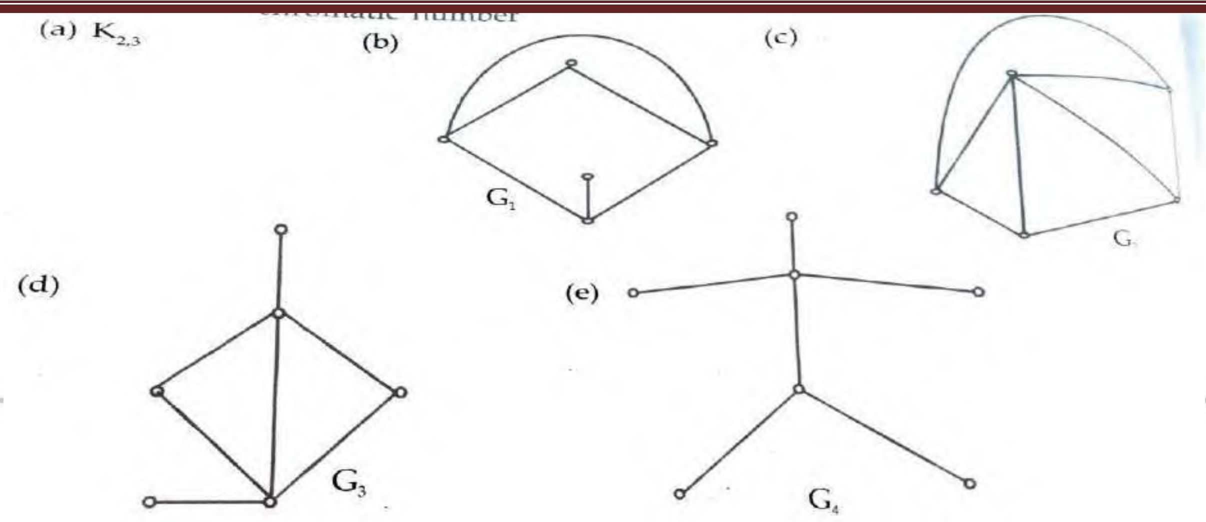
2) If $\Psi(G) = n$, then G is said to be n-chromatic graph.

3) Every non-trivial tree is 2-chromatic graph.

4) Every complete graph K_n is n-chromatic graph.

5) Every complete bipartite graph $K_{m,n}$ is 2-chromatic graph.

Ex. Determine the chromatic number of the following graphs.



Solution: The chromatic number of given graphs are

a) $\Psi(K_{2,3}) = 2$

b) $\Psi(G_1) = 3$

c) $\Psi(G_2) = 3$

d) $\Psi(G_3) = 3$

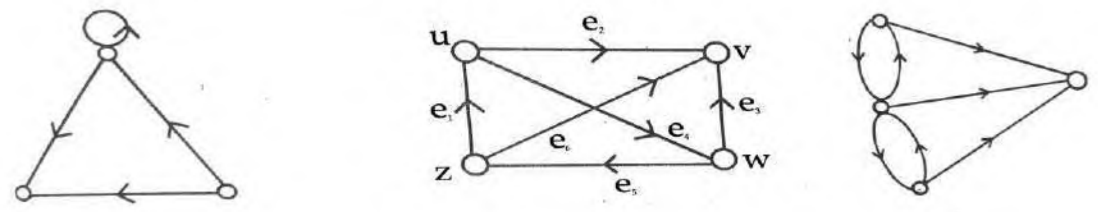
e) $\Psi(G_4) = 2$

==

UNIT-3. TREES AND DIRECTED GRAPHS

Directed graph: A pair of set of vertices V and set of directed edges or arcs E i.e G = (V, E) is called a directed graph or digraph.

e.g.



are the digraphs.

Initial Vertex and Terminal Vertex: If e is an edge directed from vertex u to v i.e. e = $\overline{uv}$ then vertex u is called initial vertex and v is called terminal vertex of an edge e.

Self-loop: An edge having initial vertex and terminal vertex same is called self-loop or loop.

Multiple or Parallel Edges or Arcs: If a certain pair of vertices are joined by two or more arcs with same directions, then such arcs are called multiple or parallel edges or arcs.

Simple digraph: A digraph without self-loops and parallel edges is called simple digraph.

Multiple digraph: A digraph with parallel edges is called as multiple digraph.

Out Degree of a Vertex: The number of edges incident out of a vertex v is called the out- degree of vertex v. Denoted by $d^+(v)$ or od(v)

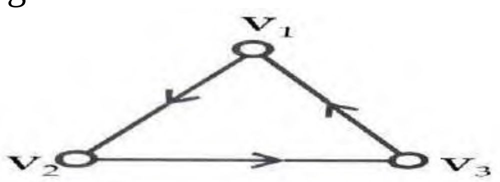
In Degree of a Vertex: The number of edges incident into a vertex v is called the in-degree of vertex v. Denoted by $d^-(v)$ or id(v).

Pendent Vertex: A vertex v of a digraph is called as pendent vertex if $d^+(v)+ d^-(v) = 1$.

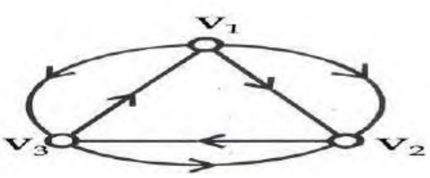
Isolated Vertex: A vertex v of a digraph is called an isolated vertex if $d^+(v)+ d^-(v) = 0$.

Complete digraph: A simple digraph in which there is exactly one edge directed from every vertex to every other vertex is called complete digraph.

e.g.



D_1



D_2

Here D_1 is not the complete digraph but D_2 is the complete digraph

. ∵ there is exactly one edge directed from every vertex to every other vertex in D_2

Balanced digraph: A digraph is called balanced digraph if for every vertex v,

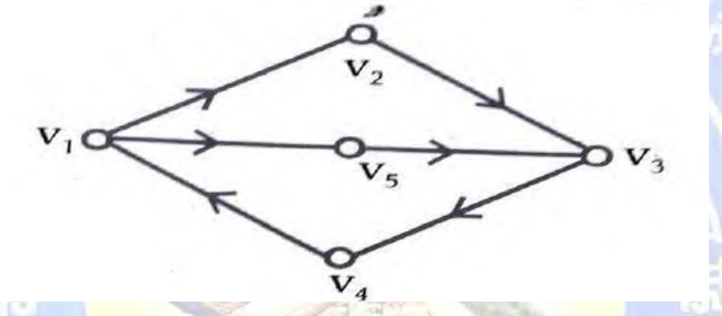
$d^+(v) = d^-(v)$.

Regular digraph: A balanced digraph is called regular digraph if every vertex has the same in-degree and out degree as every other vertex.

Connected digraph: A digraph is called connected digraph if the undirected graph obtained from it by ignoring the directions of the edges is connected.

Strongly Connected digraph: A digraph is called strongly connected digraph if there is directed path between every pair vertices.

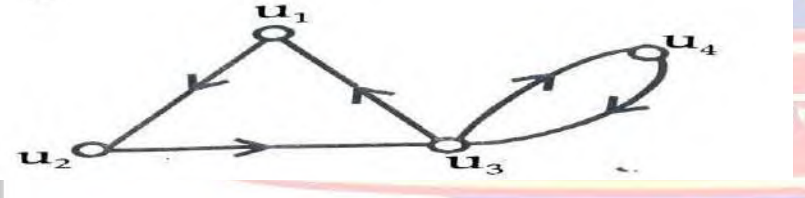
e.g.



is a strongly connected digraph. ∵ there is directed path between every pair vertices.

Euler digraph: A digraph which has an Eulerian circuit which traverses every edge of it exactly once is called Euler digraph.

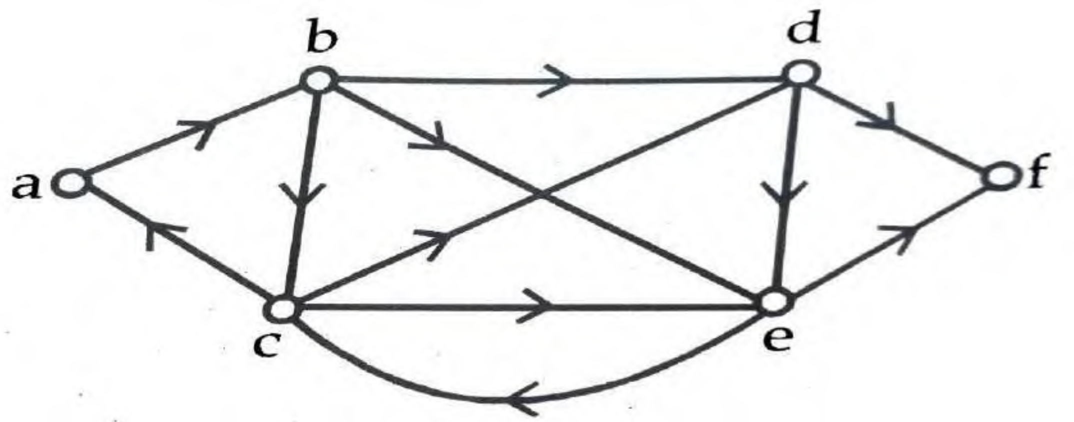
e.g.



is an Euler digraph. ∵ $u_1 \longrightarrow u_2 \longrightarrow u_3 \longrightarrow u_4 \longrightarrow u_3 \longrightarrow u_1$ is an Eulerian circuit.

==

Ex. In a given digraph below find all directed paths from a to f and directed circuits starting from d.



Solution: From given diagraph 1) All directed paths from a to f are as follows:

i) $a \longrightarrow b \longrightarrow d \longrightarrow f$

ii) $a \longrightarrow b \longrightarrow e \longrightarrow f$

iii) $a \longrightarrow b \longrightarrow c \longrightarrow d \longrightarrow f$

iv) $a \longrightarrow b \longrightarrow c \longrightarrow e \longrightarrow f$

v) $a \longrightarrow b \longrightarrow d \longrightarrow e \longrightarrow f$

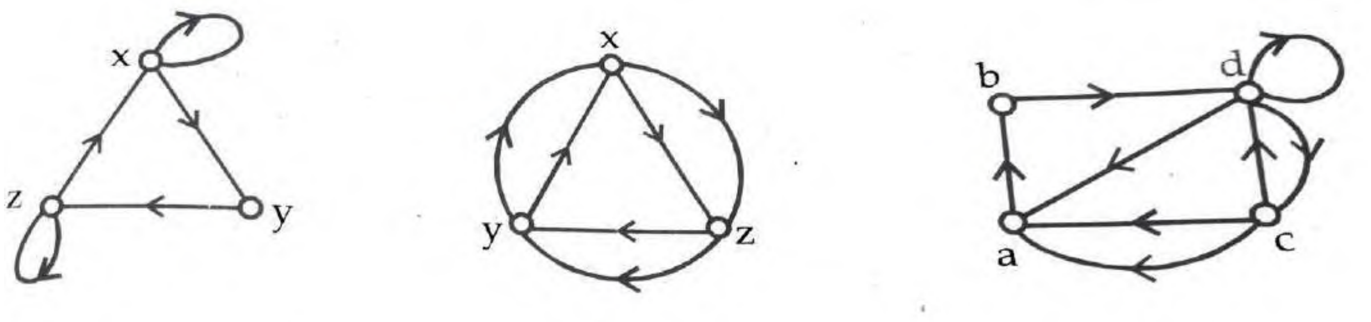
2) All directed circuits from d are as follows:

i) $d \longrightarrow e \longrightarrow c \longrightarrow d$

ii) $d \longrightarrow e \longrightarrow c \longrightarrow a \longrightarrow b \longrightarrow d$

===

Ex. Which of the following digraphs are regular, balanced or both?



D_1 D_2 D_3

Solution: From given diagraphs we observe that:

i) In D_1: $d^+(x)= d^-(x) = 2$, $d^+(y)= d^-(y) = 1$, $d^+(z)= d^-(z) = 2$

$\therefore$ Given digraph D_1 is balanced but not regular.

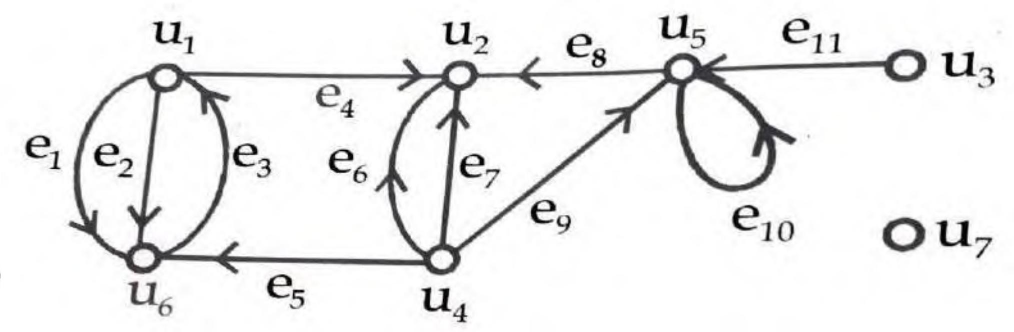
ii) In D_2: $d^+(x)= d^-(x) = 2$, $d^+(y)= d^-(y) = 2$, $d^+(z)= d^-(z) = 2$

$\therefore$ Given digraph D_2 is both balanced and regular.

iii) In D_3: $d^+(a) = 1$ & $d^-(a) = 3$ i.e. $d^+(a) \neq d^-(a)$

$\therefore$ Given digraph D_3 is neither balanced and nor regular.

Ex. Find in-degrees, out-degrees and verify that $\sum d^{-}(u_i) = \sum d^{+}(u_i)$ for the digraph.



Solution: In-degrees and out-degrees of given digraph are as follows

$d^-(u_1) = 1$ $\quad$ $d^+(u_1) = 3$

$d^-(u_2) = 4$ $\quad$ $d^+(u_2) = 0$

$d^-(u_3) = 0$ $\quad$ $d^+(u_3) = 1$

$d^-(u_4) = 0$ $\quad$ $d^+(u_4) = 4$

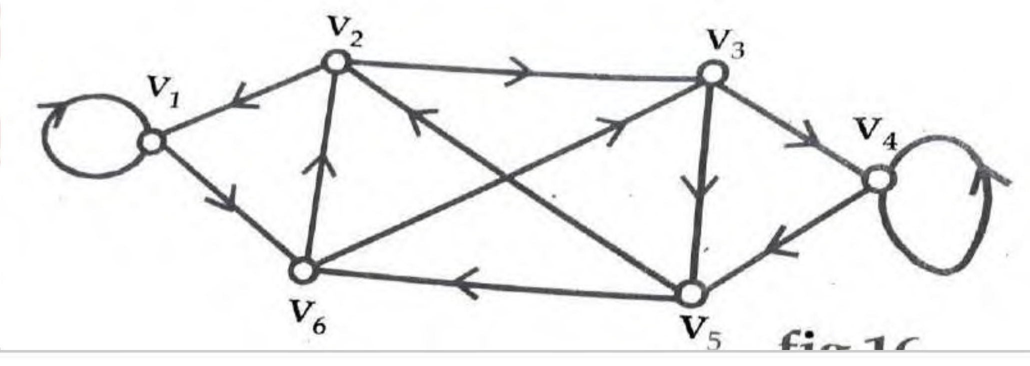
$d^-(u_5) = 3$ $\quad$ $d^+(u_5) = 2$

$d^-(u_6) = 3$ $\quad$ $d^+(u_6) = 1$

$d^-(u_7) = 0$ $\quad$ $d^+(u_7) = 0$

$\sum d^{-}(u_i) = 11$ $\quad$ $\sum d^{+}(u_i) = 11$

Ex. Find in-degrees, out-degrees and verify that $\sum d^{-}(v_i) = \sum d^{+}(v_i)$ for the digraph.



Solution: In-degrees and out-degrees of given digraph are as follows

$d^-(v_1) = 2$ $\quad$ $d^+(v_1) = 2$

$d^-(v_2) = 2$ $\quad$ $d^+(v_2) = 2$

$d^-(v_3) = 2$ $\quad$ $d^+(v_3) = 2$

$d^-(v_4) = 2$ $\quad$ $d^+(v_4) = 2$

$d^-(v_5) = 2$ $\quad$ $d^+(v_5) = 2$

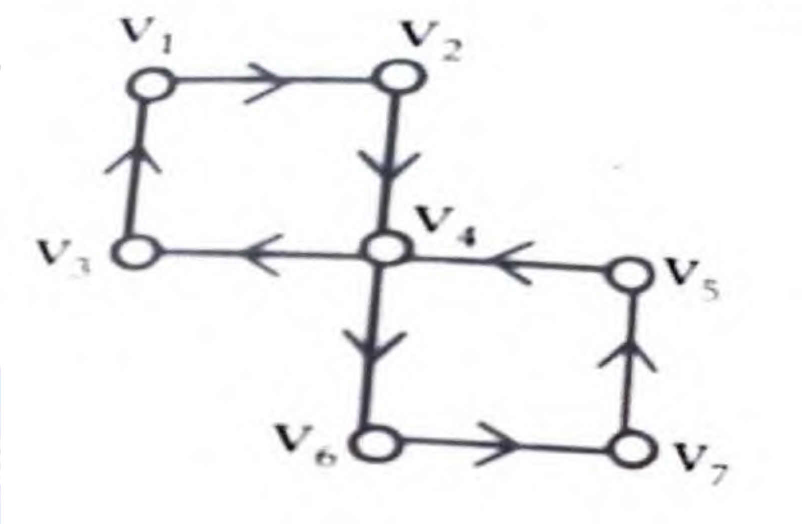
$d^-(v_6) = 2$ $\qquad$ $d^+(v_6) = 2$

============================

$\sum d^-(v_i) = 12$ $\qquad$ $\sum d^+(v_i) = 12$

==

Ex. Find in-degrees, out-degrees and verify that $\sum d^-(v_i) = \sum d^+(v_i)$ for the digraph.



Solution: In-degrees and out-degrees of given digraph are as follows

$d^-(v_1) = 1$ $\qquad$ $d^+(v_1) = 1$

$d^-(v_2) = 1$ $\qquad$ $d^+(v_2) = 1$

$d^-(v_3) = 1$ $\qquad$ $d^+(v_3) = 1$

$d^-(v_4) = 2$ $\qquad$ $d^+(v_4) = 2$

$d^-(v_5) = 1$ $\qquad$ $d^+(v_5) = 1$

$d^-(v_6) = 1$ $\qquad$ $d^+(v_6) = 1$

$d^-(v_7) = 1$ $\qquad$ $d^+(v_7) = 1$

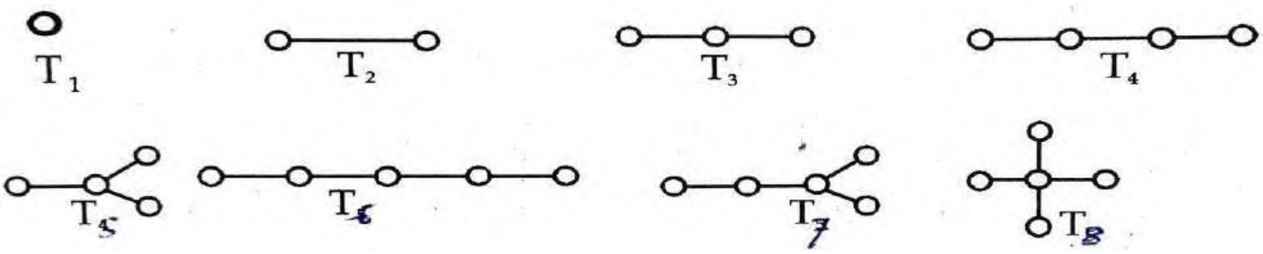
============================

$\sum d^-(v_i) = 8$ $\qquad$ $\sum d^+(v_i) = 8$

==

Tree: A connected graph without circuit is called a tree.

e.g. Following graphs are trees



Trivial Tree: A tree with 1 vertex is called a trivial tree.

Non-Trivial Tree: A tree with more than two vertices is called a non-trivial tree.

Forest: A collection of disjoint trees is called a forest.

Leaf or Terminal Node: A vertex of degree 1 in a tree is called a leaf or terminal node.

Branch Node: A vertex of degree 2 or more in a tree is called a branch node.

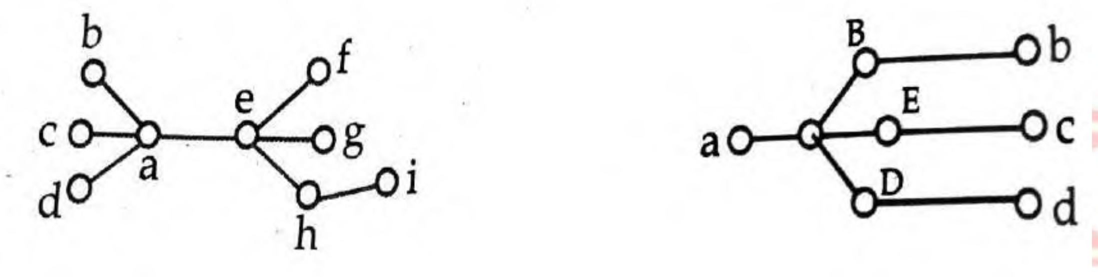
==

Properties of Trees:

1) Every tree is a simple graph.
2) A tree on n vertices has (n-1) edges.
3) There is unique path between every pair of vertices in a tree.
4) A tree with two or more vertices has at least two leaves (pendent vertices).
5) A complete graph K_n is a tree iff n = 1 or n = 2.
6) Every tree is a bipartite graph.
7) A complete bipartite graph $K_{m,n}$ is tree iff either m =1 or n =1.

==

Ex. Find the leaves (or pendent vertices or terminal nodes) and branch nodes of the following trees.



T_1 T_2

Solution: i) In tree T_1 leaves are b, c, d, f, g & i.

and branch nodes are a, e & h.

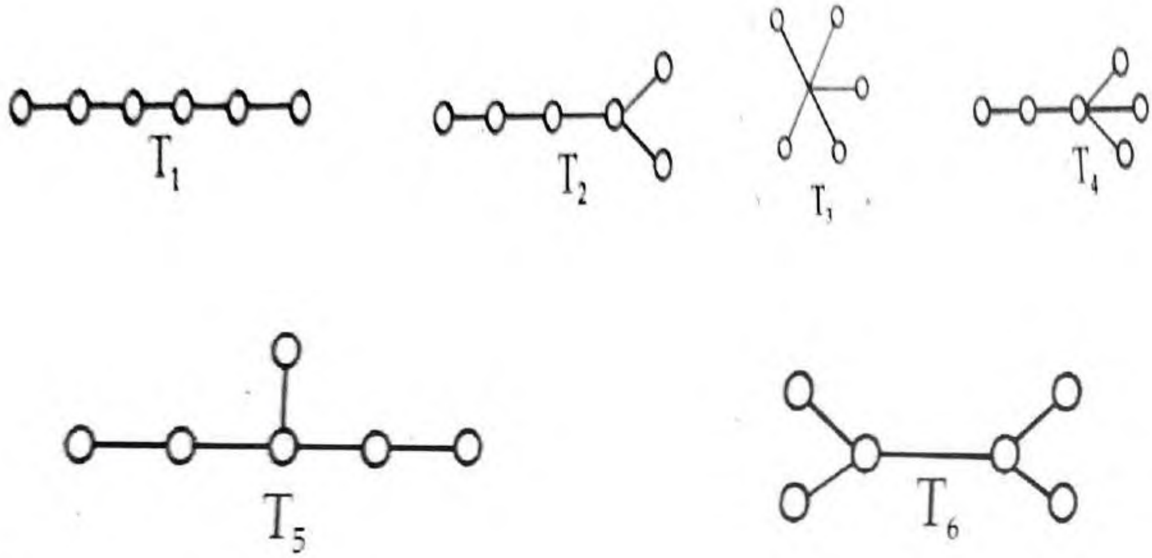
ii) In tree T_2 leaves are a, b, c & d.

and branch nodes are A, B, E & D.

==

Ex. Draw all non-isomorphic trees on six vertices.

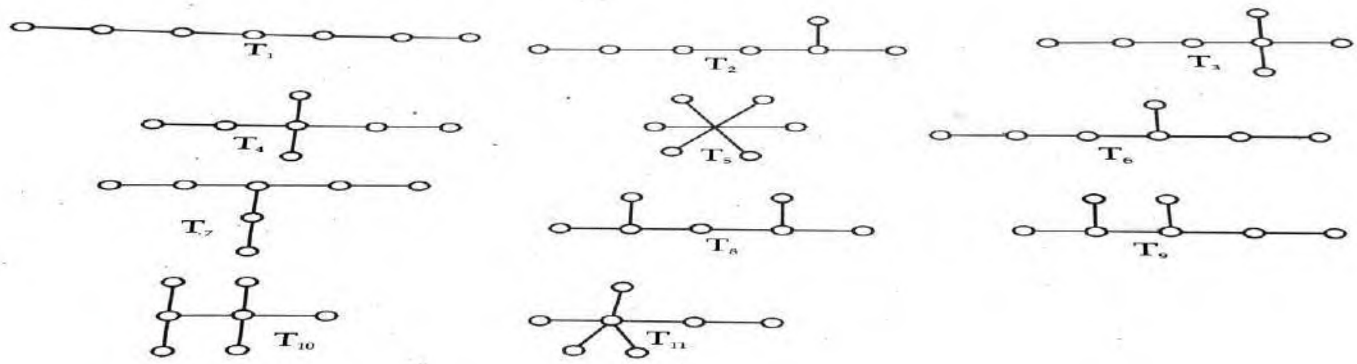
Solution: All non-isomorphic trees on six vertices are drawn as follows



==

Ex. Draw all non-isomorphic trees on seven vertices.

Solution: All non-isomorphic trees on seven vertices are drawn as follows



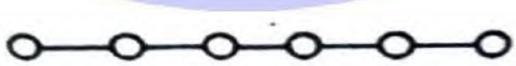
==

Ex. Construct a trees on 6 vertices having

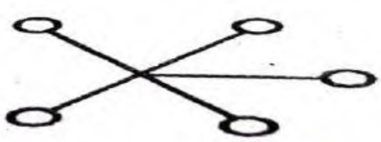
i) minimum number of pendent vertices.

ii) maximum number of pendent vertices.

Solution: i) A trees on 6 vertices having minimum number of pendent vertices is



ii) A trees on 6 vertices having maximum number of pendent vertices is



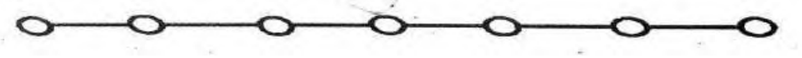
==

Ex. Construct a trees on 7 vertices having

i) minimum number of pendent vertices.

ii) maximum number of pendent vertices.

Solution: i) A trees on 7 vertices having minimum number of pendent vertices is



ii) A trees on 7 vertices having maximum number of pendent vertices is

==

Ex. A tree has 2n pendent vertices, 3n vertices of degree 2 and n having vertices of degree 3. Determine number of vertices and edges in a tree.

Solution: Let T be a tree has 2n pendent vertices, 3n vertices of degree 2 and n having vertices of degree 3.

∴ Number of vertices in a tree = 2n + 3n + n = 6n

& Number of edges in a tree = 6n -1

∴ By Hand shaking Lemma $\sum d(v) = 2|E|$, we get,

$1 \times 2n + 2 \times 3n + 3 \times n = 2(6n - 1)$

∴ 2n + 6n + 3n = 12n - 2

∴ 12n – 2 = 11n

∴ n = 2

∴ Number of vertices in a tree = 6 × 2 = 12

& Number of edges in a tree = 6 × 2 -1 = 11.

==

Ex. A tree has two vertices of degree 2, one vertex of degree 3 and three vertices of degree 4. How many number of vertices of degree 1 does it have?

Solution: Let T be a tree with two vertices of degree 2, one vertex of degree 3, three vertices of degree 4 and p vertices of degree 1.

∴ Number of vertices in a tree = 2 + 1 + 3 + p = p + 6

& Number of edges in a tree = p + 6 -1 = p + 5

∴ By Hand shaking Lemma $\sum d(v) = 2|E|$, we get,

$2 \times 2 + 1 \times 3 + 3 \times 4 + 1 \times p = 2(p + 5)$

∴ 4 + 3 + 12 + p = 2p + 10

$\therefore 2p + 10 = p + 19$

$\therefore p = 9$

$\therefore$ There are 9 vertices of degree 1.

==

Eccentricity of a vertex: The distance from vertex v to the farthest vertex from v in a graph is called eccentricity e(v) of a vertex v. i.e. $e(v) = \text{Max } d(v, u)$ for $u \in V(G)$.

Centre: A vertex in a graph G with minimum eccentricity is called a centre of graph G.

Radius: An eccentricity of a centre of a graph is called a radius of graph G. Denoted by r(G).

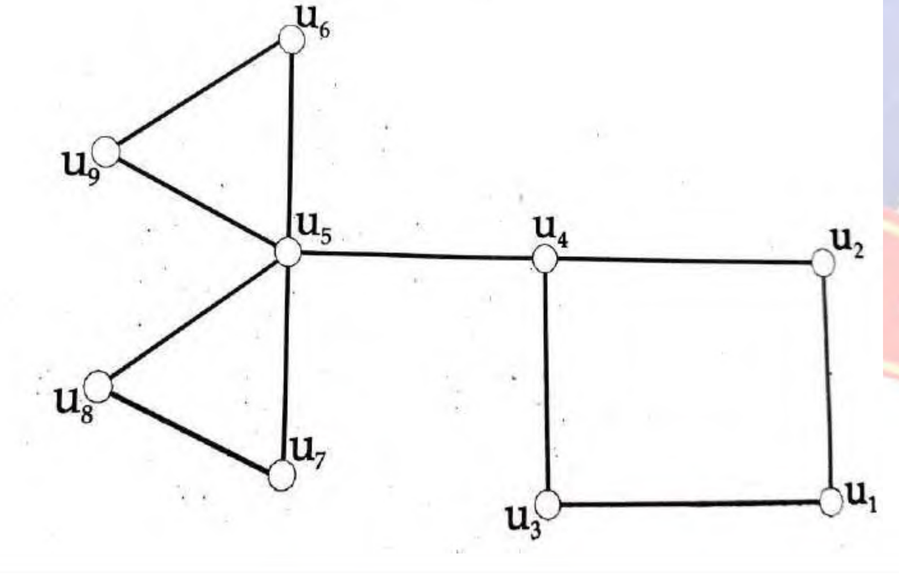
Diameter: A maximum eccentricity of a vertex of a graph is called a diameter of graph G. Denoted by $\text{diam}(G) = \text{Max. } e(v)$ for $v \in V(G)$.

Remark: 1) $\text{diam}(G) \leq 2r(G)$.

2) Every tree has either one or two centres.

==

Ex. Find eccentricity of each vertex of the following graph. Find radius, diameter and centre of G



Solution: From given graph G we have

$e(u_1) = 4$, $e(u_2) = 3$, $e(u_3) = 3$, $e(u_4) = 2$, $e(u_5) = 3$,

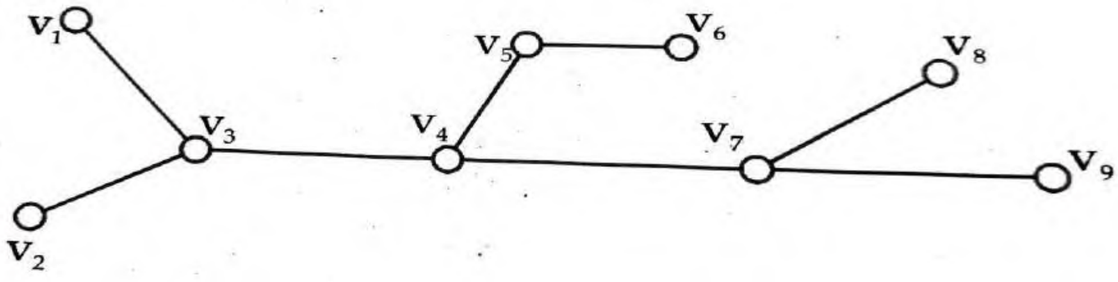
$e(u_6) = 4$, $e(u_7) = 4$, $e(u_8) = 4$ & $e(u_9) = 4$

Here u_4 has minimum eccentricity 2.

$\therefore$ Vertex u_4 is the centre of a given graph G with r(G) = radius of G = 2.

diam(G) = diameter of G = 4.

Ex. Find eccentricity of every vertex of the following tree T.



Hence find its centre, radius and diameter. Is the diameter twice its radius?

Solution: From given tree T we have

$e(v_1) = 4$, $e(v_2) = 4$, $e(v_3) = 3$, $e(v_4) = 2$, $e(v_5) = 3$,

$e(v_6) = 4$, $e(v_7) = 3$, $e(v_8) = 4$ & $e(v_9) = 4$

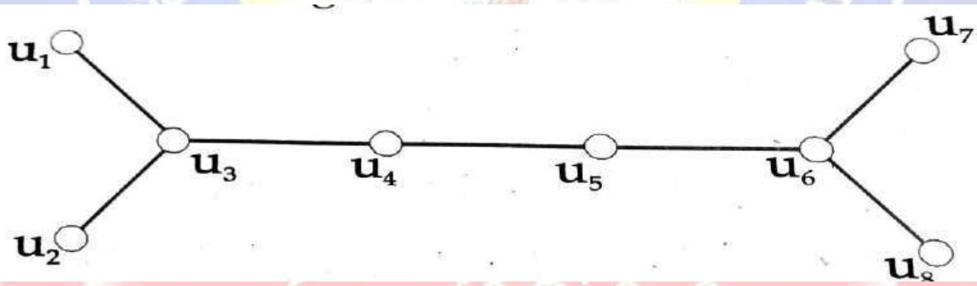
Here v_4 has minimum eccentricity 2.

$\therefore$ Vertex v_4 is the centre of a given tree G with $r(T)$ = radius of T = 2.

diam(T) = diameter of T = 4 = 2r(T).

Ex. Construct a tree in which its diameter is not equal to twice the radius of it.

Solution: Consider the tree T



In which $e(u_1) = 5$, $e(u_2) = 5$, $e(u_3) = 4$, $e(u_4) = 3$, $e(u_5) = 3$, $e(u_6) = 4$,

$e(u_7) = 5$ & $e(u_8) = 5$.

Here u_4 & u_5 has minimum eccentricity 3.

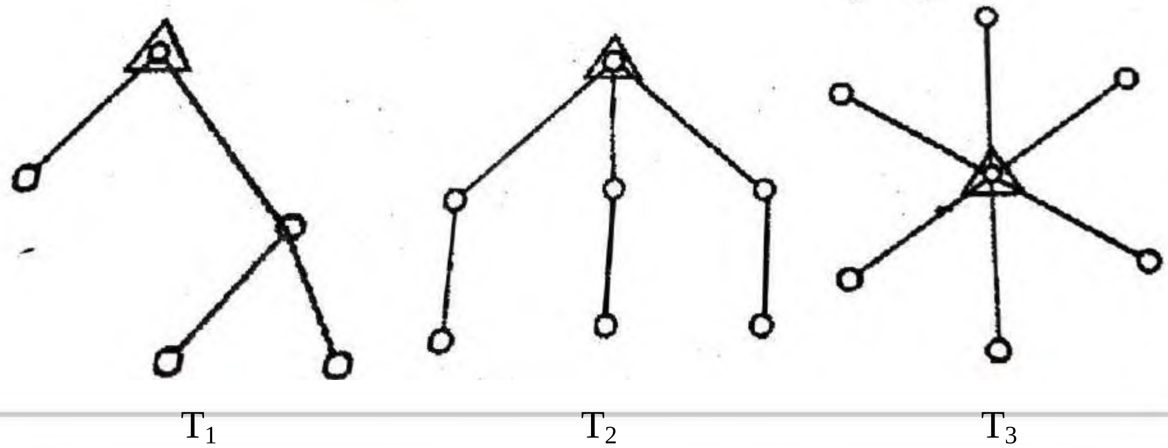
$\therefore$ Vertices u_4 & u_5 are centres of a tree T with $r(T)$ = radius of T = 3.

and diam(T) = diameter of T = 5.

$\therefore$ diam(T) $\neq$ 2 r(T) as $5 \neq 6$.

Rooted Tree: A tree in which one vertex (called the root) is distinguished from all others is called a rooted tree.

e.g.

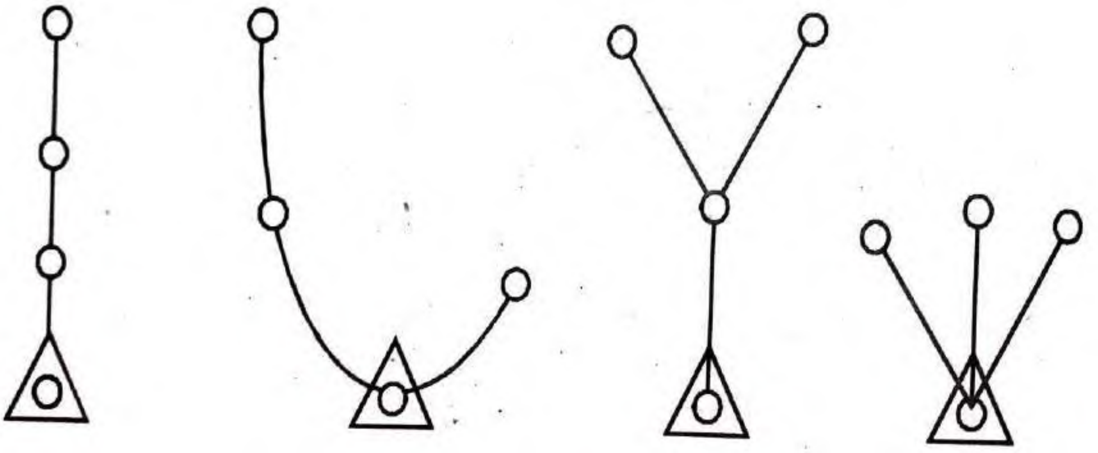


Trees T_1, T_2 and T_3 are rooted tree.

==

Ex. Find all rooted trees with four vertices.

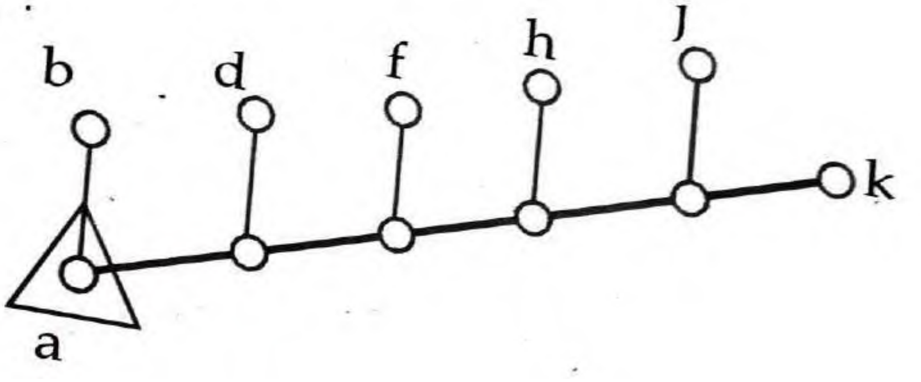
Solution: All rooted tree with 4 vertices are as follows



==

Binary Tree: If a tree contain exactly one vertex of degree two and all other vertices of have degree either one or three, then it said to be binary tree.

e.g.



is binary tree with six pendent vertices.

==

Ex. Prove that in a binary trees with n vertices.

i) The number of vertices is odd.

ii) p = $\frac{(n+1)}{2}$, where p is number of pendent vertices.

iii) q = p – 1, where q is number of non-pendent vertices.

Proof: Let T be a binary tree with n vertices, out of which one vertex of degree two, p number of pendent vertices and m vertices of degree 3.

∴ n = 1 + p + m ………. (1)

i) As every graph has vertices of odd degree is always even.

∴ p + m is an even number.

Hence n = 1 + p + m is odd number.

ii) As T is tree with n vertices.

∴ number of edges in T are (n-1).

By Handshaking Lemma $\sum d(v) = 2|E|$, we get,

$2 \times 1 + 1 \times p + 3 \times m = 2(n - 1)$

$\Rightarrow 2 + p + 3(n - 1 - p) = 2n - 2$ by (1)

$\Rightarrow 2 + p + 3n - 3 - 3p = 2n - 2$

$\Rightarrow n + 1 = 2p$

$\Rightarrow p = \frac{(n+1)}{2}$ ……… (2)

iii) Let q is number of non-pendent vertices in T, then from (2) we have,

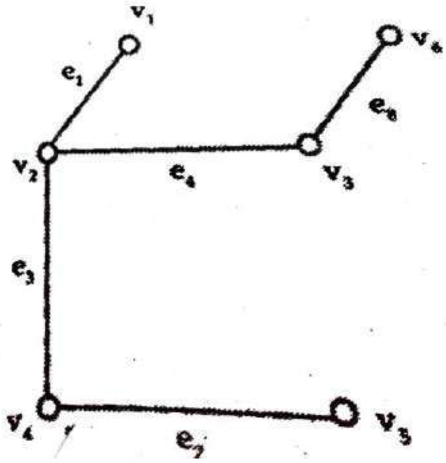
q = n – p = (2p – 1) – p = p – 1 by (2)

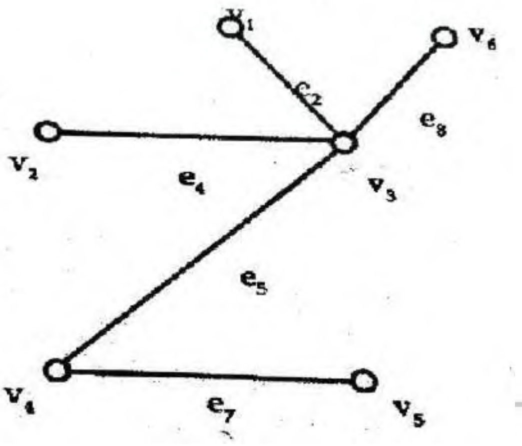
Hence proved.

==

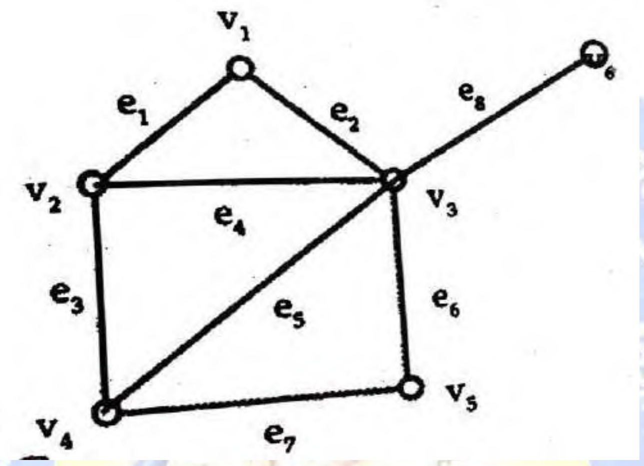
Spanning Tree: A subgraph T of a connected graph G is said to be spanning tree of graph G if T is tree and V(G) = V(T).

e.g.



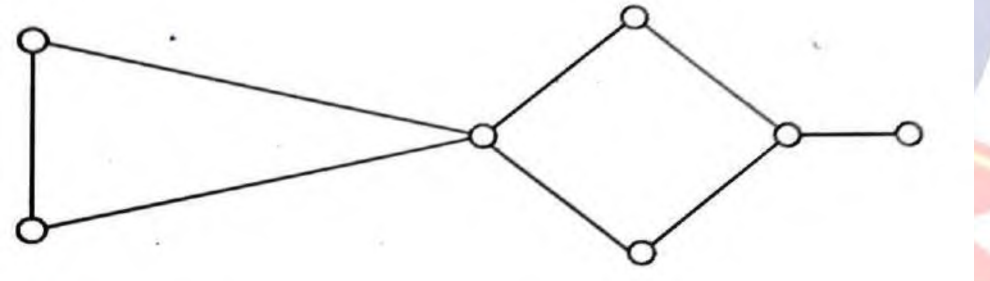


are the spanning trees of following graph.

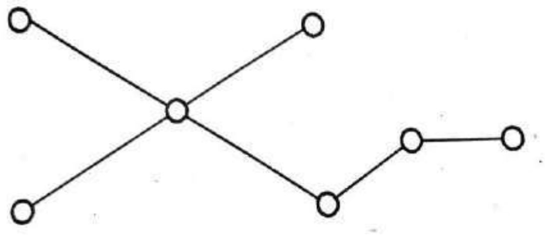


==

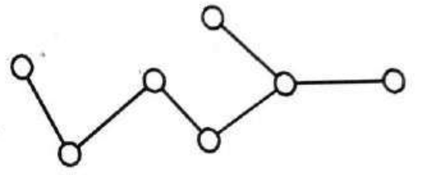
Ex. Find any four spanning trees of a following graph.



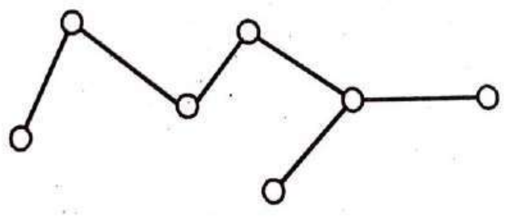
Solution: Four spanning trees of given graph are as follows



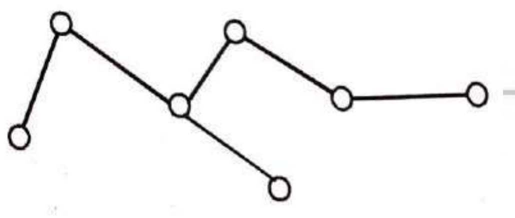
T_1



T_2

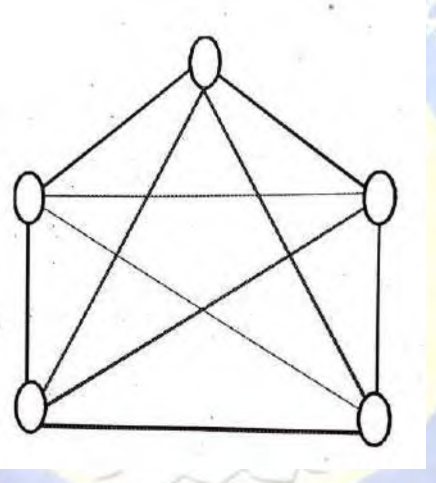


T_3

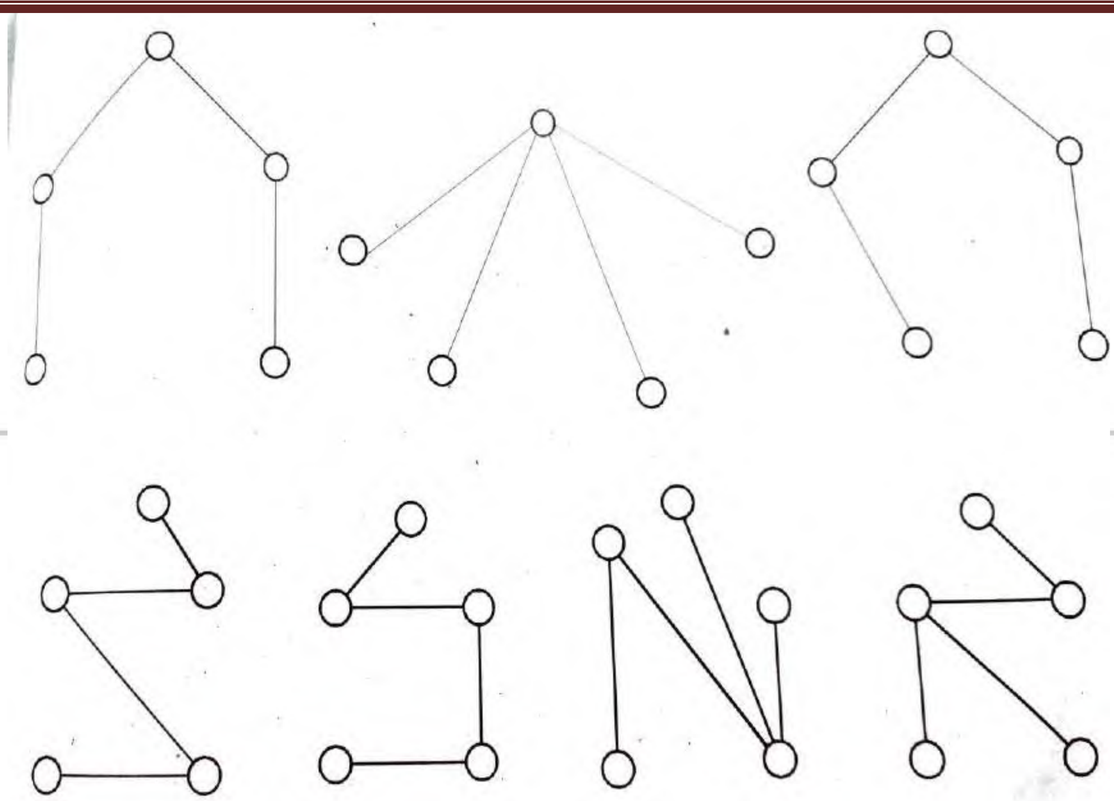


T_4

Ex. Find any seven spanning trees of K_5



Solution: Seven spanning trees of given graph are as follows



Branch: Let T be a spanning tree of a connected graph G, then an edge in the tree T is called branch of T.

Rank of G: Let T be a spanning tree of a connected graph G of n vertices, then number of edges (n-1) in the tree T i.e. number of branches is called rank of G.

Chord: Let T be a spanning tree of a connected graph G, then an edge in the graph G which is not in T is called chord w.r.t. spanning tree T.

Nullity of G: Let T be a spanning tree of a connected graph G of n vertices and q edges, then number of edges (q-n+1) in the graph G which are not in T i.e. number of chords is called nullity of G.

Remark: 1) A complete graph K_n has n^{n-2} different spanning trees.

2) If G is disconnected graph, then every component has a spanning tree.

3) If G is a graph on p vertices with q edges and k components, then rank of G = p – k and nullity of G = q – p + k.

4) If G is a tree and u, v are non-adjacent vertices of G then G + uv contain exactly one cycle.

Fundamental Cycle: Let T be a spanning tree of a connected graph G, then cycle formed by adding one chord to T is called fundamental cycle w.r.t T.

Fundamental Cutset: Let T be a spanning tree of a connected graph G, then a cutset which contain exactly one branch of T is called fundamental cutset w.r.t T.

==

Ex. Find the rank and nullity of the following graphs.

a) K_n, b) K_7, c) $K_{3,3}$ d) N_6

Solution: The rank and nullity of the given graphs are as follows.

a) K_n is connected graph with n vertices and $q = \frac{n(n-1)}{2}$ edges.

$\therefore$ rank of $K_n = n - 1$. & nullity of $K_n = q - n + 1 = \frac{n(n-1)}{2} - n + 1 = \frac{n^2-3n+2}{2}$

b) K_7 is connected graph with 7 vertices and $q = \frac{7(7-1)}{2} = 21$ edges.

$\therefore$ rank of $K_7 = 7 - 1 = 6$. & nullity of $K_7 = 21 - 7 + 1 = 15$

c) $K_{3,3}$ is connected graph with $3 + 3 = 6$ vertices and $q = 3 \times 3 = 9$ edges.

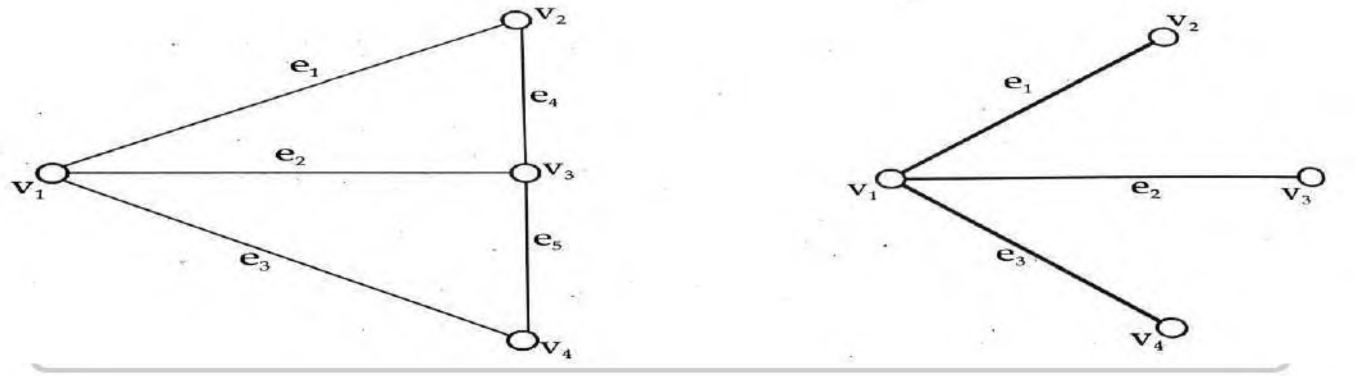
$\therefore$ rank of $K_{3,3} = 6 - 1 = 5$. & nullity of $K_{3,3} = 9 - 6 + 1 = 4$

d) N_6 is disconnected graph with $p = 6$ vertices, $k = 6$ components and $q = 0$ edges.

$\therefore$ rank of $N_6 = p - k = 6 - 6 = 0$. & nullity of $N_6 = q - p + k = 0 - 6 + 6 = 0$

==

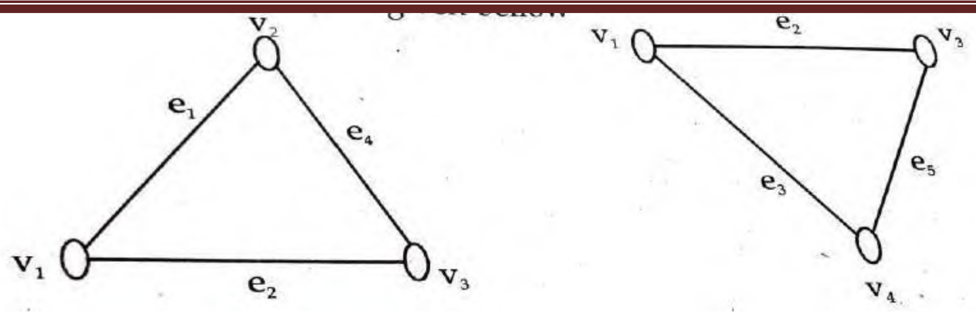
Ex. Find the fundamental cycles and fundamental cutsets of the following graph G with respect to spanning tree T.



Solution: Let T be a spanning trees of given graph G.

$\therefore$ Chords of G are e_4 & e_5. Branches of T are e_1, e_2 & e_3.

$\therefore$ Number of fundamental cycles corresponding to 2 chords are as follows

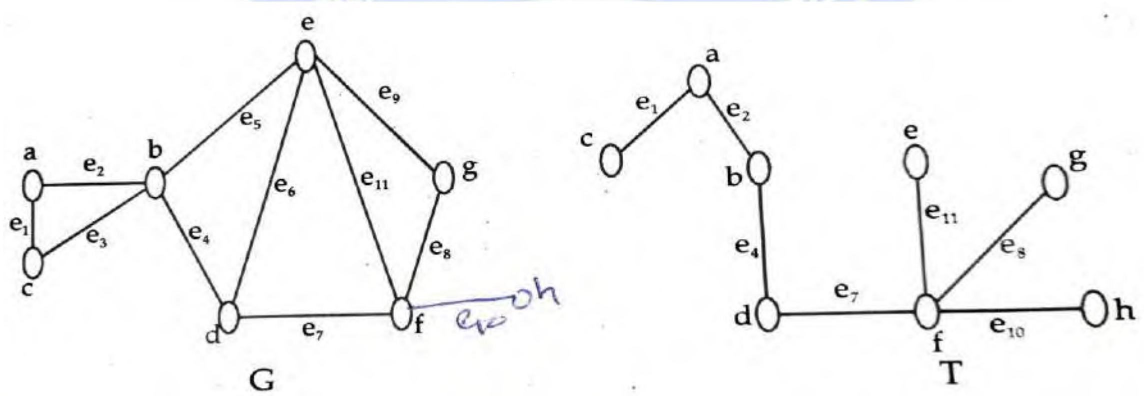


There are 3 fundamental cutsets which contain one of the branch as follows.

$\{e_1, e_4\}$, $\{e_2, e_4, e_5\}$ and $\{e_3, e_5\}$.

==

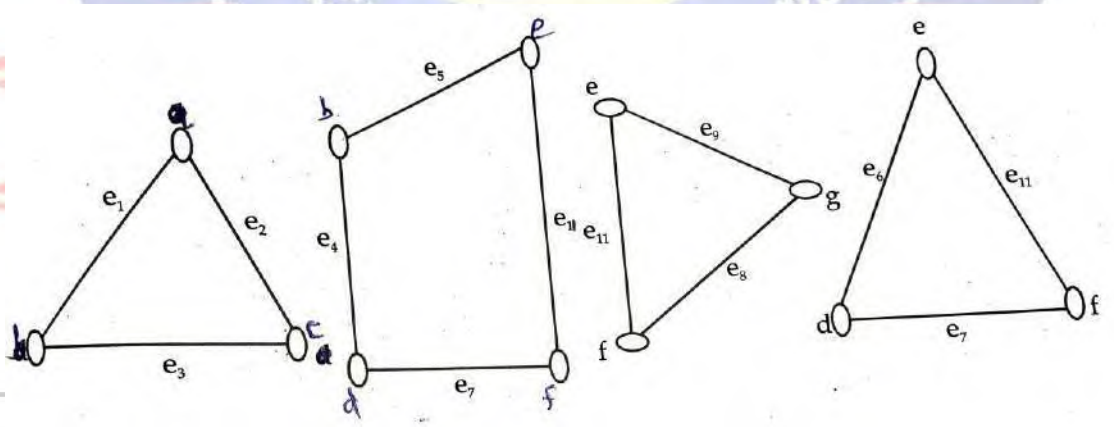
Ex. Find the fundamental cycles and fundamental cutsets of the following graph G with respect to spanning tree T.



Solution: Let T be a spanning trees of given graph G.

∴ Chords of G are e_3, e_5, e_6 & e_9. Branches of T are e_1, e_2, e_4, e_7, e_8, e_{10} & e_{11}.

∴ Number of fundamental cycles corresponding to 4 chords are as follows



There are 7 fundamental cutsets which contain one of the branch as follows.

$\{e_1, e_3\}$, $\{e_2, e_3\}$, $\{e_4, e_5\}$, $\{e_5, e_6, e_9, e_{11}\}$, $\{e_5, e_6, e_7\}$, $\{e_8, e_9\}$ and $\{e_{10}\}$.

==

UNIT-4. APPLICATIONS OF THE GRAPHS

Ex. Does there exist a regular graph of degree 5 on 7 vertices? Justify.

Solution: Let G be a regular graph of degree 5 on 7 vertices and say q edges.

∴ every vertex of regular graph G is of degree 5.

By Hand Shaking Lemma $\sum_{i=1}^{7} d(v_i) = 2q$

$\therefore 2q = \sum_{i=1}^{7} 5$

$\Longrightarrow 2q = 5 \times 7$

$\Longrightarrow q = \frac{35}{2}$ which is impossible

Hence there does not exist a regular graph of degree 5 on 7 vertices

Ex. Does there exist a graph on 5 vertices whose degrees are 1, 2, 3, 4 and 5? Justify. (Mar.2019)

Solution: Let G be a graph on 5 vertices v_1, v_2, v_3, v_4 and v_5 whose degrees are 1, 2, 3, 4 and 5 respectively and say of q edges.

By Hand Shaking Lemma $\sum_{i=1}^{5} d(v_i) = 2q$

$\therefore 2q = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$

$= 1 + 2 + 3 + 4 + 5$

$\Longrightarrow 2q = 15$

$\Longrightarrow q = \frac{15}{2}$ which is impossible

Hence there does not exist a graph on 5 vertices whose degrees are 1, 2, 3, 4 and 5.

Ex. Does there exist a graph on 9 vertices in which two vertices of degree 2, three vertices of degree 3 and four vertices of degree 4? Justify. (Oct.2018)

Solution: Let G be a graph on 9 vertices with two vertices of degree 2, three vertices of degree 3 and four vertices of degree 4 and say of q edges.

By Hand Shaking Lemma $\sum_{i=1}^{9} d(v_i) = 2q$

$\therefore 2q = 2 \times 2 + 3 \times 3 + 4 \times 4$

$= 4 + 9 + 16$

$\Longrightarrow 2q = 29$

$\Longrightarrow q = \frac{29}{2}$ which is impossible

Hence there does not exist a graph on 9 vertices with two vertices of degree 2, three vertices of degree 3 and four vertices of degree 4

==

Colouring of Graph: An assignment of colours to the vertices of a graph such that no two adjacent vertices have a same colour.

Colour Class: A set of all vertices having the same colours is called colour class.

Chromatic Number: The minimum number of colours required to colour a graph G is called the chromatic number. Denoted by $\Psi(G)$.

Note: 1) Graph is said to be n-colouring graph if it required n colours to colouring it.

2) If $\Psi(G) = n$, then G is said to be n-chromatic graph.

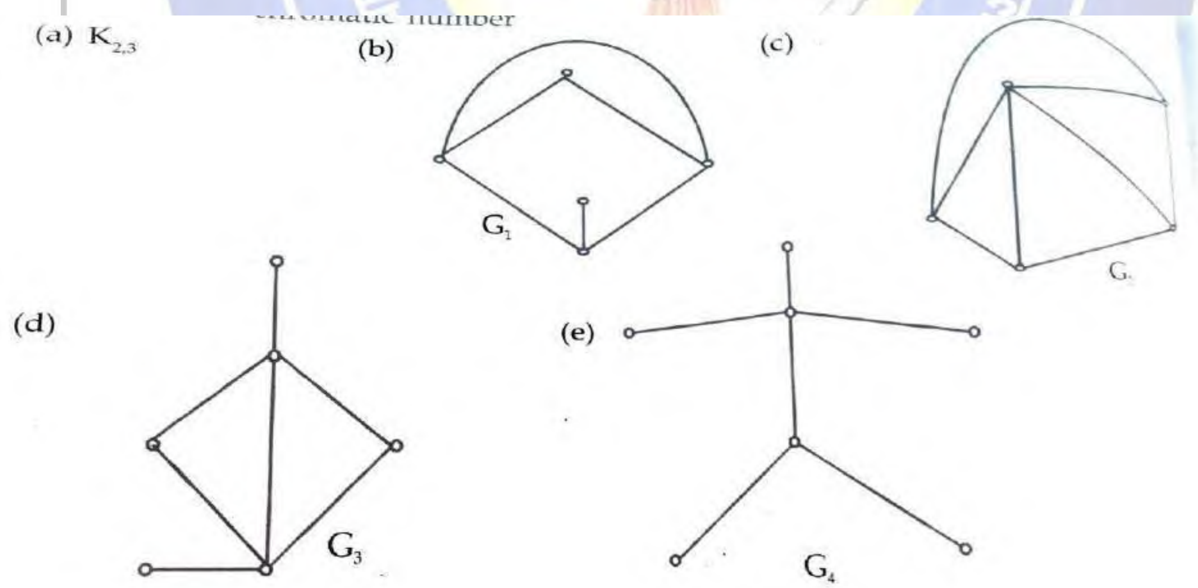
3) Every non-trivial tree is 2-chromatic graph.

4) Every complete graph K_n is n-chromatic graph.

5) Every complete bipartite graph $K_{m,n}$ is 2-chromatic graph.

==

Ex. Determine the chromatic number of the following graphs.



Solution: The chromatic number of given graphs are

a) $\Psi(K_{2,3}) = 2$

b) $\Psi(G_1) = 3$

c) $\Psi(G_2) = 3$

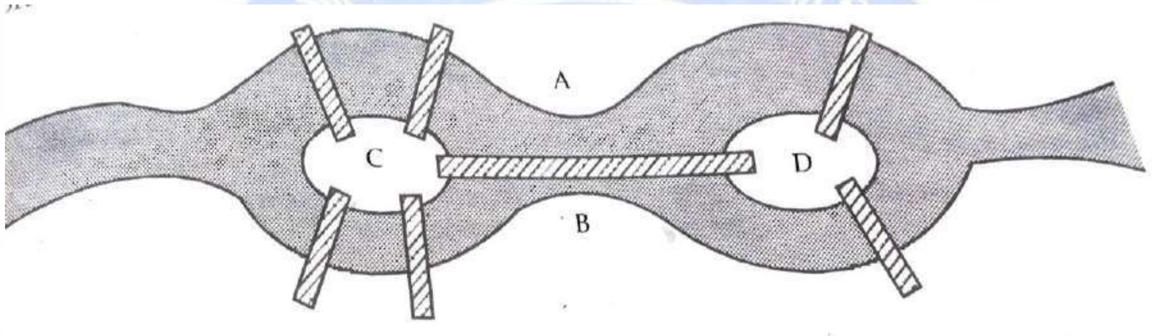
d) $\Psi(G_3) = 3$

e) $\Psi(G_4) = 2$

==

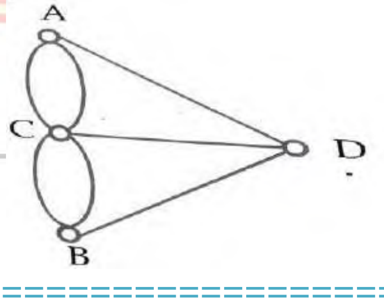
Konisgberg's Seven Bridge Problem: Konisgberg was the capital of old East Prusia, and was founded by the tetonic knigts in 1254.

The river Pregel flowed through the city forming two islands say C and D and there were seven bridges connecting the islands and two banks A and B of the river as shown in figure



The problem was start at any of the four land areas of the city say A, B, C and D, walk over each of the seven bridges exactly once and return to the starting point.

Euler presented a paper which described the solution of problem. Euler observed that such a continuous walk over the seven bridges was impossible. This paper of Euler is considered as the origin of graph theory. Euler represented the arrangement of the river and its bridges by means of a graph. In the graph land areas were shown by vertices and bridges by edges as follows.



==

Traveling Salesman Problem: The problem is stated as "A salesman is required to visit a number of cities during his trip. Given the distance between the cities, in what order should he travel so as to visit every city precisely once and return home, with the minimum mileage traveled?"

In this problem represent the cities by vertices, roads between them by edges and associate every edge e_i by a real number $w(e_i)$ called weight which is distance between cities. If there is road between every pair of cities. Then we find Hamiltonian circuit having minimum weight. Which gives solution of problem.

==

Kruskal's Algorithm: This algorithm is used to obtain a minimal spanning tree (M.S.T) in a connected weighted graph. In this procedure at every stage, a decision is to be made to select the best possible edge from the graph for inclusion as an edge in minimal spanning tree after making such that the selection of an edge does not create a cycle in the sub graph already constructed. The input is the set of edges in a weighted graph with n vertices. The output is either a report that the graph is not connected or the set of edges in a minimal spanning tree T in the network. There are four steps in this algorithm.

Step-1: Arrange the edges of the weighted graph in non-decreasing order of their weights as a list L and set T to be the empty set.

Step-2: Select the first edge from L and include that in T.

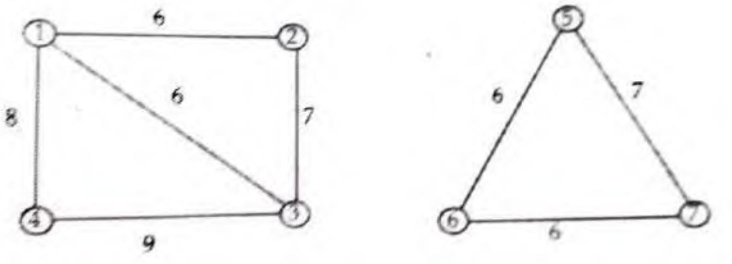
Step-3: If every edge in L examined for possible inclusion in T, stop and report that G is not connected. Otherwise, take the first unexamined edge in L. If it does not form a cycle in T.

and go to step-4. Otherwise discard that edge as an examined but not selected edge and repeat step 3.

Step-4: Stop T has (n-1) edges. Otherwise go to empty step 3.

==

Ex. Show by using Kruskal's algorithm that the network in the figure below is a disconnected weighted graph.



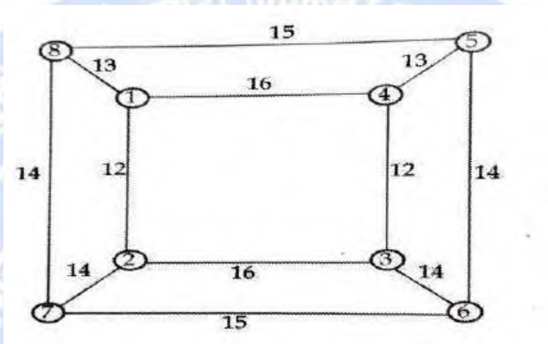
Proof: Given graph contain 7 vertices. The set L of edges in non-decreasing order is

L = {(1, 2), (1, 3), (5, 6), (6, 7), (2, 3), (5, 7), (1, 4), (3, 4)}

By using Kruskal's algorithm we select edges (1, 2), (1, 3), (5, 6) and (6, 7). Then we examined an edge (2, 3) and not selected as it creates cycle, similarly edge (5, 7) and not selected. The edge (1, 4) examined and selected. Finally, the edge (3, 4) examined and not selected. At this stage all the edges in L are examined. Here selected edges are 5 and not 6.

So the graph is not connected.

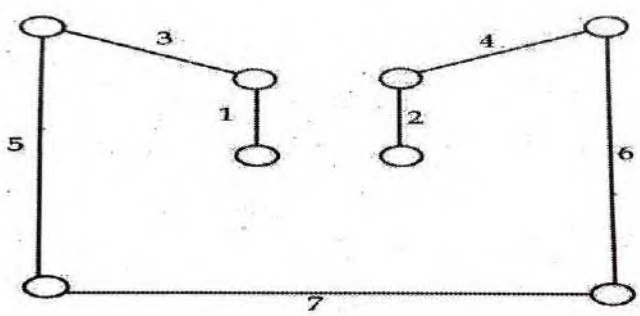
Ex. Obtain a minimal spanning tree of electrical network given below. Use Kruskal's algorithm.



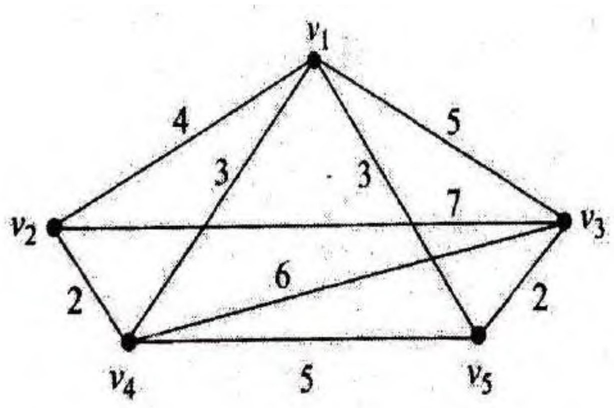
Solution: Given graph contain 8 vertices. The set L of edges in non-decreasing order is

L = {(1, 2), (3, 4), (1, 8), (4, 5), (5, 6), (7, 8), (3, 6), (2, 7), (6, 7), (5, 8), (1, 4), (2, 3)}

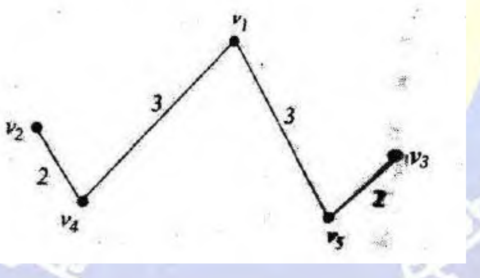
By using Kruskal's algorithm we select edges (1, 2), (3, 4), (1, 8), (4, 5), (5, 6) and (7, 8) in the non-decreasing order of their weights. Then we examined an edge (3, 6) and not selected as it creates cycle, similarly edge (2, 7) and not selected. The edge (6, 7) examined and selected. At this stage, the number of edges selected for inclusion in the tree is 7, which is the number of edges in a spanning tree. This selected 7 edges form a minimal spanning tree T in the given network with w(T) = 93 as shown in the figure below.



Ex. Use Kruskal's algorithm to determine a minimal spanning tree of the connected weighted graph G shown in the figure below.



Solution: Given graph contain 5 vertices. The set L of edges in non-decreasing order is
L = {(v_2, v_4), (v_3, v_5), (v_1, v_4), (v_1, v_5), (v_1, v_2), (v_1, v_3), (v_4, v_5), (v_3, v_4), (v_2, v_3),}
By using Kruskal's algorithm we select edges (v_2, v_4), (v_3, v_5), (v_1, v_4) and (v_1, v_5) in the non-decreasing order of their weights. At this stage, the number of edges selected for inclusion in the tree is 4, which is the number of edges in a spanning tree. This selected 4 edges form a minimal spanning tree T in the given network with w(T) = 10 as shown in the figure below.

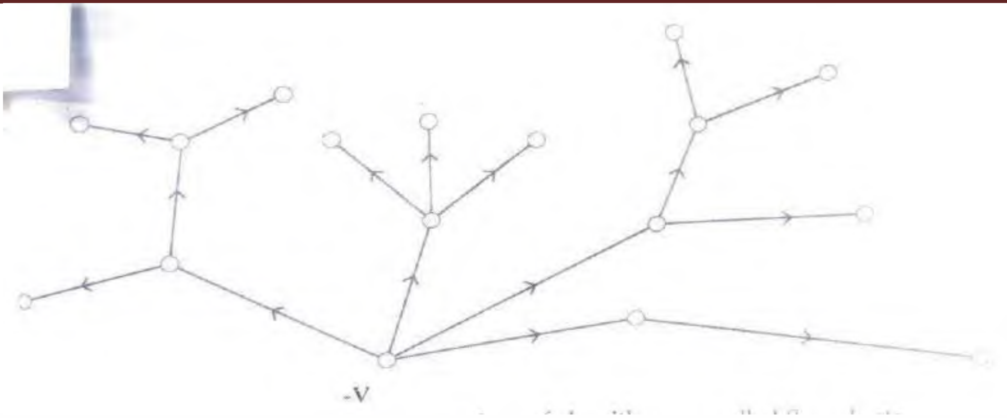


==

Arborescence: A rooted tree with directed edges and root is a vertex of in-degree 0 is called a arborescence.

Note: 1) An arborescence is rooted tree with directed edges and has exactly one vertex of in-degree 0 and all other vertices of in-degree 1 but out-degree is not fixed.

e.g.



be an arborescence, it is rooted tree with directed edges and has exactly one vertex of in-degree 0 and all other vertices of in-degree 1.

Length of Path: The number of edges on the path is called length of path.

Shortest Path Problem: The problem of finding shortest paths in networks is called shortest path problem.

Note: There are two algorithms to find shortest paths.

1) Dijkstra's algorithm, 2) Warshall's algorithm.

==

Dijkstra's algorithm: Let G = (V, E), where V = {1, 2, 3, …, n} and weight of every edge is non-negative. This algorithm can be used to find the shortest path (SP) and shortest distance (SD) from any fixed vertex say 1 to vertex i, if there is a directed path from 1 to i. Let $\omega(i, j)$ denote the weight of the edge from i to j. If there is no edge from i to j, we define $\omega(i, j) = \infty$.

Each vertex assigned a label that is either permanent or tentative. The permanent lable L(i) is the SD from 1 to i. The tentative label $L'(i)$ is an upper bound of L(i). At each stage of the algorithm, P is the set of vertices with permanent labels and T is the set of vertices with tentative labels. Initially P is the set {1} with L(1) = 0 and $L'(j) = \omega(i, j)$ for all j. The procedure terminate when P = V. Each iteration consists of two steps:

Step-1: Find a vertex k in T for which $L'(k)$ is finite and minimum. If there is no such k, stop. In that case there is no path from 1 to any unlabeled vertex. Otherwise declare k to be permanently labeled, and adjoin k to P.

Stop if P = V. Label an edge (i, k) where i is the labeled vertex that determines the minimum value of $L'(k)$.

Step-2: Replace $L'(j)$ by min{ $L'(j)$, $L(k) + \omega(k, j)$}, for each $j \in T$. and go to step-1.

Note that if G has n vertices and it is possible to obtain the SD from the starting vertex to every other vertex, then the set of (n-1) edges obtained by this method will form an arborescence rooted at v that will give both the SD and SP from v to every other vertex.

==

Remark: Dijkstra's algorithm is not applicable for negative weights of edges.

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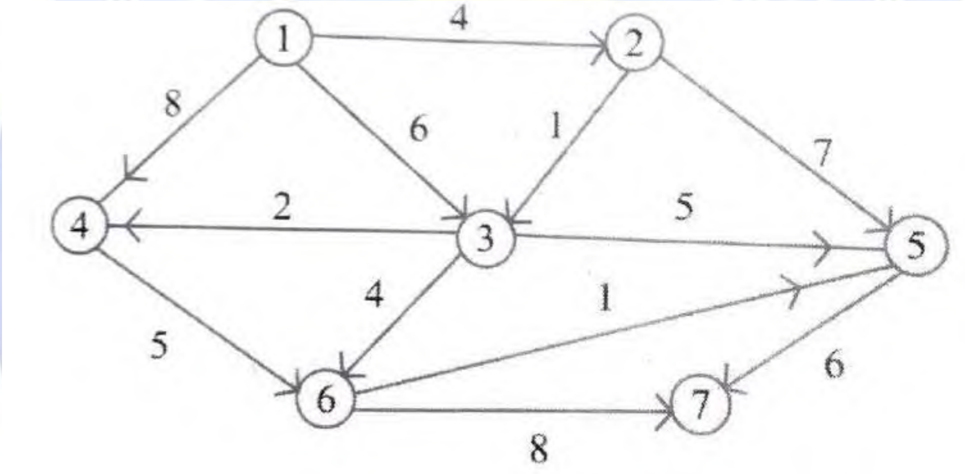
Ex. Obtain the SD and SP from vertex 1 to every other vertex in the network given below. Use Dijkstra's algorithm



Solution: By using Dijkstra's algorithm we have:

Iteration 1:

Step 1: P = {1}, $L(1) = 0$, $L'(2) = 4$, $L'(3) = 6$ and $L'(4) = 8$.

Adjoin vertex 2 to P and label the edge (1, 2).

Step 2: P = {1, 2} and $L(2) = 4$,

Now $L'(3) = \min\{6, L(2) + \omega(2, 3)\}$

$= \min\{6, 4+1\} = \min\{6, 5\} = 5$

$L'(4) = \min\{8, L(2) + \omega(2, 4)\}$

$= \min\{8, 4+\infty\} = \min\{8, \infty\} = 8$

$L'(5) = \min\{\infty, L(2) + \omega(2, 5)\}$

$= \min\{\infty, 4+7\} = \min\{\infty, 11\} = 11$

$L'(6) = \min\{\infty, L(2) + \omega(2, 6)\}$

$= \min\{\infty, 4+\infty\} = \min\{\infty, \infty\} = \infty$

$L'(7) = \min\{\infty, L(3) + \omega(2, 7)\} = \min\{\infty, \infty\} = \infty$.

Iteration 2:

Step 1: $P = \{1, 2\}$, $L(1) = 0$, $L(2) = 4$ and $L'(3) = 5$ and $L'(4) = 8$ and $L'(5) = 11$, Adjoin vertex 3 to P and label the edge (2, 3).

Step 2: $P = \{1, 2, 3\}$ and $L(3) = 5$,

Now $L'(4) = \min\{8, L(3) + \omega(3, 4)\}$

$= \min\{8, 5+2\} = \min\{8, 7\} = 7$

$L'(5) = \min\{11, L(3) + \omega(3, 5)\}$

$= \min\{11, 5+5\} = \min\{11, 10\} = 10$

$L'(6) = \min\{\infty, L(3) + \omega(3, 6)\}$

$= \min\{\infty, 5+4\} = \min\{\infty, 9\} = 9$

$L'(7) = \min\{\infty, L(3) + \omega(3, 7)\} = \min\{\infty, 5+\infty\} = \min\{\infty, \infty\} = \infty$

Iteration 3:

Step 1: $P = \{1, 2, 3\}$, $L(1) = 0$, $L(2) = 4$, $L(3) = 5$ and $L'(4) = 7$ and $L'(5) = 10$, $L'(6) = 9$, Adjoin vertex 4 to P and label the edge (3, 4).

Step 2: $P = \{1, 2, 3, 4\}$ and $L(4) = 7$,

Now $L'(5) = \min\{10, L(4) + \omega(4, 5)\}$

$= \min\{10, 7+\infty\} = \min\{10, \infty\} = 10$

$L'(6) = \min\{9, L(4) + \omega(4, 6)\}$

$= \min\{9, 7+5\} = \min\{9, 12\} = 9$

$L'(7) = \min\{\infty, L(4) + \omega(4, 7)\} = \min\{\infty, 7+\infty\} = \min\{\infty, \infty\} = \infty$

Iteration 4:

Step 1: $P = \{1, 2, 3, 4\}$, $L(1) = 0$, $L(2) = 4$, $L(3) = 5$ and $L(4) = 7$ and $L'(5) = 10$, $L'(6) = 9$, Adjoin vertex 6 to P and label the edge (3, 6).

Step 2: $P = \{1, 2, 3, 4, 6\}$ and $L(6) = 9$,

Now $L'(5) = \min\{10, L(6) + \omega(6, 5)\}$

$= \min\{10, 9 + 1\} = \min\{10, 10\} = 10$

$L'(7) = \min\{\infty, L(6) + \omega(6, 7)\} = \min\{\infty, 9+8\} = \min\{\infty, 17\} = 17$

Iteration 5:

Step 1: P = {1, 2, 3, 4, 6}, L(1) = 0, L(2) = 4, L(3) = 5, L(4) = 7, L(6) = 9, $L'(5)$ = 10,

and $L'(7)$ = 17, Adjoin vertex 5 to P and label the edge (3, 5).

Step 2: P = {1, 2, 3, 4, 6, 5} and L(5) = 10,

Now $L'(7) = \min\{17, L(5) + \omega(5, 7)\} = \min\{17, 10+6\} = \min\{17, 16\} = 16$

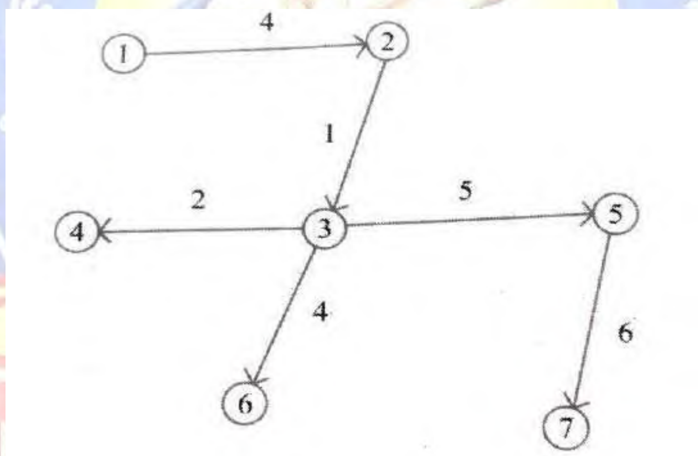
Iteration 6:

Step 1: P = {1, 2, 3, 4, 6, 5}, L(1) = 0, L(2) = 4, L(3) = 5, L(4) = 7, L(6) = 9, L(5) = 10,

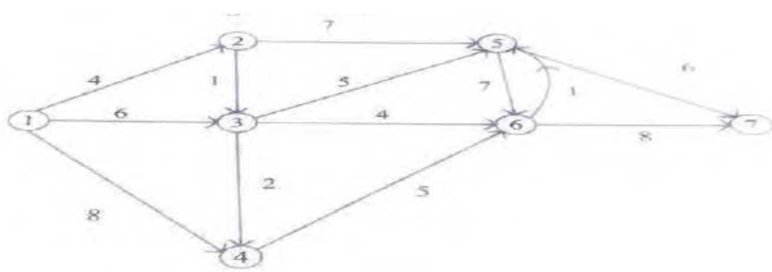
$L'(7)$ = 16, Adjoin vertex 7 to P and label the edge (5, 7).

Step 2: P = {1, 2, 3, 4, 6, 5, 7} and L(7) = 16, Adjoin vertex 7 to P and label the edge (5, 7).

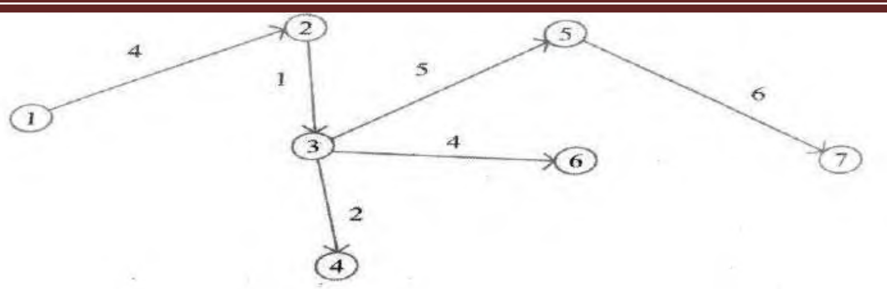
Now at this stage, we have P = V and the labeled edges (1, 2), (2, 3), (3, 4), (3, 5), (3, 6) and (5, 7) constitute a shortest path arborescence rooted at a vertex 1 as shown in the figure below.



Ex. Using Dijkstra's algorithm, find a shortest distance arborescence rooted at a vertex 1 of the directed network given below.



Solution: By using Dijkstra's algorithm shortest distance arborescence rooted at a vertex 1 of the directed network is as below.



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Warshall's algorithm (or Floyd Warshall's algorithm): The weight matrix W of a network G = (V, E), V = {1, 2, 3, ……, n}, is defined as in the case of Dijkstra's algorithm. The initial path of matrix P is the $n \times n$ matrix $[p(i, j)]$, where $p(i, j) = j$, There are n iterations in the execution of the algorithm. The j^{th} iteration based at vertex 1 begins with two matrices W_{j-1} and P_{j-1} and ends with A_j and $P_{j,}$ $j = 1, 2, \ldots, n$. Initially $W_o = W$ and $P_o = P$. The (u, v) entries of vertices in A_j and P_j are denoted by ω_i (u, v), the weight of an edge (u, v) and p_j (u, v) respectively. For fixed j, the matrices Wj and Pj are obtained from W_{j-1} and P_{j-1} by applying the following rules known as the triangle operation (or triangle inequality):

If $\omega_{j-1} (u, v) \leq \omega_{j-1} (u, j) + \omega_{j-1}(j, v)$ then $\omega_j (u, v) = \omega_{j-1} (u, v)$ and $p_j (u, v) = p_{j-1} (u, v)$, otherwise, $\omega_{j-1} (u, v) = \omega_{j-1} (u, j)+\omega_{j-1}(j, v)$ and $p_j (u, v) = p_{j-1}(u, j)$.

When the algorithm terminates, we are left with the SD matrix W_n and the SP matrix P_n. The (u, v) entry in the first matrix is the shortest distance between these two vertices and the (u, v) entry in the second matrix denotes the first vertex (after u) in a shortest path from u to v.

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Note: The Floyd-Warshall's Algorithm using the triangle operation correctly solves the SD and SP problem.

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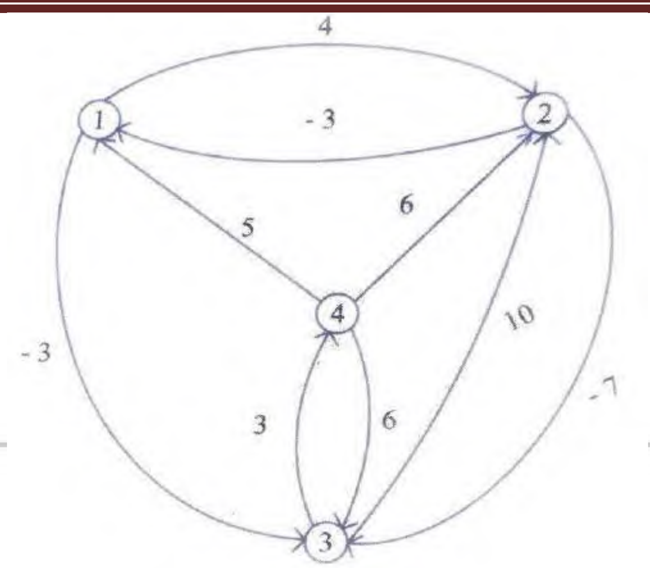
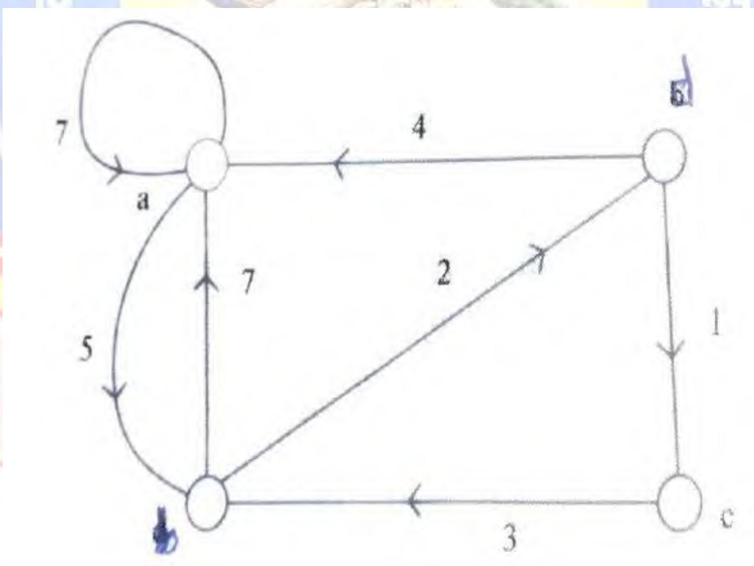
Ex.: Obtain the SD matrix and SP matrix of the network given below. Use Warshall's algorithms.



Solution: Using Warshall's Algorithm, we get SD matrix and SP matrix as follows

$$W_4 = \begin{bmatrix} 0 & 4 & -3 & 0 \\ -3 & 0 & -7 & -4 \\ 6 & 9 & 0 & 3 \\ 3 & 6 & -1 & 0 \end{bmatrix} \qquad P_4 = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 3 \\ 4 & 4 & 3 & 4 \\ 2 & 2 & 2 & 4 \end{bmatrix}$$

Ex.: Find the SD matrix and SP matrix for the following network. Use Warshals algorithm.



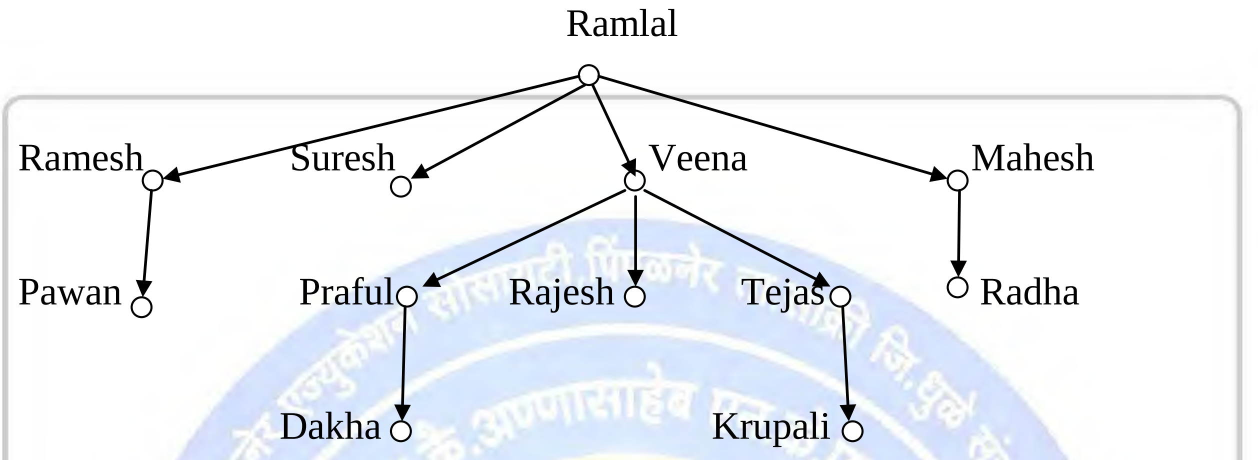
Solution: Using Warshall's Algorithm, we get SD matrix and SP matrix as follows

$$W_4 = \begin{bmatrix} 7 & 5 & 8 & 7 \\ 6 & 6 & 3 & 2 \\ 9 & 3 & 6 & 5 \\ 4 & 4 & 1 & 6 \end{bmatrix} \qquad P_4 = \begin{bmatrix} a & \mathrm{b} & \mathrm{b} & b \\ d & \mathrm{d} & \mathrm{d} & \mathrm{d} \\ b & \mathrm{b} & \mathrm{b} & \mathrm{b} \\ a & c & c & c \end{bmatrix}$$

Flow Chart: The diagrammatic representations of algorithms are called flow charts.

Ex. The father of Ramesh, Suresh, Veena & Mahesh is Ramlal. Pawan is son of Ramesh. Suresh has no child while Veena has three son's Praful, Rajesh and Tejas. Radha is daughter of Mahesh. If son of Praful is Daksha & daughter of Tejas is Krupali then find flow chart of the family using graph theory. (Oct. 2018)

Solution: Using graph theory the flow chart of family is as follows.



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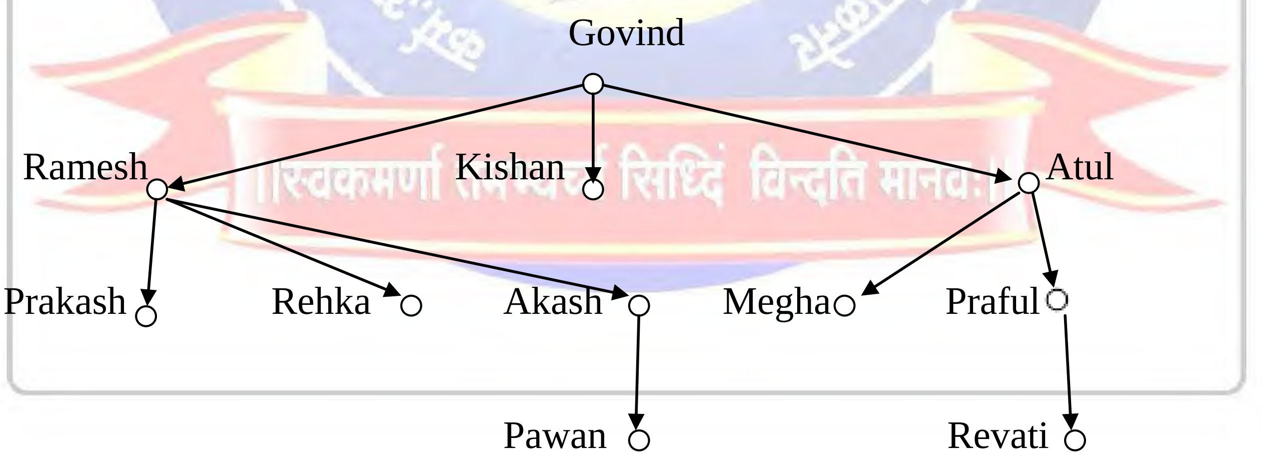
Ex. If father of Ramesh, Kishan and Atul is Govind. Where Ramesh has three children Prakash, Rekha and Akash. Kishan has no child. Megha and Praful are son and daughter of Atul. Pawan is son of Akash and Revati is daughter of Praful, then draw the flow chart of the family. (Mar. 2019)

Solution: Using graph theory the flow chart of family is as follows.



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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."