Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: F.Y.B.SC SEM.-I

SUBJECT: MTH-102: CALCULUS OF SINGLE VARIABLE

PREPARED BY: PROF. K. D. KADAM

MTH 102: CALCULUS OF SINGLE VARIABLE

UNIT-I: Limits and Continuity: **(Marks 15, 08hours)**

a) Epsilon-delta definition of limit of a function.

b) Basic properties of limits, Indeterminate forms & L-Hospitals rule.

c) Continuous functions. Properties of continuous functions on closed and bounded intervals.

d) Theorems on Boundedness of continuous functions, including Intermediate value theorem.

e) Uniform continuity.

UNIT-II: Mean Value Theorems: **(Marks 15, 08hours)**

a) Differentiability.

b) Rolle's Theorem.

c) Lagrange's Mean Value Theorem.

d) Cauchy's Mean Value Theorem.

e) Geometrical interpretation and applications.

UNIT-III: Successive Differentiation: **(Marks 15, 07hours)**

a) The nth derivative of some standard functions: e^{ax+b}, $(ax+b)^m$, x^m, $\frac{1}{ax+b}$, log(ax+b), sin(ax+b), cos(ax+b), e^{ax}sin(ax+b) and e^{ax}cos(ax+b).

b) Leibnitz's theorem & Examples.

UNIT-IV: Applications of Calculus **(Marks 15, 07hours)**

a) Taylor's theorem with Lagrange's form of remainder and related examples.

b) Maclaurin' theorem with Lagrange's form of remainder and related examples

c) Reduction Formulae: i) $\int_0^{\pi/2} \sin^n x\, dx$, ii) i) $\int_0^{\pi/2} \cos^n x\, dx$, iii) $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$, iv) $\int \frac{sinnx}{six} dx$.

REFERENCE BOOKS:

1. Theory and Problems of Advanced Calculus, by Robert Wrede and Murray R. Spiegel, McGraw- Hill Company, New York, Second Edition, 2002.

2. Text Book on Differential calculus, by Gorakh Prasad, Pothishala Private Ltd., Allahabad, 1959.

3. Integral calculus, by Gorakh Prasad, Pothishala Private Ltd., Allahabad.

Learning Outcomes: Upon successful completion of this course the student will be able to:

a) understand basic concepts on limits and continuity.

b) understand use of differentiations in various theorems.

c) know the Mean value theorems and its applications.

d) make the applications of Taylor's, Maclaurin's theorem.

e) know the applications of calculus.

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UNIT-1 LIMITS AND CONTINUITY

Limit of a function:

If for small ε > 0, there exist $\delta > 0$ depends on ε such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$. Then l is said to be limit of f(x) as x ⟶ a. Denoted by $\lim_{x\to a} f(x) = l$.

Algebra of Limits:

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then

i) $\lim_{x\to a}[f(x) \pm g(x)] = l \pm m$

ii) $\lim_{x\to a}[f(x)g(x)] = lm$

iii) $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{l}{m}$ provided m ≠ 0

iv) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{l}$

Right hand limit:

If for small ε > 0, there exist $\delta > 0$ depends on ε such that $|f(x) - l| < \varepsilon$ whenever $\forall\, x \in (a, a + \delta)$. Then l is said to be right hand limit of f(x) as x ⟶ a. Denoted by $\lim_{x\to a^+} f(x) = l$.

Left hand limit:

If for small ε > 0, there exist $\delta > 0$ depends on ε such that $|f(x) - l| < \varepsilon$ whenever $\forall\, x \in (a - \delta,\ a)$. Then l is said to be left hand limit of f(x) as x ⟶ a. Denoted by $\lim_{x\to a^-} f(x) = l$.

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0.∞ or ∞.0, ∞ - ∞, 0^0, ∞^0 and 1^∞ are the indeterminate forms.

Indeterminate forms $\frac{0}{0}$:

L' Hospital's Rule:

If f(x) and g(x) are two real valued functions such that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and f '(a) and g'(a) exist where g'(a) ≠ 0 then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$

Generalized L' Hospital's Rule:

If f(x) and g(x) are two real valued functions such that $\lim_{x\to a} f(x) = \lim_{x\to a} f'(x) = \lim_{x\to a} f''(x) = \ldots \lim_{x\to a} f^{(n-1)}(x) = 0$ and $\lim_{x\to a} g(x) = \lim_{x\to a} g'(x) = \lim_{x\to a} g''(x) = \ldots \lim_{x\to a} g^{(n-1)}(x) = 0$ and $f^{(n)}(a)$ and $g^{(n)}(a)$ exist where $g^{(n)}(a) \neq 0$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$

Ex.: Evaluate $\lim_{x\to 1}\frac{\log x}{x-1}$ (Oct. 2018)

Sol. Let $L = \lim_{x\to 1}\frac{\log x}{x-1}$ ($\frac{0}{0}$ form)

By L' Hospital's rule

$L = \lim_{x\to 1}\frac{\frac{1}{x}}{1}$

$= 1$

Ex.: Evaluate $\lim_{x \to 5} \frac{x^2-4x-5}{x^2+2x-35}$ (Mar. 2019)

Sol. Let L = $\lim_{x \to 5} \frac{x^2-4x-5}{x^2+2x-35}$ $\left(\frac{0}{0}\text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 5} \frac{2x-4}{2x+2}$$

$$= \frac{10-4}{10+2}$$

$$= \frac{6}{12}$$

$$= \frac{1}{2}$$

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Ex.: Evaluate $\lim_{x \to 7} \frac{x^2-10x+21}{x^2-12x+35}$

Sol. Let L = $\lim_{x \to 7} \frac{x^2-10x+21}{x^2-12x+35}$ $\left(\frac{0}{0}\text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 7} \frac{2x-10}{2x-12}$$

$$= \frac{14-10}{14-12}$$

$$= \frac{4}{2}$$

$$= 2$$

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Ex.: Evaluate $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$ (Mar. 2019)

Sol. Let L = $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{\sec^2 x - 1}{1-\cos x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \tan x}{\sin x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\sec^2 x \sin x}{\sin x \cos x}$$

$$= \lim_{x \to 0} 2\sec^3 x$$

$$= 2(1)^3$$

$$= 2$$

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Ex.: Evaluate $\lim_{x \to 0} \frac{e^x-e^{-x}-2x}{x^2 \sin x}$

Sol. Let L = $\lim_{x \to 0} \frac{e^x-e^{-x}-2x}{x^2 \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{e^x+e^{-x}-2}{2xsinx+x^2 \cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$L = \lim_{x \to 0} \frac{e^x-e^{-x}}{2sinx+2xcosx+2x\cos x-x^2 \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$\text{i.e } L = \lim_{x \to 0} \frac{e^x-e^{-x}}{2sinx+4xcosx-x^2 \sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$L = \lim_{x \to 0} \frac{e^x+e^{-x}}{2cosx+4cosx-4xsinx-2x\sin x-x^2 \cos x}$$

$$= \frac{1+1}{2+4-0}$$

$$= \frac{1}{3}$$

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Ex.: Evaluate $\lim_{x \to 0} \frac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$

Sol. Let $L = \lim_{x \to 0} \frac{e^x-e^{-x}-2\log(1+x)}{x\sin x} \quad \left(\frac{0}{0} \text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{e^x+e^{-x}-\frac{2}{1+x}}{\sin x+x\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$L = \lim_{x \to 0} \frac{e^x-e^{-x}+\frac{2}{(1+x)^2}}{\cos x+\cos x-x\sin x}$$

$$= \frac{1-1+2}{1+1-0}$$

$$= 1$$

==

Ex.: Evaluate $\lim_{x \to 0} \frac{e^x+e^{-x}-2\cos x}{x\sin x}$

Sol. Let $L = \lim_{x \to 0} \frac{e^x+e^{-x}-2\cos x}{x\sin x} \quad \left(\frac{0}{0} \text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{e^x-e^{-x}+2\sin x}{\sin x+x\cos x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$L = \lim_{x \to 0} \frac{e^x+e^{-x}+2\cos x}{\cos x+\cos x-x\sin x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{1+1+2}{1+1-0}$$

$$= 2$$

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Indeterminate forms $\frac{\infty}{\infty}$: In this form we use L' Hospital's Rule.

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Ex.: Evaluate $\lim_{x \to \infty} \frac{\log x}{x}$

Sol. Let $L = \lim_{x \to \infty} \frac{\log x}{x} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$L = 0$$

Ex.: Evaluate $\lim_{x\to\pi}\frac{\log(\pi-x)}{\cot x}$

Sol. Let L = $\lim_{x\to\pi}\frac{\log(\pi-x)}{\cot x}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$$L = \lim_{x\to\pi}\frac{-\frac{1}{\pi-x}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to\pi}\frac{\sin^2 x}{\pi-x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to\pi}\frac{2\sin x\cos x}{-1}$$

L = 0

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Ex.: Evaluate $\lim_{x\to 0}\frac{\operatorname{cosec}x}{\log x}$

Sol. Let L = $\lim_{x\to 0}\frac{\operatorname{cosec}x}{\log x}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$$L = \lim_{x\to 0}\frac{-\operatorname{cosec}x\cot x}{\frac{1}{x}}$$

$$= \lim_{x\to 0}\frac{-x}{\sin x\tan x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-1}{\cos x\tan x+\sin x\sec^2 x}$$

$$= -\infty$$

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Ex.: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\log(x-\frac{\pi}{2})}{\tan x}$ (Mar. 2019)

Sol. Let L = $\lim_{x\to\frac{\pi}{2}}\frac{\log(x-\frac{\pi}{2})}{\tan x}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$$L = \lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{(x-\frac{\pi}{2})}}{\sec^2 x}$$

i.e. $L = \lim_{x\to\frac{\pi}{2}}\frac{\cos^2 x}{(x-\frac{\pi}{2})}$ ($\frac{0}{0}$ form)

$$\therefore L = \lim_{x\to\frac{\pi}{2}}\frac{-2\cos x\sin x}{1}$$

$\therefore$ L = 0.

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Ex.: Evaluate $\lim_{x\to 0}\frac{\log x}{\cot x}$

Sol. Let L = $\lim_{x\to 0}\frac{\log x}{\cot x}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{\frac{1}{x}}{-cosec^2 x}$$

i.e. $L = \lim_{x \to 0} \frac{-sin^2 x}{x}$ $\left(\frac{0}{0} \text{ form}\right)$

$\therefore L = \lim_{x \to 0} \frac{-2sinxcosx}{1}$

$\therefore L = 0.$

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Indeterminate forms 0×∞: In this case we convert the given indeterminate form 0×∞ into the indeterminate form $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ and use L' Hospital's rule.

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Ex.: Evaluate $\lim_{x \to 0} x \log x$

Sol. Let $L = \lim_{x \to 0} x \log x$ $(0.\infty \text{ form})$

i.e. $L = \lim_{x \to 0} \frac{logx}{\frac{1}{x}}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

i.e. $L = \lim_{x \to 0} -x$

$\therefore L = 0.$

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Ex.: Evaluate $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi}{2}x\right)$ (Oct.2018)

Sol. Let $L = \lim_{x \to 1} (1-x) \tan\left(\frac{\pi}{2}x\right)$ $(0.\infty \text{ form})$

i.e. $L = \lim_{x \to 1} \frac{1-x}{\cot(\frac{\pi}{2}x)}$ $\left(\frac{0}{0} \text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 1} \frac{-1}{-\frac{\pi}{2}cosec^2\left(\frac{\pi}{2}x\right).}$$

i.e. $L = \lim_{x \to 1} \left(\frac{2}{\pi}\right) sin^2\left(\frac{\pi}{2}x\right)$

$\therefore L = \frac{2}{\pi}.$

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Ex.: Evaluate $\lim_{x \to 0} (1-\cos x)(\cot x)$

Sol. Let $L = \lim_{x \to 0} (1-\cos x)(\cot x)$ $(0.\infty \text{ form})$

i.e. $L = \lim_{x \to 0} \frac{1-cosx}{\tan x}$ $\left(\frac{0}{0} \text{ form}\right)$

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{\sin x}{\sec^2 x}$$

$\therefore L = \frac{0}{1}$

$\therefore L = 0.$

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Ex.: Evaluate $\lim_{x \to 0} \sin x \log x$

Sol. Let $L = \lim_{x \to 0} \sin x \log x$ $\quad$ ($0.\infty$ form)

i.e. $L = \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x}$ $\quad$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}$$

i.e. $L = \lim_{x \to 0} \frac{-\sin x \tan x}{x}$ $\quad$ ($\frac{0}{0}$ form)

i.e. $L = \lim_{x \to 0} -\left(\frac{\sin x}{x}\right) \tan x$

$\therefore L = -(1).0$

$\therefore L = 0.$

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Ex.: Evaluate $\lim_{x \to 0} \tan x \log x$ (Oct.2018)

Sol. Let $L = \lim_{x \to 0} \tan x \log x$ $\quad$ ($0.\infty$ form)

i.e. $L = \lim_{x \to 0} \frac{\log x}{\cot x}$ $\quad$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$$L = \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x}$$

i.e. $L = \lim_{x \to 0} \frac{-\sin^2 x}{x}$ $\quad$ ($\frac{0}{0}$ form)

i.e. $L = \lim_{x \to 0} -\left(\frac{\sin x}{x}\right) \sin x$

$\therefore L = -(1).0$

$\therefore L = 0.$

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Ex.: Evaluate $\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \sec x$

Sol. Let $L = \lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \sec x$ $\quad$ ($0.\infty$ form)

i.e. $L = \lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos x}$ $\quad$ ($\frac{0}{0}$ form)

By L' Hospital's rule

$L = \lim_{x \to \frac{\pi}{2}} \frac{1}{-sinx}$

$\therefore L = \frac{1}{-1}$

$\therefore L = -1.$

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Indeterminate forms ∞ - ∞: In this case we convert the given indeterminate form ∞ - ∞ into the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use L' Hospital's rule.

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Ex.: Evaluate $\lim_{x \to 0}$ (cosecx-cotx)

Sol. Let L = $\lim_{x \to 0}$ (cosecx - cotx)(∞-∞ form)

i.e. L = $\lim_{x \to 0}[\frac{1}{\text{sinx}} - \frac{cosx}{\text{sinx}}]$

i.e. L = $\lim_{x \to 0}[\frac{1-cosx}{\text{sinx}}]$($\frac{0}{0}$ form)

By L' Hospital's rule

$L = \lim_{x \to 0} \frac{sinx}{cosx}$

$\therefore L = \frac{0}{1}.$

$\therefore L = 0.$

==

Ex.: Evaluate $\lim_{x \to \frac{\pi}{2}}$ (secx - tanx)

Sol. Let L = $\lim_{x \to \frac{\pi}{2}}$ (secx - tanx) (∞-∞ form)

i.e. L = $\lim_{x \to \frac{\pi}{2}}[\frac{1}{\text{cosx}} - \frac{sinx}{\text{cosx}}]$

i.e. L = $\lim_{x \to \frac{\pi}{2}}[\frac{1-sinx}{\text{cosx}}]$ ($\frac{0}{0}$ form)

By L' Hospital's rule

$L = \lim_{x \to \frac{\pi}{2}} \frac{-cosx}{-sinx}$

$\therefore L = \frac{0}{1} = 0.$

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Ex.: Evaluate $\lim_{x \to 0} (\frac{1}{x} - \text{cotx})$ (Oct. 2018)

Sol. Let L = $\lim_{x \to 0} (\frac{1}{x} - \text{cotx})$ (∞-∞ form)

i.e. L = $\lim_{x \to 0}(\frac{1}{\text{x}} - \frac{cosx}{\text{sinx}})$

i.e. L = $\lim_{x \to 0}(\frac{sinx-xcosx}{\text{xsinx}})$ ($\frac{0}{0}$ form)

By L' Hospital's rule

$$L = \lim_{x\to 0} \frac{cosx - cosx + xsinx}{sinx + xcosx} \qquad (\tfrac{0}{0} \text{ form})$$

$$\therefore L = \lim_{x\to 0} \frac{sinx + xcosx}{cosx + cosx - xsinx}$$

$$\therefore L = \frac{0}{1+1-0} = 0.$$

Ex.: Evaluate $\lim_{x\to 0} \left(cosecx - \frac{1}{x} \right)$

Sol. Let $L = \lim_{x\to 0} \left(cosecx - \frac{1}{x} \right)$ (∞-∞ form)

$$\text{i.e. } L = \lim_{x\to 0} \left(\frac{1}{sinx} - \frac{1}{x} \right)$$

$$\text{i.e. } L = \lim_{x\to 0} \left(\frac{x - sinx}{xsinx} \right) \qquad (\tfrac{0}{0} \text{ form})$$

By L' Hospital's rule

$$L = \lim_{x\to 0} \frac{1 - cosx}{sinx + xcosx} \qquad (\tfrac{0}{0} \text{ form})$$

$$\therefore L = \lim_{x\to 0} \frac{sinx}{cosx + cosx - xsinx}$$

$$\therefore L = \frac{0}{1+1-0}.$$

$$\therefore L = 0.$$

Ex.: Evaluate $\lim_{x\to 1} \left(\frac{1}{logx} - \frac{1}{x-1} \right)$

Sol. Let $L = \lim_{x\to 1} \left(\frac{1}{logx} - \frac{1}{x-1} \right)$ (∞-∞ form)

$$\text{i.e. } L = \lim_{x\to 1} \left[\frac{x - 1 - logx}{(x-1)logx} \right] \qquad (\tfrac{0}{0} \text{ form})$$

By L' Hospital's rule

$$L = \lim_{x\to 1} \left[\frac{1 - \frac{1}{x}}{logx + (x-1)\frac{1}{x}} \right]$$

$$\text{i.e. } L = \lim_{x\to 1} \left[\frac{x-1}{xlogx + (x-1)} \right] \qquad (\tfrac{0}{0} \text{ form})$$

$$\therefore L = \lim_{x\to 1} \frac{1}{logx + 1 + 1}$$

$$\therefore L = \frac{1}{0+1+1}.$$

$$\therefore L = \frac{1}{2}.$$

Indeterminate forms 0^0 or 1^∞ or ∞^0: In this case denote $L = \lim_{x\to a} f(x)^{g(x)}$ and taking log on both sides we get $\log L = \lim_{x\to a} g(x) \log[f(x)]$ which in the indeterminate form $0\times\infty$ convert it into the indeterminate form $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ and using L' Hospital's rule we get log L = l which gives $L = \lim_{x\to a} f(x)^{g(x)} = e^{l}$

Ex.: Evaluate $\lim_{x\to 0} x^x$

Sol. Let L = $\lim_{x\to 0} x^x$ $(0^0$ form)

By taking log on both sides, we get,

log L = $\lim_{x\to 0} x\log x$ ($0.\infty$ form)

i.e. log L = $\lim_{x\to 0} \frac{\log x}{\frac{1}{x}}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

log L = $\lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$

i.e. log L = $\lim_{x\to 0} -x$

$\therefore$ log L = 0

$\therefore$ L = e^o

$\therefore$ L = 1

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Ex.: Evaluate $\lim_{x\to a} (x-a)^{(x-a)}$ (Oct. 2018)

Sol. Let L = $\lim_{x\to a} (x-a)^{(x-a)}$ $(0^0$ form)

By taking log on both sides, we get,

log L = $\lim_{x\to a}$ (x-a) log(x-a) ($0.\infty$ form)

i.e. log L = $\lim_{x\to a} \frac{\log(x-a)}{\frac{1}{x-a}}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

log L = $\lim_{x\to a} \frac{\frac{1}{x-a}}{\frac{-1}{(x-a)^2}}$

i.e. log L = $\lim_{x\to a} -(x\text{-a})$

$\therefore$ log L = 0

$\therefore$ L = e^o

$\therefore$ L = 1

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Ex.: Evaluate $\lim_{x\to \frac{\pi}{2}} (sinx)^{tanx}$ (Mar. 2019)

Sol. Let L = $\lim_{x\to \frac{\pi}{2}} (sinx)^{tanx}$ $(1^\infty$ form)

By taking log on both sides, we get,

log L = $\lim_{x\to \frac{\pi}{2}}$ tanx log(sinx)($\infty.0$ form)

i.e. log L = $\lim_{x\to \frac{\pi}{2}} \frac{\log(sinx)}{cotx}$ ($\frac{0}{0}$ form)

By L' Hospital's rule

$\log L = \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

i.e. $\log L = \frac{0}{-1}$

∴ $\log L = 0$

∴ $L = e^o$

∴ $L = 1$

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Ex.: Evaluate $\lim_{x \to 0} (\cos x)^{\cot x}$

Sol. Let $L = \lim_{x \to 0} (\cos x)^{\cot x}$ $\quad (1^\infty$ form)

By taking log on both sides, we get,

$\log L = \lim_{x \to 0} \cot x \log(\cos x)$ $\quad (\infty . 0$ form)

i.e. $\log L \lim_{x \to 0} \frac{\log(\cos x)}{\tan x}$ $\quad (\frac{0}{0}$ form)

By L' Hospital's rule

$\log L = \lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{\sec^2 x}$

i.e. $\log L = \frac{0}{1}$

∴ $\log L = 0$

∴ $L = e^o$

∴ $L = 1$

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Ex.: Evaluate $\lim_{x \to \infty} (1 + x)^{\frac{1}{x}}$

Sol. Let $L = \lim_{x \to \infty} (1 + x)^{\frac{1}{x}}$ $\quad (\infty^0$ form)

By taking log on both sides, we get,

$\log L = \lim_{x \to \infty} \frac{1}{x} \log(1+x)$ $\quad (0.\infty$ form)

i.e. $\log L = \lim_{x \to \infty} \frac{\log(1+x)}{x}$ $\quad (\frac{\infty}{\infty}$ form)

By L' Hospital's rule

$\log L = \lim_{x \to \infty} \frac{\frac{1}{1+x}}{1}$

i.e. $\log L = \frac{1}{\infty}$

∴ $\log L = 0$

∴ $L = e^o$

∴ $L = 1$

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Ex.: Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^{tanx}$

Sol. Let L = $\lim_{x\to 0}\left(\frac{1}{x}\right)^{tanx}$ (∞^0 form)

By taking log on both sides, we get,

log L = $\lim_{x\to 0} tanx \log(\frac{1}{x})$ ($\infty.0$ form)

i.e. log L = $\lim_{x\to 0}\frac{-\log x}{cotx}$ ($\frac{\infty}{\infty}$ form)

By L' Hospital's rule

log L = $\lim_{x\to 0}\frac{\frac{-1}{x}}{-cosec^2x}$

i.e. log L = $\lim_{x\to 0}\frac{sin^2x}{x}$

i.e. log L = $\lim_{x\to 0}\left(\frac{sinx}{x}\right)sinx$

log L = $(1).0$

∴ log L = 0

∴ L = e^o

∴ L = 1

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Continuity of a function at a point:

A function f (x) is said to be continuous at a point x = a if f (a) is defined, $\lim_{x\to a}$f (x) is exist and $\lim_{x\to a}$f (x) = f (a).

Remark:

i) The limit of the function at a point x = a exists if both the left hand limit and right hand limit are equal i.e. $\lim_{x\to a^-}$ f (x) = $\lim_{x\to a^+}$ f (x) and is denoted by $\lim_{x\to a}$f (x)

ii) The function is discontinuous at a point x = a if f (a) is not defined or $\lim_{x\to a}$f (x) does not exists i.e. $\lim_{x\to a^-}$ f (x) ≠ $\lim_{x\to a^+}$ f (x) or $\lim_{x\to a}$f (x) exists but $\lim_{x\to a}$f (x) ≠ f (a)

Removable discontinuity:

A function f (x) is said to have removable discontinuity at x = a if $\lim_{x\to a}$f (x) ≠ f (a) and the discontinuity can be removed by giving the value to f (a) as $\lim_{x\to a}$f (x).

Irremovable discontinuity:

A function f(x) is said to have an irremovable discontinuity at x = a if $\lim_{x\to a^-}$ f (x) ≠ $\lim_{x\to a^+}$ f (x)

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Ex. Examine the continuity of the function f (x) at x = a. (March 2019)

where f (x) = $\frac{x^2}{a}$ – a for 0 < x < a

= 0 for x = a

= a - $\frac{a^3}{x^2}$ for x > a

Sol. Let f (x) = $\frac{x^2}{a}$ – a for 0 < x < a
 = 0 for x = a
 = a - $\frac{a^3}{x^2}$ for x > a

Here f (a) = 0 ……..(1)

Now we find left hand & right hand limits of a given function as follows

$\lim_{x \to a^-} f(x) = \lim_{x \to a}(\frac{x^2}{a} - a) = \frac{a^2}{a} - a = a - a = 0$ &

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+}(a - \frac{a^3}{x^2}) = a - \frac{a^3}{a^2} = a - a = 0$

∴ $\lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = 0$

∴ $\lim_{x \to a} f(x)$ is exist and $\lim_{x \to a} f(x) = 0 = f(a)$ by (1)

∴ f (x) is continuous at x = a.

==

Ex. Examine the continuity of the function f (x) at x = 2. (March 2019)

where f (x) = $\frac{x^2}{2}$ – 2 for 0 < x < 2
 = 2 for x = 2
 = 2 - $\frac{8}{x^2}$ for x > 2

Sol. Let f (x) = $\frac{x^2}{2}$ – 2 for 0 < x < 2
 = 2 for x = 2
 = 2 - $\frac{8}{x^2}$ for x > 2

Here f (2) = 2 ……..(1)

Now we find left hand & right hand limits of a given function as follows

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2}(\frac{x^2}{2} - 2) = \frac{4}{2} - 2 = 2 - 2 = 0$ &

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2}(2 - \frac{8}{x^2}) = 2 - \frac{8}{4} = 2 - 2 = 0$

∴ $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 0$

∴ $\lim_{x \to 2} f(x)$ is exist and $\lim_{x \to 2} f(x) = 0 \neq f(2)$ by (1)

∴ f (x) is not continuous at x = 2.

==

Ex. Show that the function f (x) where f (x) = $\frac{x^2}{4}$ – 4 for 0 < x < 4
 = 0 for x = 4
 = 4 - $\frac{64}{x^2}$ for x > 4

is continuous at x = 4.

Proof. Let f (x) = $\frac{x^2}{4} - 4$ for 0 < x < 4

= 0 for x = 4

= 4 - $\frac{64}{x^2}$ for x > 4

Here f (4) = 0 ……..(1)

Now we find left hand & right hand limits of a given function as follows

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4}\left(\frac{x^2}{4} - 4\right) = \frac{16}{4} - 4 = 4 - 4 = 0$ &

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4}\left(4 - \frac{64}{x^2}\right) = 4 - \frac{64}{16} = 4 - 4 = 0$

∴ $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = 0$

∴ $\lim_{x\to 4} f(x)$ is exist and $\lim_{x\to 4} f(x) = 0 = f(4)$ by (1)

Hence f (x) is continuous at x = 4 is proved.

==

Ex. Discuss the continuity of function f (x) at x = 5

where f (x) = $\frac{x^2}{5} - 5$ if 0 < x < 5

= 0 if x = 5

= 5 - $\frac{125}{x^2}$ if x > 5

Proof. Let f (x) = $\frac{x^2}{5} - 5$ if 0 < x < 5

= 0 if x = 5

= 5 - $\frac{125}{x^2}$ if x > 5

Here f (5) = 0 ……..(1)

Now we find left hand & right hand limits of a given function as follows

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5}\left(\frac{x^2}{5} - 5\right) = \frac{25}{5} - 5 = 5 - 5 = 0$ &

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5}\left(5 - \frac{125}{x^2}\right) = 5 - \frac{125}{25} = 5 - 5 = 0$

∴ $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 0$

∴ $\lim_{x\to 5} f(x)$ is exist and $\lim_{x\to 5} f(x) = 0 = f(5)$ by (1)

Hence f (x) is continuous at x = 5 is proved.

==

Ex. Examine the continuity of the function f (x) at x = 3.

where f (x) = $\frac{x^2-9}{x-3}$ for 0 ≤ x < 3

= 6 for x = 3

= 8 - $\frac{18}{x^2}$ for x > 3

Sol. Let f (x) = $\frac{x^2-9}{x-3}$ for 0 ≤ x < 3

$= 6$ for x = 3

$= 8 - \frac{18}{x^2}$ for x > 3

Here f (3) = 6 ……..(1)

Now we find left hand & right hand limits of a given function as follows

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3}(x+3) = 3 + 3 = 6$ &

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3}\left(8 - \frac{18}{x^2}\right) = 8 - \frac{18}{9} = 8 - 2 = 6$

$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 6$

$\therefore \lim_{x\to 3} f(x)$ is exist and $\lim_{x\to 3} f(x) = 6 = f(3)$ by (1)

$\therefore$ f (x) is continuous at x = 3.

==

Ex. If the following function is continuous at x = 0, then find the values of a and b

where $f(x) = \frac{\sin 2x}{3x} + a$ if x > 0

$= 3 + b$ if x < 0

$= \frac{3}{2}$ if x = 0

Sol. Let $f(x) = \frac{\sin 2x}{3x} + a$ if x > 0

$= 3 + b$ if x < 0

$= \frac{3}{2}$ if x = 0

be the given function which is continuous at x = 0.

$\therefore \lim_{x\to 0} f(x) = f(0)$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \frac{3}{2}$

$\therefore \lim_{x\to 0}(3+b) = \frac{3}{2}$ & $\lim_{x\to 0}\left(\frac{\sin 2x}{3x} + a\right) = \frac{3}{2}$

$\therefore 3+b = \frac{3}{2}$ & $\lim_{x\to 0^+} \frac{2}{3}\left(\frac{\sin 2x}{2x}\right) + a = \frac{3}{2}$

$\therefore b = \frac{3}{2} - 3$ & $\frac{2}{3}(1) + a = \frac{3}{2}$

$\therefore b = -\frac{3}{2}$ & $a = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$

$\therefore a = \frac{5}{6}$ & $b = -\frac{3}{2}$.

==

Ex. If the following function is continuous at x = 0, then find the values of a and b

where $f(x) = \frac{\sin 5x}{6x} + a$ if x > 0

$= x+5 - b$ if x < 0

$= 1$ if x = 0

Sol. Let $f(x) = \frac{\sin 5x}{6x} + a$ if x > 0

= x+5 - b if x < 0

= 1 if x = 0

be the given function which is continuous at x = 0.

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1$

$\therefore \lim_{x \to 0}(x+5-b) = 1$ & $\lim_{x \to 0}(\frac{sin5x}{6x} + a) = 1$

$\therefore 5 - b = 1$ & $\lim_{x \to 0^+} \frac{5}{6}(\frac{sin5x}{5x}) + a = 1$

$\therefore b = 5 - 1$ & $\frac{5}{6}(1) + a = 1$

$\therefore b = 4$ & $a = 1 - \frac{5}{6} = \frac{1}{6}$

$\therefore a = \frac{1}{6}$ & b = 4.

==

Ex. Find the value of m, such that the function f (x) = m (x^2-2x) if x < 0

= cosx if x ≥ 0

is continuous at x = 0.

Sol. Let f (x) = m (x^2-2x) if x < 0

= cosx if x ≥ 0

is continuous at x = 0

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \cos 0 = 1$

$\therefore \lim_{x \to 0} m(x^2-2x) = 1$

$\therefore$ 0 = 1 which is not possible.

$\therefore$ Value of m cannot be determined.

==

Ex. For what value of k, the function f (x) = $\frac{\sin 2x}{x}$ if x ≠ 0

= k if x = 0

is continuous at x = 0 ?

Sol. Let f (x) = $\frac{\sin 2x}{x}$ if x ≠ 0

= k if x = 0

is continuous at x = 0 if $\lim_{x \to 0} f(x) = f(0)$

i. e. $\lim_{x \to 0} \frac{\sin 2x}{x} = k$

i. e. $\lim_{x \to 0} 2(\frac{\sin 2x}{2x}) = k$ i.e. k = 2(1) = 2

Hence for k = 2 the given function is continuous at x = 0.

==

Ex. For what value of k, the function f (x) = $\frac{\tan 2x}{x}$ if $x \neq 0$

= k if x = 0

is continuous at x = 0 ?

Sol. Let f (x) = $\frac{\tan 2x}{x}$ if $x \neq 0$

= k if x = 0

is continuous at x = 0 if $\lim_{x \to 0} f(x) = f(0)$

i.e. $\lim_{x \to 0} \frac{\tan 2x}{x} = k$

i.e. $\lim_{x \to 0} 2\left(\frac{\tan 2x}{2x}\right) = k$

i.e. k = 2(1) = 2

Hence for k = 2 the given function is continuous at x = 0.

==

Ex. If the function f (x) = $\frac{(5^{\sin x}-1)^2}{x\log(1+2x)}$ is continuous at x = 0, then show that f (0) = $\frac{(\log 5)^2}{2}$.

Proof. Let f (x) = $\frac{(5^{\sin x}-1)^2}{x\log(1+2x)}$ is continuous at x = 0.

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\therefore f(0) = \lim_{x \to 0} \frac{(5^{\sin x}-1)^2}{x\log(1+2x)}$

$= \lim_{x \to 0} \left(\frac{5^{\sin x}-1}{\sin x}\right)^2 \times \left(\frac{\sin x}{x}\right)^2 \times \left(\frac{1}{\frac{1}{x}\log(1+2x)}\right)$

$= \lim_{x \to 0} \left(\frac{5^{\sin x}-1}{\sin x}\right)^2 \times \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \times \lim_{x \to 0} \left(\frac{1}{\frac{1}{x}\log(1+2x)}\right)$

$= (\log 5)^2 \times (1)^2 \times \lim_{x \to 0} \left(\frac{1}{\log(1+2x)^{\frac{1}{x}}}\right)$

$= (\log 5)^2 \times \lim_{x \to 0} \left(\frac{1}{\log(1+2x)^{\frac{2}{2x}}}\right)$

$= (\log 5)^2 \times \lim_{x \to 0} \left(\frac{1}{2\log[(1+2x)^{\frac{1}{2x}}]}\right)$

$= (\log 5)^2 \times \left(\frac{1}{2\log e}\right)$

$\therefore f(0) = \frac{(\log 5)^2}{2} \quad \because \log e = 1$

Hence Proved.

==

Ex. Show that the function f (x) = 2x - $|x|$ is continuous at x = 0. (Oct.2018)

Proof. Now $|x| = \begin{cases} x & for\ x > 0 \\ -x & for\ x < 0 \end{cases}$ gives

f (x) = 2x - $|x|$ = x for x > 0

= 0 for x = 0

= 3x for x< 0

Here f (0) = 0 ……..(1)

Now we find left hand & right hand limits of a given function as follows

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(3x) = 0$ &

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(x) = 0$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$

$\therefore \lim_{x\to 0} f(x)$ is exist and $\lim_{x\to 0} f(x) = 0 = f(0)$ by (1)

Hence f (x) is continuous at x = 0 is proved.

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Properties of continuous functions:

i) If f(x) and g(x) are two continuous functions then f (x) + g(x), f (x) - g(x) and f (x).g(x) and $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ are continuous.

ii) A polynomial function is always continuous at every point in its domain.

iii) A rational function is continuous at every point in its domain.

Theorem: If f (x) is continuous in [a, b], then interval [a, b] is divided into a finite number of subintervals such that, given $\epsilon > 0$, $| f(x_1) - f(x_2) | < \epsilon$ where x_1, x_2 are any points in the same subinterval

Supremum of a function: Least among all upper bounds of a function is called supremum of a function or least upper bound (l.u.b.). Denoted by Sup.f (x).

Infimum of a function: Greatest among all lower bounds of a function is called infimum of a function or greatest lower bound (g.l.b.). Denoted by Inf.f (x).

Note: Every bounded function has l.u.b. & g.l.b.i.e. every bounded function has supremum & infimum.

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Property-1: Every continuous function on closed and bounded interval is bounded.

Proof: Let f (x) be continuous function on closed and bounded interval [a, b] .....(1)

∴ interval [a, b] is divided into a finite number of subintervals

$[a, a_1], [a_1, a_2], [a_2, a_3], \ldots\ldots[a_{n-1}, b]$, such that, given $\epsilon > 0$,

$| f(x_1) - f(x_2) | < \epsilon$ …(1)

where x_1, x_2 are any points in the same subinterval.

For $x \in [a, a_1]$ we have, $|f(x) - f(a)| < \epsilon$

$|f(x)| = |f(x) - f(a) + f(a)|$

$\leq |f(x) - f(a)| + |f(a)|$

$\therefore |f(x)| \leq \epsilon + |f(a)|$ by (1)

$\therefore |f(a_1)| \leq \epsilon + |f(a)|$ …….(2) $\because a_1 \in [a, a_1]$

For $x \in [a_1, a_2]$ we have, $|f(x) - f(a_1)| < \epsilon$

$|f(x)| = |f(x) - f(a_1) + f(a_1)|$

$\leq |f(x) - f(a_1)| + |f(a_1)|$

$\therefore |f(x)| \leq \epsilon + |f(a_1)|$ by (1)

$\therefore |f(x)| \leq \epsilon + \epsilon + |f(a)|$ by (2)

$\therefore |f(x)| \leq 2\epsilon + |f(a)|$

Similarly $|f(x)| \leq 3\epsilon + |f(a)|$ for x ∈ [a_2, a_3] and continuing in this way for x ∈ [a_{n-1}, b] , we get, $|f(x)| \leq n\epsilon + |f(a)|$

i.e. f (a) – $n\epsilon \leq |f(x)| \leq$ f (a) + $n\epsilon$

i.e. f (x) is bounded on [a, b]. Hence proved.

==

Property-2: Every continuous function on closed and bounded interval attains its bounds.

Proof: Let f(x) be continuous function on [a , b].

∴ f (x) is bounded.

Supremum and infinum of f (x) exist say sup.f (x) = M and inf.f (x) = m.

i.e. $m \leq |f(x)| \leq$ M …….(1)

To prove f(x) attains its bounds, we have to prove there exists points c and d in [a, b] Such that f (c) = M and f (d) = m.

Now if possible assume that f(x) ≠ M = sup. f(x) ∀ x ∈ [a, b]

⟹ M - f(x) ≠ 0 ∀ x ∈ [a, b] ……..(2)

Consider $\emptyset(x) = \frac{1}{M - f(x)}$ ∀ x ∈ [a, b]

Here clearly ∅(x) is continuous on [a, b]

∴ sup. ∅(x) exist in [a, b] and is say k. and k > 0

∴ sup. ∅(x) = k ∀ x ∈ [a, b]

∴ ∅(x) ≤ k ∀ x ∈ [a, b]

$\therefore \frac{1}{M - f(x)} \leq k$ ∀ x ∈ [a, b]

$\therefore \frac{1}{K} \leq M - f(x)$ ∀ x ∈ [a, b]

$\therefore f(x) \leq M - \frac{1}{K}$ ∀ x ∈ [a, b]

Which is contradiction that M = sup.f{x)

∴ our assumption is wrong

∴ ∃ c ∈ [a, b] such that M = f (c)

Similarly we can show that ∃ d ∈ [a, b] such that f (d) = m.

Hence f (x) attains its bounds is proved.

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Bolzano's Theorem: (Intermediate Value Theorem): If f(x) is continuous on closed and bounded interval [a, b] and f (a) and f (b) have opposite signs then f (x) = 0 for some x ∈ [a, b]

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Property-3: If f(x) is continuous on [a, b] and f (a) ≠ f (b) then f (x) assumes every value between f (a) and f (b).

Proof: Let f (x) be continuous function on [a, b] and f (a) ≠ f (b) …..(1)

To prove f (x) assumes every value between f (a) and f (b)

Let λ be any value between f (a) and f (b).

Define g(x) = f (x) – λ   ∀ x ∈ [a, b]

By (1), g(x) is continuous function on [a, b] and g(a) = f (a) – λ & g(b) = f (b) – λ have an opposite sighs.

By Bolzano's theorem

g(x) = 0 for some x ∈ [a, b].

i.e. f (x) – λ = 0 for some x ∈ [a, b].

∴ f (x) = λ for some x ∈ [a, b]

Hence f (x) assumes every value between f (a) and f (b) is proved.

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Property-4: If f (x) is continuous in [a, b] then f(x) assumes every value between inf.f (x) = m and sup. f (x) = M

Proof: Let f (x) be continuous on [a, b].

∴ f (x) is bounded and attends its bounds in [a, b]

i. e. M= sup f(x) = f (c) and m = inf f (x) = f (d) for some c and d ∈ [a, b]

As $m \le |f(x)| \le$ M and m ≠ M i.e. f (c) ≠ f (d) with [c, d] ⊂ [a, b]

∴ f(x) assume every value between f (c) & f (d) is proved.

==

Uniform Continuity:

A function f (x) defined on an interval I is said to be uniformly continuous on I if given $\epsilon > 0$, $\exists\ \delta > 0$ depending on ϵ such that $|f(x_1) - f(x_2)| < \epsilon$ when ever $|x_1 - x_2| < \delta$. where $x_1, x_2 \in I$.

Remark: Uniform continuity of a function is a global property and continuity is a local property.

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UNIT-1 LIMITS AND CONTINUITY [MCQ'S]

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1) If f(x) and g(x) are two real valued functions such that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and f '(a) and g'(a) exist where g'(a) ≠ 0 then $\lim_{x\to a} \frac{f(x)}{g(x)} =$ ……..

a) $\frac{f'(a)}{g'(a)}$ b) $\frac{f(a)}{g(a)}$ c) $\frac{g'(a)}{f'(a)}$ d) $\frac{f''(a)}{g''(a)}$

2) If f(x) and g(x) are two real valued functions such that $\lim_{x \to a} f(x) = \lim_{x \to a} f'(x) = \lim_{x \to a} f''(x) = \ldots \lim_{x \to a} f^{(n-1)}(x) = 0$ and $\lim_{x \to a} g(x) = \lim_{x \to a} g'(x) = \lim_{x \to a} g''(x) = \ldots \lim_{x \to a} g^{(n-1)}(x) = 0$ and $f^{(n)}(a)$ and $g^{(n)}(a)$ exist where $g^{(n)}(a) \neq 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)}$ = …………..

a) $\frac{g'(a)}{f'(a)}$ b) $\frac{f^{(n)}(a)}{g^{(n)}(a)}$ c) $\frac{f'(a)}{g'(a)}$ d) $\frac{f''(a)}{g''(a)}$

3) $\lim_{x \to 1} \frac{\log x}{x-1}$ is ………..

a) 0   b) 1   c) 2   d) 3

4) $\lim_{x \to 5} \frac{x^2-4x-5}{x^2+2x-35}$ is ………..

a) 0   b) 5   c) $\frac{1}{2}$ d) 2

5) $\lim_{x \to 7} \frac{x^2-10x+21}{x^2-12x+35}$ is ………..

a) 7   b) 2   c) 0   d) 1

6) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$ is ………..

a) 0   b) 2   c) 0   d) 1

7) $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^2 \sin x}$ is ………..

a) 0   b) $\frac{1}{3}$ c) 1   d) 2

8) $\lim_{x \to \infty} \frac{\log x}{x}$ is ………..

a) 0   b) 1   c) -1   d) 2

9) $\lim_{x \to \pi} \frac{\log(\pi - x)}{\cot x}$ is ………..

a) 0   b) π c) $-\pi$ d) 1

10) $\lim_{x \to 0} \frac{\operatorname{cosec} x}{\log x}$ is ……….

a) 0   b) ∞   c) - ∞   d) 1

11) $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \frac{\pi}{2})}{\tan x}$ is ………..

a) 0   b) $\frac{\pi}{2}$ c) $-\frac{\pi}{2}$ d) 2

12) $\lim_{x \to 0} \frac{\log x}{\cot x}$ is ………..

a) 0   b) ∞   c) $\frac{\pi}{2}$ d) 1

13) $\lim_{x \to 0} x \log x$ is ……

a) 0   b) 1   c) ∞   d) 2

14) $\lim_{x \to 0} (1-\cos x)(\cot x)$ is ………..

a) 0   b) 1   c) -1   d) 3

15) $\lim_{x\to 0}$ sinx logx is ……….

a) -1 b) 0 c) 1 d) 2

16) $\lim_{x\to 0}$ tanx logx is ……….

a) 1 b) -1 c) 0 d) 2

17) $\lim_{x\to \frac{\pi}{2}}$ (x - $\frac{\pi}{2}$) secx is ……….

a) 1 b) -1 c) 0 d) 2

18) $\lim_{x\to 0}$ (cosecx - cotx) is ……….

a) 1 b) -1 c) 0 d) 2

19) $\lim_{x\to \frac{\pi}{2}}$ (secx - tanx) is ……….

a) 0 b) 1 c) -1 d) -2

20) $\lim_{x\to 0}$ ($\frac{1}{x} -$ cotx) is ……….

a) 1 b) 0 c) -1 d) -3

21) $\lim_{x\to 0}$ (cosecx - $\frac{1}{x}$) is ……….

a) 1 b) -1 c) 0 d) 2

22) $\lim_{x\to 0} x^x$ is ……….

a) 1 b) -1 c) 0 d) -2

23) $\lim_{x\to a} (x-a)^{(x-a)}$ is ……….

a) 1 b) a c) 0 d) -a

24) $\lim_{x\to \frac{\pi}{2}} (sinx)^{tanx}$ is ……….

a) 0 b) $\frac{\pi}{2}$ c) 1 d) 2

25) $\lim_{x\to 0} (cosx)^{cotx}$ is ……….

a) 0 b) 1 c) 2 d) 3

26) $\lim_{x\to \infty} (1+x)^{\frac{1}{x}}$ is ……….

a) 0 b) ∞ c) 1 d) 4

27) A function f (x) is said to be continuous at a point x = a if ………

a) $\lim_{x\to a}$ f (x) = f(a) b) $\lim_{x\to a}$ f (x) ≠ f '(a) c) None of these

28) Limit of the function at a point x = a exists if …….

a) $\lim_{x\to a^-}$ f (x) = $\lim_{x\to a^+}$ f (x) b) $\lim_{x\to a^-}$ f (x) ≠ $\lim_{x\to a^+}$ f (x) c) None of these

29) Every continuous function on closed and bounded interval is…….

a) bounded b) not bounded c) None of these

30) Every continuous function on closed and bounded interval………

a) attains its bounds b) not attains its bounds c) None of these

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UNIT-2: MEAN VALUE THEOREMS

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Derivative of a function:

A function f (x) is said to be derivable at x if $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Derivative of a function at point x = a:

A function f (x) is said to be derivable at point x = a if

$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ or $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.

Left Hand Derivative of a function:

If $f_-'(a) = \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$ or $\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$ exists,

then is called left hand derivative of f (x) at x = a.

Right Hand Derivative of a function:

If $f_+'(a) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$ or $\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$ exists,

then is called right hand derivative of f (x) at x = a.

Remark: A function f (x) is said to be derivable at point x = a iff $f_-'(a) = f_+'(a)$

==

Theorem: Every differentiable function is continuous. (Oct. 2018)

Proof. Let f (x) is any function differentiable at point x = a.

$\therefore\ f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ ……. (1) is exists.

Consider $\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \times (x-a)$

$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \times \lim_{x \to a}(x-a)$

$= f'(a) \times 0$

$\therefore\ \lim_{x \to a} f(x) - f(a) = 0$

$\therefore\ \lim_{x \to a} f(x) = f(a)$

i.e. f (x) is continuous at point x = a.

Hence every differentiable function is continuous is proved.

==

Remark: Every continuous function may not be differentiable.

==

Ex.: Show that the function f (x) = |x| is continuous but not be differentiable at x = 0.

Proof. Let f (x) = |x|

$\therefore f(0) = |0| = 0$

and $\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = |0| = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

i. e. f (x) is continuous at x = 0.

Now $f_-'(0) = \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h \to 0^-} \frac{|h|-|0|}{h}$

$= \lim_{h \to 0} \frac{-h-0}{h} \because |h| = -h$ for $h < 0$

$= -1$

& $f_+'(0) = \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h \to 0^+} \frac{|h|-|0|}{h}$

$= \lim_{h \to 0} \frac{h-0}{h} \because |h| = h$ for $h > 0$

$= 1$

Here $f_-'(0) \neq f_+'(0)$

$\therefore f'(0)$ does not exists.

Hence f (x) is continuous but not differentiable at x = 0 is proved.

==

Ex.: Show that the function f (x) = |x − a| is continuous at x = a but not be differentiable at x = a.

Proof. Let f (x) = |x − a|

$\therefore$ f (a) = |0| = 0

and $\lim_{x \to a} f(x) = \lim_{x \to a} |x-a| = |0| = 0$

$\therefore \lim_{x \to a} f(x) = f(a)$

i. e. f (x) is continuous at x = a.

Now $f_-'(a) = \lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$

$= \lim_{h \to 0^-} \frac{|h|-|0|}{h}$

$= \lim_{h \to 0} \frac{-h-0}{h} \because |h| = -h$ for $h < 0$

$= -1$

& $f_+'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

$= \lim_{h \to 0^+} \frac{|h|-|0|}{h}$

$= \lim_{h \to 0} \frac{h-0}{h} \because |h| = h$ for $h > 0$

$= 1$

Here $f_-'(a) \neq f_+'(a)$

$\therefore f'(a)$ does not exists.

Hence f (x) is continuous at x = a but not derivable at x = a is proved.

==

Ex.: Show that the function f (x) = |x − 1| is continuous at x = 1 but not be differentiable at x = 1.

Proof. Let f (x) = |x − 1|

$\therefore$ f (1) = |0| = 0

and $\lim_{x \to 1} f(x) = \lim_{x \to 1} |x - 1| = |0| = 0$

$\therefore \lim_{x \to 1} f(x) = f(1)$

i. e. f (x) is continuous at x = 1.

Now $f_-'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0^-} \frac{|h| - |0|}{h}$

$= \lim_{h \to 0} \frac{-h-0}{h} \because |h| = -h \text{ for } h < 0$

$= -1$

$\& \ f_+'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0^+} \frac{|h| - |0|}{h}$

$= \lim_{h \to 0} \frac{h-0}{h} \because |h| = h \text{ for } h > 0$

$= 1$

Here $f_-'(1) \neq f_+'(1)$

$\therefore f'(1)$ does not exists.

Hence f (x) is continuous at x = 1 but not derivable at x = 1 is proved.

==

Ex.: Show that the function $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and f (0) = 0 is continuous but not be derivable at x = 0.

Proof. Let $f(x) = x \sin\frac{1}{x}$ for $x \neq 0$ and f (0) = 0

As $-1 \leq \sin\frac{1}{x} \leq 1$

$\therefore -|x| \leq x \sin\frac{1}{x} \leq |x|$

$\lim_{x \to 0} -|x| \leq \lim_{x \to 0} x \sin\frac{1}{x} \leq \lim_{x \to 0} |x|$

$\therefore 0 \leq \lim_{x \to 0} x \sin\frac{1}{x} \leq 0$

$\therefore \lim_{x \to 0} f(x) = 0 = f(0)$

i. e. f (x) is continuous at x = 0.

Now $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{h \sin\frac{1}{h}}{h}$

$= \lim_{h \to 0} \sin\frac{1}{h}$

$\therefore f'(0)$ does not exists.

Hence f (x) is continuous but not differentiable at x = 0 is proved.

==

Rolle's Theorem: If a function f (x) defined on [a, b] is
i) continuous in [a, b], ii) derivable in (a, b) and iii) f (a) = f (b).
Then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Proof: Let f (x) is continuous in [a, b].
∴ f (x) is bounded and it attains its bounds in [a, b]
i.e. there exists some $c, d \in [a, b]$ with sup.f = f (c) = M & inf.f = f (d) = m.
Case i) If M = m, then f (x) is constant $\forall x \in [a, b]$.
∴ $f'(x) = 0, \forall x \in [a, b]$
In particular, $f'(c) = 0$ for some $c \in (a, b)$ is proved.
Case ii) If $M \neq m$, then one of M or m is differ from f (a) and hence from f (b).
Suppose $M \neq f(a)$, $M \neq f(b)$ i. e. $f(c) \neq f(a)$, $f(c) \neq f(b)$ i.e. $a \neq c$ & $b \neq c$
∴ $a < c < b$ i.e. $c \in (a, b)$
As f (x) is derivable in (a, b) and hence derivable at x = c.
∴ $f_-'(c) = f_+'(c) = f'(c)$ ……..(1)
Now $f(x) \leq M, \forall x \in [a, b] \Longrightarrow f(c+h) \leq f(c)$
∴ $f(c+h) - f(c) \leq 0$
∴ $\frac{f(c+h)-f(c)}{h} \geq 0$ if h < 0 & $\frac{f(c+h)-f(c)}{h} \leq 0$ if h > 0
∴ $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$ & $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$
i.e. $f_-'(c) \geq 0$ & $f_+'(c) \leq 0$
∴ $f'(c) = 0$ by (1)
Hence $f'(c) = 0$ for some $c \in (a, b)$ is proved.

==

Alternative Form of Rolle's Theorem:
If a function f (x) defined on [a, a+h] is
i) continuous in [a, a+h], ii) derivable in (a, a+h) and iii) f (a) = f (a+h).
Then there exists some real number $\theta \in (0, 1)$ such that $f'(a+\theta h) = 0$.

==

Ex. Verify Rolle's theorem for the function $f(x) = x^2-1$ in [-1, 1].

Proof. Let $f(x) = x^2-1$
∴ $f'(x) = 2x$
Which exists for very value of x in (-1, 1)
i.e. f (x) is derivable in (-1, 1).
As every derivable function is continuous ⟹ f (x) is continuous in [-1, 1].
Also f (-1) = 0 = f (1).
i.e. f (x) satisfies all conditions of Rolle's theorem.
∴ Rolle's Theorem is applicable.
i.e. there exist some $c \in (-1, 1)$ such that $f'(c) = 0$. i.e. 2c = 0.
∴ $c = 0 \in (-1, 1)$.
Hence Rolle's Theorem is verified.

==

Ex. Verify Rolle's theorem for the function f (x) = x^2-6x+5 in [1, 5].

Proof. Let f (x) = x^2-6x+5 ∴ f '(x) = 2x-6

Which exists for very value of x in (1, 5)

i.e. f (x) is derivable in (1, 5).

As every derivable function is continuous ⟹ f (x) is continuous in [1, 5].

Also f (1) = 0 = f (5).

i.e. f (x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (1, 5) such that f '(c) = 0.

i.e. 2c -6 = 0.

∴ c = 3 ∈ (1, 5).

Hence Rolle's theorem is verified.

==

Ex. Verify Rolle's theorem for the function f (x) = x^2+2x-8 in [-4, 2].

Proof. Let f (x) = x^2+2x-8

∴ f '(x) = 2x+2

Which exists for very value of x in (-4, 2)

i.e. f (x) is derivable in (-4, 2).

As every derivable function is continuous ⟹ f (x) is continuous in [-4, 2].

Also f (-4) = 0 = f (2).

i.e. f (x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (-4, 2) such that f '(c) = 0.

i.e. 2c +2 = 0.

∴ c = -1 ∈ (-4, 2).

Hence Rolle's theorem is verified.

==

Ex. Verify Rolle's theorem for the function f (x) = x (x+3) $e^{\frac{-x}{2}}$ in [-3, 0].

Proof. Let f (x) = x (x+3) $e^{\frac{-x}{2}}$ = $e^{\frac{-x}{2}}$(x^2+3x)

$$\therefore f'(x) = \frac{-1}{2}e^{\frac{-x}{2}}(x^2+3x) + e^{\frac{-x}{2}}(2x+3)$$

$$= \frac{-1}{2}e^{\frac{-x}{2}}(x^2+3x-4x-6)$$

$$= \frac{-1}{2}e^{\frac{-x}{2}}(x^2-x-6)$$

Which exists for very value of x in (-3, 0)

i.e. f (x) is derivable in (-3, 0).

As every derivable function is continuous ⟹ f (x) is continuous in [-3, 0].

Also f (-3) = 0 = f (0).

i.e. f (x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (-3, 0) such that f '(c) = 0.

i.e. $\frac{-1}{2} e^{\frac{-c}{2}}(c^2-c-6) = 0$.

∴ $c^2-c-6 = 0$. ∵ $\frac{-1}{2} e^{\frac{-c}{2}} \neq 0$.

∴ (c-3) (c+2) = 0.

∴ c = 3 or c = -2.

Here c = -2 ∈ (-3, 0).

Hence Rolle's theorem is verified.

===

Ex. Verify Rolle's theorem for the function f (x) = $(x-a)^m(x-b)^n$ in [a, b], m, n ∈ I+

Proof. Let f (x) = $(x-a)^m(x-b)^n$

∴ $f'(x) = m\,(x-a)^{m-1}(x-b)^n + n\,(x-a)^m(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}(mx-mb+nx-na)$

$= (x-a)^{m-1}(x-b)^{n-1}\,[(m+n)x-(mb+na)]$

Which exists for very value of x in (a, b)

i.e. f (x) is derivable in (a, b).

As every derivable function is continuous ⟹ f (x) is continuous in [a, b].

Also f (a) = 0 = f (b).

i.e. f (x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (a, b) such that f '(c) = 0.

i.e.$(c-a)^{m-1}(c-b)^{n-1}\,[(m+n)c-(mb+na)] = 0$.

∴ (c-a) = 0 or (c-b) = 0 or [(m+n)c-(mb+na)] = 0

∴ c = a or c = b or c = $\frac{mb+na}{m+n}$

Here c = $\frac{mb+na}{m+n}$ ∈ (a, b).

Hence Rolle's theorem is verified.

===

Ex. Verify Rolle's theorem for the function f (x) = $(x-3)^2(x-5)^3$ in [3, 5]

Proof. Let f (x) = $(x-3)^2(x-5)^3$

∴ $f'(x) = 2\,(x-3)(x-5)^3 + 3\,(x-3)^2(x-5)^2$

$= (x-3)(x-5)^2(2x-10+3x-9)$

$= (x-3)(x-5)^2(5x-19)$

Which exists for very value of x in (3, 5)

i.e. f (x) is derivable in (3, 5).

As every derivable function is continuous ⟹ f (x) is continuous in [3, 5].

Also f (3) = 0 = f (5).

i.e. f (x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (3, 5) such that f '(c) = 0.

i.e. $(c-3)(c-5)^2(5c-19) = 0$.

∴ (c-3) = 0 or (c-5) = 0 or (5c-19) = 0

∴ c = 3 or c = 5 or c = $\frac{19}{5}$

Here c = $\frac{19}{5}$ ∈ (3, 5).

Hence Rolle's theorem is verified.

==

Ex. Verify Rolle's theorem for the function f (x) = log $[\frac{x^2+ab}{x(a+b)}]$ in [a, b], 0 ∉ [a, b].(Oct.2018)

Proof. Let f (x) = log $[\frac{x^2+ab}{x(a+b)}] = \log[x^2+ab] - \log x - \log(a+b)$

$\therefore f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} - 0 = \frac{2x^2 - x^2 - ab}{x(x^2+ab)} = \frac{x^2-ab}{x(x^2+ab)}$

Which exists for very value of x in (a, b)

i.e. f (x) is derivable in (a, b).

As every derivable function is continuous ⟹ f (x) is continuous in [a, b].

Also f (a) = 0 = f (b).

i.e. f (x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (a, b) such that f '(c) = 0.

i.e. $\frac{c^2-ab}{c(c^2+ab)} = 0$.

$\therefore c^2 - ab = 0$

$\therefore c^2 = ab$

$\therefore c = \pm\sqrt{ab}$

Here c = $\sqrt{ab}$ ∈ (a, b).

Hence Rolle's theorem is verified.

==

Lagrange's Mean Value Theorem: If a function f (x) defined on [a, b] is

i) continuous in [a, b] and ii) derivable in (a, b).

Then there exists some c ∈ (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b-a}$.

Proof: Let us define F(x) = f (x) + Ax……..(1)

Where A is constant such that F(a) = F(b).

Now F(a) = F(b) gives

f (a) + Aa = f (b) + Ab

i.e. –Ab + Aa = f (b) - f (a)

i.e. –A(b – a) = f (b) - f (a)

i.e. $-A = \frac{f\,(b) - f\,(a)}{b-a}$ ……….(2)

As A is constant and f (x) is continuous in [a, b] & derivable in (a, b).

∴ F(x) is i) continuous in [a, b],

ii) derivable in (a, b) with F'(x) = f '(x) + A

and iii) F(a) = F(b)

i.e. F(x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (a, b) such that F'(c) = 0.

i.e. f '(c) + A = 0

i.e. f '(c) = - A

i.e. $f'(c) = \frac{f\,(b) - f\,(a)}{b-a}$ by (2)

Hence proved.

==

Alternative Form of Lagrange's M.V.T.: If a function f (x) defined on [a, a+h] is i) continuous in [a, a+h] and ii) derivable in (a, a+h).

Then there exists some $\theta \in (0, 1)$ such that $f'(a+\theta h) = \frac{f\,(a+h) - f\,(a)}{h}$.

==

Ex.: Discuss the applicability of Mean Value Theorem for the function f (x) = 1- x^2 in [1, 2].

Sol.: Let f (x) = 1-x^2

∴ f '(x) = -2x

Which exists for very value of x in (1, 2)

i.e. f (x) is derivable in (1, 2).

As every derivable function is continuous ⟹ f (x) is continuous in [1, 2].

i.e. f (x) satisfies both conditions of Lagrange's M.V.T.

∴ Lagrange's M.V.T. is applicable.

i.e. there exist some c ∈ (1, 2) such that $f'(c) = \frac{f\,(2) - f\,(1)}{2-1}$.

i.e. $-2c = \frac{(1-4)-(1-1)}{1} = \frac{-3-0}{1} = -3$

∴ 2c = 3

∴ $c = \frac{3}{2} \in (1, 2)$.

==

Ex. Verify Lagrange's theorem for the function f (x) = x(x-1)(x-2) in $[0, \frac{1}{2}]$.

Proof. Let f (x) = x(x-1)(x-2) = x^3-3x^2+2x

∴ f '(x) = $3x^2-6x+2$

Which exists for very value of x in $[0, \frac{1}{2}]$

i.e. f (x) is derivable in $(0, \frac{1}{2})$.

As every derivable function is continuous $\Longrightarrow$ f (x) is continuous in $[0, \frac{1}{2}]$.

i.e. f (x) satisfies both conditions of Lagrange's M.V.T.

∴ Lagrange's M.V.T. is applicable.

i.e. there exist some $c \in (0, \frac{1}{2})$ such that $f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$.

i.e. $3c^2-6c+2 = 2[\frac{1}{2} \times \left(-\frac{1}{2}\right) \times (-\frac{3}{2})-0]$

∴ $3c^2-6c+2 = \frac{3}{4}$

∴ $12c^2-24c+8 = 3$

∴ $12c^2-24c+5 = 0$

∴ $c = \frac{24 \pm \sqrt{576-240}}{24} = \frac{24 \pm \sqrt{336}}{24}$

∴ $c = \frac{24 \pm 4\sqrt{21}}{24} = 1 \pm \frac{\sqrt{21}}{6}$

Here $c = 1 - \frac{\sqrt{21}}{6} \in (0, \frac{1}{2})$.

Hence Lagrange's M.V.T. is verified.

==

Ex. Verify Mean Value Theorem for the function $f(x) = x^2-4x-3$ in [a, b]. Where a = 1 and b = 4.

Proof. Let $f(x) = x^2-4x-3$

∴ $f'(x) = 2x-4$

Which exists for very value of x in [1, 4]

i.e. f (x) is derivable in (1, 4).

As every derivable function is continuous $\Longrightarrow$ f (x) is continuous in [1, 4].

i.e. f (x) satisfies both conditions of Lagrange's M.V.T.

∴ Lagrange's M.V.T. is applicable.

i.e. there exist some $c \in (1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{4-1}$.

i.e. $2c-4 = \frac{(16-16-3)-(1-4-3)}{3} = \frac{-3+6}{3} = 1$

∴ $2c = 5$

∴ $c = \frac{5}{2}$

Here $c = \frac{5}{2} \in (1, 4)$.

Hence M.V.T. is verified.

==

Ex. Verify Lagrange's theorem for the function f (x) = x^3-5x^2-3x in [a, b].
Where a = 1 and b = 3. Find all c ∈ (1, 3)

Proof. Let f (x) = x^3-5x^2-3x

∴ f '(x) = 3x^2-10x-3

Which exists for very value of x in (1, 3)

i.e. f (x) is derivable in (1, 3).

As every derivable function is continuous ⟹ f (x) is continuous in [1, 3].

i.e. f (x) satisfies both conditions of Lagrange's M.V.T.

∴ Lagrange's M.V.T. is applicable.

i.e. there exist some c ∈ (1, 3) such that f '(c) = $\frac{f(3)-f(1)}{3-1}$.

i.e.3c^2-10c-3 = $\frac{(27-45-9)-(1-5-3)}{2} = \frac{-27+7}{2}$ = -10

∴ 3c^2-10c+7 = 0

∴ c = $\frac{10 \pm\sqrt{100-84}}{6}$

∴ c = $\frac{10 \pm 4}{6} = \frac{7}{3}$ or 1

Here c = $\frac{7}{3}$ ∈ (1, 3).

Hence Lagrange's M.V.T. is verified.

===

Ex. Use Lagrange's Mean Value Theorem to show that $\left|\frac{\cos a\theta-\cos b\theta}{\theta}\right| \leq$ (b-a) if $\theta \neq 0$.

Proof. Let us define f (x) = cos$x\theta$

∴ f '(x) = - $\theta \sin x\theta$

Which exists for very value of x in [a, b]

i.e. f (x) is derivable in (a, b).

As every derivable function is continuous ⟹ f (x) is continuous in [a, b].

i.e. f (x) satisfies both conditions of Lagrange's M.V.T.

∴ Lagrange's M.V.T. is applicable.

i.e. there exist some c ∈ (a, b) such that f '(c) = $\frac{f(b)-f(a)}{b-a}$.

i.e. - $\theta \sin c\theta = \frac{\cos b\theta-\cos a\theta}{b-a}$

∴ $\left|\frac{\cos a\theta-\cos b\theta}{b-a}\right| = |-\theta \sin c\theta| \leq |\theta|$ ∵ $|\sin c\theta| \leq 1$ & $\theta \neq 0$

∴ $\left|\frac{\cos a\theta-\cos b\theta}{\theta}\right| \leq |b-a|$

∴ $\left|\frac{\cos a\theta-\cos b\theta}{\theta}\right| \leq (b-a)$ ∵ b≥ a.

Hence proved.

===

Ex. For 0 < a < b, prove that $1-\frac{a}{b} < \log\frac{b}{a} < \frac{b}{a}-1$ and deduce that $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$ (Oct.2018)

Proof. Let us define f (x) = log x in [a, b]

$\therefore f'(x) = \frac{1}{x}$

By Lagrange's M.V.T.

$f'(c) = \frac{f(b)-f(a)}{b-a}$ where c ∈ (a, b)

i.e. $\frac{1}{c} = \frac{\log b - \log a}{b-a} = \frac{\log\frac{b}{a}}{b-a}$……..(1)

As c ∈ (a, b) ⟹ a < c < b

$\Rightarrow \frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

$\therefore \frac{1}{b} < \frac{\log\frac{b}{a}}{b-a} < \frac{1}{a}$ by (1)

$\therefore \frac{b-a}{b} < \log\frac{b}{a} < \frac{b-a}{a}$ ………(2) ∵ 0 < a < b

i.e. $1-\frac{a}{b} < \log\frac{b}{a} < \frac{b}{a}-1$

Hence proved.

By putting a = 5 and b = 6 in (2), we get

$\frac{6-5}{6} < \log\frac{6}{5} < \frac{6-5}{5}$

i.e. $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$

Hence proved.

==

Ex. Show that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ if 0 < a < b

and deduce that $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}(\frac{4}{3}) < \frac{\pi}{4}+\frac{1}{6}$

Proof. Let us define f (x) = $\tan^{-1}x$

$\therefore f'(x) = \frac{1}{1+x^2}$

Which exists for very value of x in (a, b) if 0 < a < b

i.e. f (x) is derivable in (a, b).

As every derivable function is continuous ⟹ f (x) is continuous in [a, b].

i.e. f (x) satisfies both conditions of Lagrange's M.V.T.

∴ Lagrange's M.V.T. is applicable.

i.e. there exist some c ∈ (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

i.e. $\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$ ……..(1)

As c ∈ (a, b) ⟹ a < c < b

⟹ $a^2 < c^2 < b^2$

$\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$

$\Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$

$\therefore \frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$ by (1)

$\therefore \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ $\because 0 < a < b$

Hence proved.

By putting a = 1 and b = $\frac{4}{3}$, we get

$\frac{\frac{4}{3}-1}{1+(\frac{4}{3})^2} < \tan^{-1}(\frac{4}{3}) - \tan^{-1}1 < \frac{\frac{4}{3}-1}{1+1^2}$

$\therefore \frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{\frac{1}{3}}{2}$

$\therefore \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}$

Hence proved.

==

Ex. Show that $\frac{x}{1+x^2} < \tan^{-1}x < x$ if $x > 0$ (Mar.2019)

Proof. Let us define f (x) = $\tan^{-1}x$ in [0, x]

$\therefore f'(x) = \frac{1}{1+x^2}$

By Lagrange's M.V.T.

i.e. there exist some $\theta \in (0, 1)$ such that $f'(0+\theta x) = \frac{f(x) - f(0)}{x-0}$.

i.e. $\frac{1}{1+(\theta x)^2} = \frac{\tan^{-1}x - \tan^{-1}0}{x-0} = \frac{\tan^{-1}x}{x}$……..(1)

As $\theta \in (0, 1) \Rightarrow 0 < \theta < 1$

$\Rightarrow 0 < \theta x < x$

$\Rightarrow 0 < (\theta x)^2 < x^2$

$\Rightarrow 1 < 1+(\theta x)^2 < 1+x^2$

$\Rightarrow \frac{1}{1+x^2} < \frac{1}{1+(\theta x)^2} < 1$

$\therefore \frac{1}{1+x^2} < \frac{\tan^{-1}x}{x} < 1$ by (1)

$\therefore \frac{x}{1+x^2} < \tan^{-1}x < x$ $\because x > 0$

Hence proved.

==

Monotonic Increasing Function: A function f (x) is said to be monotonic increasing function if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Monotonic decreasing Function: A function f (x) is said to be monotonic decreasing function if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

==

Ex. Show that the function f (x) = 7x – 3 is strictly increasing function on R.

Proof. For $x_1, x_2 \in R$, suppose $x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$

Hence given function is strictly increasing function on R is proved.

==

Ex. Prove that f (x) = ax + b, where a, b are constant and a > 0 is strictly increasing function for all real values of x without using the derivative.

Proof. For $x_1, x_2 \in R$, suppose $x_1 < x_2 \Rightarrow ax_1 < ax_2 \because a > 0$

$\Rightarrow ax_1 + b < ax_2 + b$

$\Rightarrow f(x_1) < f(x_2)$

Hence given function is strictly increasing function on R is proved.

==

Ex. Show that $f(x) = x^2$ is strictly decreasing function in $(-\infty, 0)$.

Proof. For $x_1, x_2 \in (-\infty, 0)$, suppose $x_1 < x_2 < 0 \Rightarrow x_1^2 > x_2^2 > 0 \Rightarrow f(x_1) > f(x_2)$

Hence given function is strictly decreasing function on R is proved.

==

Remark: i) A function f (x) is monotonic increasing function if $f'(x) > 0\ \forall\ x \in (a, b)$

ii) A function f (x) is monotonic decreasing function if $f'(x) < 0\ \forall\ x \in (a, b)$

==

Ex. Find the least value of a, such that the function $f(x) = x^2+ax+1$ is strictly increasing function on (1, 2).

Sol. Let $f(x) = x^2+ax+1$

$\therefore f'(x) = 2x + a$

For $x \in (1, 2) \Rightarrow 1 < x < 2 \Rightarrow 2 < 2x < 4 \Rightarrow 2+a < 2x + a < 4 + a$

$\Rightarrow 2+a < f'(x) < 4 + a$

Function f (x) is strictly increasing if $f'(x) > 0$ i.e. if $2 + a > 0$ i.e. if $a > -2$.

Hence least value of a is -2.

==

Ex. Show that $\frac{x}{1+x} < \log(1+x) < x$

Proof. Let us define $f(x) = \log(1+x) - \frac{x}{1+x}$ and $g(x) = x - \log(1+x)$

Now $f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{(1+x)-1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0\ \forall\ x > 0$

$\therefore$ f (x) is increasing function $\forall\ x > 0$.

i.e. $f(x) > f(0)$

i.e. $\log(1+x) - \frac{x}{1+x} > 0$

i.e. $\frac{x}{1+x} < \log(1+x)$ ……..(1)

& $g'(x) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x} > 0\ \forall\ x > 0$

$\therefore$ g(x) is increasing function $\forall\ x > 0$

i.e. g(x) >g (0)
i.e. $x - \log(1+x) > 0$
i.e. $\log(1+x) <$ x ……..(2)
From (1) & (2) $\frac{x}{1+x} < \log(1+x) < x$ is proved.

==

Ex. Show that $1 - x < -\log x < \frac{1}{x}$ - 1, 0 < x < 1.

Proof. Let us define $f(x) = -\log x - 1 + x$ and $g(x) = \frac{1}{x} - 1 + \log x$

Now $f'(x) = \frac{-1}{x} + 1 = \frac{-1+x}{x} = \frac{x-1}{x} < 0$ for $0 < x < 1$
∴ f (x) is decreasing function for 0 < x < 1.
i.e. f (1) < f (x)
i.e. $0 < -\log x - 1 + x$
i.e. $1 - x < -\log x$ ……..(1)
& $g'(x) = \frac{-1}{x^2} + \frac{1}{x} = \frac{-1+x}{x^2} = \frac{x-1}{x^2} < 0$ for $0 < x < 1$
∴ g(x) is decreasing function for 0 < x < 1
i.e. g(1) < g (x)
i.e. $0 < \frac{1}{x} - 1 + \log x$
i.e.- $\log x < \frac{1}{x} - 1$ ……..(2)
From (1) & (2) $1 - x < -\log x < \frac{1}{x} - 1$ is proved.

==

Cauchy's Mean Value Theorem: If the functions f (x) and g(x) defined on [a, b] are i) continuous in [a, b], ii) derivable in (a, b), and iii) g'(x) ≠ 0 ∀ x ∈ (a, b).
Then there exists some c ∈ (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

Proof: Let us define F(x) = f (x) + Ag(x)……..(1)
Where A is constant such that F(a) = F(b).
Now F(a) = F(b) gives
f (a) + Ag(a) = f (b) + Ag(b)
i.e. –Ag(b) + Ag(a) = f (b) - f (a)
i.e. –A[g(b) –g(a)] = f (b) - f (a)
i.e. $-A = \frac{f(b) - f(a)}{g(b) - g(a)}$ ………..(2)
Provided $g(b) - g(a) \neq 0$.
Since if $g(b) - g(a) = 0$, then by given hypothesis g (x) is
i) continuous in [a, b], ii) derivable in (a, b), and iii) g(a) =g(b)
i.e. g(x) satisfies all conditions of Rolle's theorem.
∴ Rolle's theorem is applicable.
i.e. there exist some c ∈ (a, b) such that g'(c) = 0.
Which contradicts to g'(x) ≠ 0 ∀ x ∈ (a, b).
Hence $g(b) - g(a) \neq 0$.
As A is constant and f (x) and g(x) are continuous in [a, b] & derivable in (a, b).

∴ F(x) is i) continuous in [a, b],

ii) derivable in (a, b) with F'(x) = f '(x) +A g'(x)

and iii) F(a) = F(b)

i.e. F(x) satisfies all conditions of Rolle's theorem.

∴ Rolle's theorem is applicable.

i.e. there exist some c ∈ (a, b) such that F'(c) = 0.

i.e. f '(c) + Ag'(c) = 0

i.e. f '(c) = - Ag'(c)

i.e. $\frac{f'(c)}{g'(c)} = -A$

i.e. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$ by (2)

Hence proved.

==

Alternative Form of Cauchy's M.V.T.: If the functions f (x) and g(x) defined on [a, a+h] are i) continuous in [a, a+h] and ii) derivable in (a, a+h), and iii) g'(x) ≠ 0 ∀ x ∈ (a, a+h). Then there exists some $\theta \in (0, 1)$ such that $\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h) - f(a)}{g(a+h) - g(a)}$.

==

Ex. Verify Cauchy's Mean Value Theorem for the functions f (x) = sinx and g(x) = cosx in $0 \leq x \leq \frac{\pi}{2}$ (Mar.2019)

Proof. Let f (x) = sinx and g(x) = cosx defined in [o, $\frac{\pi}{2}$].

∴ f '(x) = cosx and g'(x) = -sinx

Which exists for very value of x in (o, $\frac{\pi}{2}$)

i.e. f (x) and g(x) are derivable in (o, $\frac{\pi}{2}$).

As every derivable function is continuous ⟹ f (x) and g(x) are continuous in [o, $\frac{\pi}{2}$].

Also g'(x) = -sinx ≠ 0 ∀ x ∈ (o, $\frac{\pi}{2}$)

i.e. f (x) and g(x) satisfies all conditions of Cauchy's M.V.T.

∴ Cauchy's M.V.T. is applicable.

i.e. there exist some c ∈ (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{g(\frac{\pi}{2}) - g(0)}$

i.e. $\frac{cosc}{-sinc} = \frac{1-0}{0-1}$

∴ $-cotc = -1$

∴ cotc = 1

∴ $c = \frac{\pi}{4}$

$Here\ c = \frac{\pi}{4} \in (o, \frac{\pi}{2}). Hence\ Cauchy's\ M.V.T. is\ verified.$

==

Ex. *Show that* $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta$, where $0<\alpha<\theta<\beta<\frac{\pi}{2}$

Proof. Let f (x) = sinx and g(x) = cosx defined in $[0, \frac{\pi}{2}]$.

∴ f '(x) = cosx and g'(x) = -sinx

As f(x) and g(x) are trigonometric functions.

∴ f(x) and g(x) are continuous and derivable.

∴ By Cauchy's M.V.T. in $[\alpha,\beta]$, where $0<\alpha<\theta<\beta<\frac{\pi}{2}$

$\frac{f'(\theta)}{g'(\theta)}=\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}$ where $\alpha<\theta<\beta$

i.e. $\frac{\cos\theta}{-\sin\theta}=\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}$ where $\alpha<\theta<\beta$

i.e. $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta$

Hence proved.

==

Ex. In Cauchy's Mean Value Theorem if f (x) = e^x and g(x) = e^{-x}.
Show that c is the arithmetic mean between a and b. (Mar.2019)

Proof. Let f (x) = e^x and g(x) = e^{-x} defined on [a, b].

∴ f '(x) = e^x and g'(x) = $-e^{-x}$

Which exists for very value of x in (a, b)

i.e. f (x) and g(x) are derivable in (a, b).

As every derivable function is continuous ⟹ f (x) and g(x) are continuous in [a, b].

Also g'(x) = $-e^{-x} \neq 0$ ∀ x ∈ (a, b)

i.e. f (x) satisfies all conditions of Cauchy's M.V.T.

∴ Cauchy's M.V.T. is applicable.

i.e. there exist some c ∈ (a, b) such that $\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

i.e. $\frac{e^c}{-e^{-c}}=\frac{e^b-e^a}{e^{-b}-e^{-a}}$

∴ $-e^{2c}=\frac{-(e^a-e^b)}{\frac{1}{e^b}-\frac{1}{e^a}}$

∴ $-e^{2c}=\frac{-(e^a-e^b)}{\frac{e^a-e^b}{e^{a+b}}}$ = $-e^{a+b}$

∴ 2c = a+b

Here c = $\frac{a+b}{2}$ ∈ (a, b).

Hence c is the arithmetic mean between a and b is proved.

==

Ex. In Cauchy's Mean Value Theorem if f (x) = $\frac{1}{x^2}$ and g(x) = $\frac{1}{x}$. Show that c is the harmonic mean between a and b. Where a, b > 0.

Proof. Let f (x) = $\frac{1}{x^2}$ = x^{-2} and g(x) = $\frac{1}{x}$ = x^{-1} defined on [a, b]. Where a, b > 0.

∴ f '(x) = $-2x^{-3}$ and g'(x) = $-x^{-2}$

Which exists for very value of x in (a, b)

i.e. f (x) and g(x) are derivable in (a, b).

As every derivable function is continuous ⟹ f (x) and g(x) are continuous in [a, b].

Also g'(x) = $-x^{-2} \neq 0\ \forall\ x \in (a, b)$

i.e. f (x) satisfies all conditions of Cauchy's M.V.T.

∴ Cauchy's M.V.T. is applicable.

i.e. there exist some c ∈ (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

i.e. $\frac{-2c^{-3}}{-c^{-2}} = \frac{b^{-2}-a^{-2}}{b^{-1}-a^{-1}} = b^{-1}+a^{-1}$

∴ $2c^{-1} = \frac{1}{b}+\frac{1}{a}$

∴ $\frac{2}{c} = \frac{a+b}{ab}$

∴ c = $\frac{2ab}{a+b} \in (a, b)$.

Hence c is the harmonic mean between a and b is proved.

==

Ex. In Cauchy's Mean Value Theorem if f (x) = $\sqrt{x}$ and g(x) = $\frac{1}{\sqrt{x}}$. Show that c is the geometric mean between a and b. Where b > a > 0.

Proof. Let f (x) = $\sqrt{x} = x^{1/2}$ and g(x) = $\frac{1}{\sqrt{x}} = x^{-1/2}$ defined on [a, b]. Where a, b > 0.

∴ f '(x) = $\frac{1}{2}x^{-1/2}$ and g'(x) = $\frac{-1}{2}x^{-3/2}$

Which exists for very value of x in (a, b)

i.e. f (x) and g(x) are derivable in (a, b).

As every derivable function is continuous ⟹ f (x) and g(x) are continuous in [a, b].

Also g'(x) = $\frac{-1}{2}x^{-3/2} \neq 0\ \forall\ x \in (a, b)$

i.e. f (x) satisfies all conditions of Cauchy's M.V.T.

∴ Cauchy's M.V.T. is applicable.

i.e. there exist some c ∈ (a, b) such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

i.e. $\frac{\frac{1}{2}c^{-1/2}}{\frac{-1}{2}c^{-3/2}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$

∴ -c = $\frac{-(\sqrt{a}-\sqrt{b})}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}$

∴ c = $\sqrt{ab} \in (a, b)$.

Hence c is the geometric mean between a, and b is proved.

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UNIT-2: MEAN VALUE THEOREMS [MCQ'S]

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1) Derivative of a function f (x) at point x = a is given by ……….

a) $f_-'(a) = \lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$ b) $f_+'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

c) $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ d) $f_+'(a) = \lim_{h \to 0^+} \frac{f(a-h)-f(a)}{h}$

2) Left hand derivative of a function f (x) at point x = a is given by ……….

a) $f_-'(a) = \lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$ b) $f_+'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

c) $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ d) $f_+'(a) = \lim_{h \to 0^-} \frac{f(a-h)-f(a)}{h}$

3) Right hand derivative of a function f (x) at point x = a is given by ……….

a) $f_-'(a) = \lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h}$ b) $f_+'(a) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

c) $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ d) $f_-'(a) = \lim_{h \to 0^+} \frac{f(a-h)-f(a)}{h}$

4) A function f (x) is said to be derivable at point x = a if and only if………

a) $f_-'(a) = f_+'(a)$ b) $f_-'(a) \neq f_+'(a)$ c) $f_-'(a) < f_+'(a)$ d) $f_-'(a) > f_+'(a)$

5) Every differentiable function is continuous is…

a) True b) False

6) Every continuous function is differentiable is…

a) True b) False

7) The function f (x) = |x| is ………at x = 0.

a) Not continuous b) continuous but not differentiable

c) neither continuous nor differentiable d) differentiable

8) The function f (x) = |x − a| is ………at x = 0.

a) Differentiable b) continuous but not differentiable

c) neither continuous nor differentiable d) Not continuous

9) The function f (x) = |x − 1| is ………at x = 0.

a) neither continuous nor differentiable b) both continuous and differentiable

c) continuous but not differentiable d) differentiable but not continuous

10) The function $f(x) = x\sin\frac{1}{x}$ is ………at x = 0.

a) both continuous and differentiable b) neither continuous nor differentiable

c) differentiable but not continuous a) continuous but not differentiable

11) By Rolle's Theorem if a function f (x) defined on [a, b] is

i) continuous in [a, b], ii) derivable in (a, b) and iii) f (a) = f (b).

Then there exists some c ∈ (a, b) such that ………...

a) f '(c) = 0 b) f '(c) ≠ 0 c) f '(c) < 0 d) f '(c) > 0

12) Using Rolle's theorem for the function f (x) = x^2-1 in [-1, 1] the value of c is…..

a) -1 b) 1 c) 0 d) 2

13) Using Rolle's theorem for the function f (x) = x^2- 6x + 5 in [1, 5] the value of c is…..

a) 3 b) 1 c) 5 d) 4

14) Using Rolle's theorem for the function f (x) = x^2 + 2x - 8 in [-4, 2] the value of c is…..

a) -4 b) 2 c) -1 d) 3

15) Using Rolle's theorem for the function f (x) = $(x-3)^2(x-5)^3$ in [3, 5] the value of c is…..

a) 3 b) $\frac{19}{5}$ c) 5 d) 4

16) Using Rolle's theorem for the function f (x) = log $[\frac{x^2+ab}{x(a+b)}]$ in [a, b] the value of c is…..

a) $\sqrt{ab}$ b) a c) b d) -$\sqrt{ab}$

17) By Lagranges's M. V. T. if a function f (x) defined on [a, b] is i) continuous in [a, b], ii) derivable in (a, b).Then there exists some c ∈ (a, b) such that …

a) f '(c) = 0. b) f '(c) = $\frac{f(b)-f(a)}{b-a}$ c) f '(c) = $f(b) - f(a)$ d) f '(c) = b-a

18) Using Lagranges's M. V. T. for the function f (x) = $1-x^2$ in [1, 2] the value of c is…..

a) 1 b) $\frac{3}{2}$ c) 2 d) 0

19) Using Lagranges's M. V. T. for the function f (x) = x^2-4x-3 in [1, 4] the value of c is…..

a) $\frac{5}{2}$ b) 1 c) 4 d) 0

20) A function f (x) is said to be monotonic increasing function if $x_1 < x_2 \Rightarrow$ ……..

a) f (x_1) < f (x_2) b) f (x_1) > f (x_2) c) f (x_1) = f (x_2) d) f $(x_1) \neq$ f (x_2)

21) A function f (x) is said to be monotonic decreasing function if $x_1 < x_2 \Rightarrow$ ……..

a) f (x_1) < f (x_2) b) f (x_1) > f (x_2) c) f (x_1) = f (x_2) d) f $(x_1) \neq$ f (x_2)

22) The function f (x) = 7x – 3 is ………..function on R.

a) strictly increasing, b) strictly decreasing, c) Neither decreasing nor increasing

23) The function f (x) = ax + b, where a, b are constant and a > 0 is ………..function on R.

a) strictly increasing, b) strictly decreasing, c) Neither decreasing nor increasing

24) The function f (x) = x^2 is ………..function in (-∞, 0).

a) strictly increasing, b) strictly decreasing, c) Neither decreasing nor increasing

25) A function f (x) is monotonic increasing function if ………∀ x ∈ (a, b)

a) f '(x) > 0 b) f '(x) < 0 c) f '(x) = 0 d) f '(x) ≠ 0

26) A function f (x) is monotonic decreasing function if ………∀ x ∈ (a, b)

a) f '(x) > 0 b) f '(x) < 0 c) f '(x) = 0 d) f '(x) ≠ 0

27) By using Cauchy's Mean Value Theorem for the functions f (x) = sinx and g(x) = cosx in $0 \leq x \leq \frac{\pi}{2}$, the value of c is ……. a) 0 b) $\frac{\pi}{2}$ c) $\frac{\pi}{4}$ d) π

28) By using Cauchy's Mean Value Theorem for the functions f (x) = e^x and g(x) = e^{-x} in [a, b], the value of c is ……. a) $\frac{a+b}{2}$ b) a c) b d) 0

29) By using Cauchy's Mean Value Theorem for the functions f (x) = $\frac{1}{x^2}$ and g(x) = $\frac{1}{x}$ in [a, b], the value of c is ……. a) $\frac{a+b}{2}$ b) $\frac{2ab}{a+b}$ c) $\sqrt{ab}$ d) 0

30) By using Cauchy's Mean Value Theorem for the functions f (x) = $\sqrt{x}$ and g(x) = $\frac{1}{\sqrt{x}}$ in [a, b], the value of c is ……. a) $\frac{a+b}{2}$ b) $\frac{2ab}{a+b}$ c) $\sqrt{ab}$ d) 0

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UNIT-3: SUCCESSIVE DIFFERENTIATION

Successive Differentiation: The process of differentiating the same function again and again is called successive differentiation.

Remark: The successive derivatives of the function y = f(x) are denoted by $y_1, y_2, y_3, \ldots\ldots y_n$. or $y', y'', y''', \ldots\ldots, y^{(n)}$ or $Dy, D^2y, D^3y, \ldots\ldots, D^ny$ or $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots\ldots, \frac{d^ny}{dx^n}$ or $f'(x), f''(x), f'''(x), \ldots\ldots, f^{(n)}(x)$ etc.

1) If $y = x^m$, then show that $y_n = \begin{cases} \frac{m!}{(m-n)!}\, x^{m-n} & if\ m > n \\ n! & if\ m = n \\ 0 & if\ m < n \end{cases}$

Proof: Let $y = x^m$

$\therefore y_1 = mx^{m-1}$

$\therefore y_2 = m(m-1)x^{m-2}$

$\therefore y_3 = m(m-1)(m-2)x^{m-3}$

………………………………….

………………………………….

$\therefore y_n = m(m-1)(m-2)(m-3)\ldots\ldots\ldots\ldots\ldots(m-n+1)x^{m-n}$

$\therefore y_n = \frac{m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)(m-n)(m-n-1)\ldots\ldots 3.2.1}{(m-n)(m-n-1)\ldots\ldots 3.2.1} x^{m-n}$

$\therefore y_n = \begin{cases} \frac{m!}{(m-n)!}\, x^{m-n} & if\ m > n \\ n! & if\ m = n \\ 0 & if\ m < n \end{cases}$

2) If $y = (ax+b)^m$, then show that $y_n = \begin{cases} \frac{m!a^n}{(m-n)!}\, (ax+b)^{m-n} & if\ m > n \\ n!\,a^n & if\ m = n \\ 0 & if\ m < n \end{cases}$

Proof: Let $y = (ax+b)^m$

$\therefore y_1 = m(ax+b)^{m-1}a$

$\therefore y_2 = m(m-1)(ax+b)^{m-2}a^2$

$\therefore y_3 = m(m-1)(m-2)(ax+b)^{m-3}a^3$

………………………………….

………………………………….

$\therefore y_n = m(m-1)(m-2)(m-3)\ldots\ldots\ldots\ldots\ldots(m-n+1)(ax+b)^{m-n}a^n$

$\therefore y_n = \frac{m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)(m-n)(m-n-1)\ldots\ldots3.2.1}{(m-n)(m-n-1)\ldots\ldots3.2.1}(ax+b)^{m-n}a^n$

$$\therefore y_n = \begin{cases} \frac{m!\,a^n}{(m-n)!}\,(ax+b)^{m-n} & if\ m > n \\ n!\,a^n & if\ m = n \\ 0 & if\ m < n \end{cases}$$

===

3) If $y = e^{ax+b}$, then show that $y_n = a^n e^{ax+b}$

Proof: Let $y = e^{ax+b}$

$\therefore y_1 = a\,e^{ax+b}$

$\therefore y_2 = a^2e^{ax+b}$

$\therefore y_3 = a^3e^{ax+b}$

……………………………………

……………………………………

$\therefore y_n = a^ne^{ax+b}$

===

4) If $y = \frac{1}{ax+b}$, then show that $y_n = \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$

Proof: Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$

$\therefore y_1 = (-1)(ax+b)^{-2}a$

$\therefore y_2 = (-1)(-2)(ax+b)^{-3}a^2$

$\therefore y_3 = (-1)(-2)(-3)(ax+b)^{-4}a^3$

……………………………………

……………………………………

$\therefore y_n = (-1)(-2)(-3)(-4)\ldots\ldots(-n)(ax+b)^{-n-1}a^n$

$\therefore y_n = \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$

===

5) If $y = \log(ax+b)$, then show that $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

Proof: Let $y = \log(ax+b)$

$\therefore y_1 = \frac{a}{ax+b} = (ax+b)^{-1}a$

$\therefore y_2 = (-1)(ax+b)^{-2}a^2$

$\therefore y_3 = (-1)(-2)(ax+b)^{-3}a^3$

……………………………………

……………………………………

$\therefore y_n = (-1)(-2)(-3)\ldots\ldots(-n+1)(ax+b)^{-n}a^n$

$\therefore y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

==

6) If $y = \sin(ax+b)$, then show that $y_n = a^n \sin(ax+b+\frac{n\pi}{2})$

Proof: Let $y = \sin(ax+b)$

$\therefore y_1 = a\cos(ax+b) = a\sin(ax+b+\frac{\pi}{2})$ $\qquad \because \cos\theta = \sin(\theta + \frac{\pi}{2})$

$\therefore y_2 = a^2 \cos(ax+b+\frac{\pi}{2}) = a^2 \sin(ax+b+\frac{2\pi}{2})$

$\therefore y_3 = a^3 \cos(ax+b+\frac{2\pi}{2}) = a^3 \sin(ax+b+\frac{3\pi}{2})$

……………………………………

……………………………………

$\therefore y_n = a^n \sin(ax+b+\frac{n\pi}{2})$

==

7) If $y = \cos(ax+b)$, then show that $y_n = a^n \cos(ax+b+\frac{n\pi}{2})$

Proof: Let $y = \cos(ax+b)$

$\therefore y_1 = -a\sin(ax+b) = a\cos(ax+b+\frac{\pi}{2})$ $\qquad \because -\sin\theta = \cos(\theta + \frac{\pi}{2})$

$\therefore y_2 = -a^2 \sin(ax+b+\frac{\pi}{2}) = a^2 \cos(ax+b+\frac{2\pi}{2})$

$\therefore y_3 = -a^3 \sin(ax+b+\frac{2\pi}{2}) = a^3 \cos(ax+b+\frac{3\pi}{2})$

……………………………………

……………………………………

$\therefore y_n = a^n \cos(ax+b+\frac{n\pi}{2})$

==

8) If $y = e^{ax} \sin(bx+c)$, then show that $y_n = (a^2+b^2)^{n/2} e^{ax} \sin(bx+c+n\tan^{-1}\frac{b}{a})$

Proof: Let $y = e^{ax} \sin(bx+c)$

$\therefore y_1 = a\,e^{ax} \sin(bx+c) + b\,e^{ax} \cos(bx+c)$

Put $a = r\cos\theta$, $b = r\sin\theta$

i.e. $r = \sqrt{a^2+b^2} = (a^2+b^2)^{1/2}$ and $\theta = \tan^{-1}\frac{b}{a}$

$\therefore y_1 = r\cos\theta\, e^{ax} \sin(bx+c) + r\sin\theta\, e^{ax} \cos(bx+c)$

$= r\,e^{ax} [\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta]$

$\therefore y_1 = r\,e^{ax} \sin(bx+c+\theta)$

Similarly by repeating the process, we get

$\therefore y_2 = r^2 e^{ax} \sin(bx+c+2\theta)$

$\therefore y_3 = r^3 e^{ax} \sin(bx+c+3\theta)$

……………………………………

……………………………………

$\therefore y_n = r^n e^{ax} \sin(bx+c+n\theta)$

Substituting the values of r and θ, we get

$y_n = (a^2+b^2)^{n/2} e^{ax} \sin(bx+c+n\tan^{-1}\frac{b}{a})$

==

9) If $y = e^{ax} \cos(bx+c)$, then show that $y_n = (a^2+b^2)^{n/2} e^{ax} \cos(bx+c+n\tan^{-1}\frac{b}{a})$

Proof: Let $y = e^{ax} \cos(bx+c)$

$\therefore y_1 = a e^{ax} \cos(bx+c) - b e^{ax} \sin(bx+c)$

Put $a = r\cos\theta$, $b = r\sin\theta$

i.e. $r = \sqrt{a^2+b^2} = (a^2+b^2)^{1/2}$ and $\theta = \tan^{-1}\frac{b}{a}$

$\therefore y_1 = r\cos\theta\, e^{ax} \cos(bx+c) - r\sin\theta\, e^{ax} \sin(bx+c)$

$= r e^{ax} [\cos(bx+c)\cos\theta - \sin(bx+c)\sin\theta]$

$\therefore y_1 = r e^{ax} \cos(bx+c+\theta)$

Similarly by repeating the process, we get

$\therefore y_2 = r^2 e^{ax} \cos(bx+c+2\theta)$

$\therefore y_3 = r^3 e^{ax} \cos(bx+c+3\theta)$

……………………………………

……………………………………

$\therefore y_n = r^n e^{ax} \cos(bx+c+n\theta)$

Substituting the values of r and θ, we get

$y_n = (a^2+b^2)^{n/2} e^{ax} \cos(bx+c+n\tan^{-1}\frac{b}{a})$

==

Ex.1) Find the n^{th} derivative of $\frac{1}{(x+2)(x-3)}$

Proof: Let $y = \frac{1}{(x+2)(x-3)}$

First we express it into partial fractions as follows

$\frac{1}{(x+2)(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$

i.e. $1 = A(x-3) + B(x+2)$…………(1)

Putting $x = -2$ in (1), we get,

$1 = -5A + 0 \quad \therefore A = \frac{-1}{5}$

Putting $x = 3$ in (1), we get,

$1 = 0 + 5B \quad \therefore B = \frac{1}{5}$

$\therefore y = \frac{\frac{-1}{5}}{(x+2)} + \frac{\frac{1}{5}}{(x-3)}$

$\therefore y = \frac{1}{5}\left[\frac{1}{(x-3)} - \frac{1}{(x+2)}\right]$

By taking n^th^ derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{5}\left[\frac{(-1)^n n!}{(x-3)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}\right]$

i.e. $y_n = \frac{(-1)^n n!}{5}\left[\frac{1}{(x-3)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$

===

Ex.2) Find the n^{th} derivative of $\frac{1}{x^2-5x+4}$

Proof: Let $y = \frac{1}{x^2-5x+4} = \frac{1}{(x-4)(x-1)}$

First we express it into partial fractions as follows

$\frac{1}{(x-4)(x-1)} = \frac{A}{(x-4)} + \frac{B}{(x-1)}$

i.e. 1 = A(x-1) + B(x-4)………….(1)

Putting x = 4 in (1), we get,

$1 = 3A + 0 \quad \therefore A = \frac{1}{3}$

Putting x = 1 in (1), we get,

$1 = 0 - 3B \quad \therefore B = \frac{-1}{3}$

$\therefore y = \frac{\frac{1}{3}}{(x-4)} + \frac{\frac{-1}{3}}{(x-1)}$

$\therefore y = \frac{1}{3}\left[\frac{1}{(x-4)} - \frac{1}{(x-1)}\right]$

By taking n^{th} derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{3}\left[\frac{(-1)^n n!}{(x-4)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}}\right]$

i.e. $y_n = \frac{(-1)^n n!}{3}\left[\frac{1}{(x-4)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$

===

Ex.3) Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Proof: Let $y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$

First we express it into partial fractions as follows

$\frac{1}{(2x-1)(3x-1)} = \frac{A}{(2x-1)} + \frac{B}{(3x-1)}$

i.e. 1 = A(3x-1) + B(2x-1)………….(1)

Putting x = $\frac{1}{2}$ in (1), we get,

$1 = \frac{1}{2}A + 0 \quad \therefore A = 2$

Putting x = $\frac{1}{3}$ in (1), we get,

$1 = 0 - \frac{1}{3}B \quad \therefore B = -3$

$\therefore y = \frac{2}{(2x-1)} + \frac{-3}{(3x-1)}$

$\therefore y = \frac{1}{(x-\frac{1}{2})} - \frac{1}{(x-\frac{1}{3})}$

By taking n^th derivative w.r.t. x, we get,

$\therefore y_n = [\frac{(-1)^n n!}{\left(x-\frac{1}{2}\right)^{n+1}} - \frac{(-1)^n n!}{\left(x-\frac{1}{3}\right)^{n+1}}]$

i.e. $y_n = (-1)^n n! [\frac{1}{\left(x-\frac{1}{2}\right)^{n+1}} - \frac{1}{\left(x-\frac{1}{3}\right)^{n+1}}]$

==

Ex.4) Find the n^th derivative of $\frac{2x-1}{(x-2)(x+1)}$

Proof: Let y = $\frac{2x-1}{(x-2)(x+1)}$

First we express it into partial fractions as follows

$\frac{2x-1}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$

i.e. 2x-1 = A(x+1) + B(x-2)…………(1)

Putting x = 2 in (1), we get,

$3 = 3A + 0 \quad \therefore A = 1$

Putting x = -1 in (1), we get,

$-3 = 0 - 3B \quad \therefore B = 1$

$\therefore y = \frac{1}{(x-2)} + \frac{1}{(x+1)}$

By taking n^th derivative w.r.t. x, we get,

$\therefore y_n = [\frac{(-1)^n n!}{(x-2)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}}]$

i.e. $y_n = (-1)^n n! [\frac{1}{(x-2)^{n+1}} + \frac{1}{(x+1)^{n+1}}]$

==

Ex.5) Find the n^th derivative of $\frac{x^2+1}{(x-1)(x-2)(x-3)}$

Proof: Let y = $\frac{x^2+1}{(x-1)(x-2)(x-3)}$

First we express it into partial fractions as follows

$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

i.e. $x^2 + 1$ = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)…………(1)

Putting x = 1 in (1), we get,

2 = 2A + 0 + 0 ∴ A = 1

Putting x = 2 in (1), we get,

5 = 0 - B + 0 ∴ B = -5

Putting x = 3 in (1), we get,

10 = 0 + 0 + 2C ∴ C = 5

∴ y = $\frac{1}{(x-1)} + \frac{-5}{(x-2)} + \frac{5}{(x-3)}$

∴ y = $\frac{1}{(x-1)} - \frac{5}{(x-2)} + \frac{5}{(x-3)}$

By taking n^{th} derivative w.r.t. x, we get,

∴ $y_n = \frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{5(-1)^n n!}{(x-2)^{n+1}} + \frac{5(-1)^n n!}{(x-3)^{n+1}}$

i.e. $y_n = (-1)^n n! \left[\frac{1}{(x-1)^{n+1}} - \frac{5}{(x-2)^{n+1}} + \frac{5}{(x-3)^{n+1}}\right]$

==

Ex.6) Find the n^{th} derivative of $\frac{1}{x^3+6x^2+11x+6}$

Proof: Let y = $\frac{1}{x^3+6x^2+11x+6} = \frac{1}{(x+1)(x+2)(x+3)}$

First we express it into partial fractions as follows

$\frac{1}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}$

i.e. 1 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)…………(1)

Putting x = -1 in (1), we get,

1 = 2A + 0 + 0 ∴ A = $\frac{1}{2}$

Putting x = -2 in (1), we get,

1 = 0 - B + 0 ∴ B = -1

Putting x = -3 in (1), we get,

1 = 0 + 0 + 2C ∴ C = $\frac{1}{2}$

∴ y = $\frac{\frac{1}{2}}{(x+1)} + \frac{-1}{(x+2)} + \frac{\frac{1}{2}}{(x+3)}$

∴ y = $\frac{1}{2}\left[\frac{1}{(x+1)} - \frac{2}{(x+2)} + \frac{1}{(x+3)}\right]$

By taking n^{th} derivative w.r.t. x, we get,

∴ $y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{2(-1)^n n!}{(x+2)^{n+1}} + \frac{(-1)^n n!}{(x+3)^{n+1}}\right]$

i.e. $y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{(x+1)^{n+1}} - \frac{2}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}}\right]$

==

Ex.7) Find the n^{th} derivative of $\log\sqrt{\frac{5x+3}{3x-2}}$

Proof: Let $y = \log\sqrt{\frac{5x+3}{3x-2}} = \frac{1}{2}[\log(5x+3) - \log(3x-2)]$

By taking n^{th} derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{2}\left[\frac{(-1)^{n-1}(n-1)!5^n}{(5x+3)^n} - \frac{(-1)^{n-1}(n-1)!3^n}{(3x-2)^n}\right]$

i.e. $y_n = \frac{(-1)^{n-1}(n-1)!}{2}\left[\frac{5^n}{(5x+3)^n} - \frac{3^n}{(3x-2)^n}\right]$

==

Ex.8) Find the n^{th} derivative of sin2x cos3x

Proof: Let y = sin2x cos3x

$= \frac{1}{2}$ [2cos3x sin2x]

$= \frac{1}{2}$ [sin(3x+2x)-sin(3x-2x)]

$= \frac{1}{2}$ [sin5x-sinx]

By taking n^{th} derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{2}[\,5^n \sin(5x+\frac{n\pi}{2}) - \sin(x+\frac{n\pi}{2})]$

==

Ex.9) Find the n^{th} derivative of $\cos^4 x$

Proof: Let $y = \cos^4 x = (\cos^2 x)^2$

$= (\frac{1+\cos 2x}{2})^2$

$= \frac{1}{4}(1+2\cos 2x+\cos^2 2x)$

$= \frac{1}{4}[1+2\cos 2x+(\frac{1+\cos 4x}{2})]$

$= \frac{1}{8}[2+4\cos 2x+1+\cos 4x]$

$= \frac{1}{8}[3+4\cos 2x+\cos 4x]$

By taking n^{th} derivative w.r.t. x, we get,

$y_n = \frac{1}{8}[\,0+4.2^n \cos(2x+\frac{n\pi}{2}) + 4^n \cos(4x+\frac{n\pi}{2})]$

$\therefore y_n = \frac{1}{8}[\,2^{n+2} \cos(2x+\frac{n\pi}{2}) + 4^n \cos(4x+\frac{n\pi}{2})]$

==

Ex.10) Find the n^{th} derivative of $\sin^4 x$

Proof: Let $y = \sin^4 x = (\sin^2 x)^2$

$= (\frac{1-\cos 2x}{2})^2$

$= \frac{1}{4}(1-2\cos 2x+\cos^2 2x)$

$= \frac{1}{4}[1-2\cos 2x+(\frac{1+\cos 4x}{2})]$

$= \frac{1}{8}[2-4\cos 2x+1+\cos 4x]$

$= \frac{1}{8}[3-4\cos 2x+\cos 4x]$

By taking n^{th} derivative w.r.t. x, we get,

$y_n = \frac{1}{8}[\,0-4.2^n \cos(2x+\frac{n\pi}{2}) +4^n \cos(4x+\frac{n\pi}{2})]$

$\therefore y_n = \frac{1}{8}[4^n \cos(4x+\frac{n\pi}{2}) - 2^{n+2} \cos(2x+\frac{n\pi}{2})]$

==

Ex.11) Find the n^{th} derivative of cosx cos2x cos3x

Proof: Let $y = \cos x \cos 2x \cos 3x$

$= \frac{1}{2}\cos x[2\cos 3x \cos 2x]$

$= \frac{1}{2}\cos x[\cos(3x+2x)+\cos(3x-2x)]$

$= \frac{1}{2}\cos x[\cos 5x+\cos x]$

$= \frac{1}{2}[\cos 5x\cos x+\cos^2 x]$

$= \frac{1}{4}[2\cos 5x\cos x+2\cos^2 x]$

$= \frac{1}{4}[\cos(5x+x)+\cos(5x-x)+1+\cos 2x]$

$= \frac{1}{4}[\cos 6x+\cos 4x+\cos 2x+1]$

By taking n^{th} derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{2}[\,6^n \cos(6x+\frac{n\pi}{2}) + 4^n \cos(4x+\frac{n\pi}{2}) + 2^n \cos(2x+\frac{n\pi}{2})]$

==

Ex.12) Find the n^{th} derivative of e^{ax} sinbx coscx

Proof: Let $y = e^{ax} \sin bx \cos cx$

$= \frac{1}{2} e^{ax}[2\sin bx \cos cx]$

$= \frac{1}{2} e^{ax} [\sin(bx+cx)+\sin(bx-cx)]$

$= \frac{1}{2} [e^{ax} \sin(b+c)x+ e^{ax} \sin(b-c)x]$

By taking n^{th} derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{2}\{ [a^2+(b+c)^2]^{n/2} e^{ax} \sin[(b+c)x + n \tan^{-1} (\frac{b+c}{a})]$

$+ [a^2+(b-c)^2]^{n/2} e^{ax} \sin[(b-c)x+ n \tan(\frac{b-c}{a})]\}$

==

Ex.13) Find the n^{th} derivative of $e^x \cos^2x \cos2x$

Proof: Let $y = e^x \cos^2x \cos2x$

$= e^x[(\frac{1+\cos2x}{2}) \cos2x]$

$= \frac{1}{2} e^x [\cos2x+\cos^2 2x]$

$= \frac{1}{2} e^x [\cos2x+\frac{1+\cos4x}{2}]$

$= \frac{1}{4} e^x [2\cos2x+1+\cos4x]$

$= \frac{1}{4} [2 e^x \cos2x+ e^x \cos4x+ e^x]$

By taking n^{th} derivative w.r.t. x, we get,

$\therefore y_n = \frac{1}{4}\{2 (1^2+2^2)^{n/2} e^x \cos[2x+n \tan^{-1} (\frac{2}{1})$

$+ [1^2+4^2]^{n/2} e^x \cos(4x+ n \tan^{-1}(\frac{4}{1})]+e^x\}$

$\therefore y_n = \frac{e^x}{4}\{2 (5)^{n/2} \cos(2x+n \tan^{-1}2)+ (17)^{n/2} \cos(4x+ n \tan^{-1}4)+1\}$

==

Ex.14) If $y = e^x (\sin x + \cos x)$, then prove that $y_n = 2^{\frac{n+1}{2}} e^x \sin[x+(n+1) \frac{\pi}{4}]$

Proof: Let $y = e^x (\sin x + \cos x)$

$= \sqrt{2}e^x[\sin(x + \frac{\pi}{4})]$

By taking n^{th} derivative w.r.t. x, we get,

$y_n = \sqrt{2} (1^2+1^2)^{n/2} e^x \sin[x+ \frac{\pi}{4}+n \tan^{-1} (\frac{1}{1})]$

$\therefore y_n = 2^{1/2} 2^{n/2} e^x \sin[x+ \frac{\pi}{4}+n \frac{\pi}{4}]$

$\therefore y_n = 2^{\frac{n+1}{2}} e^x \sin[x+ (n+1) \frac{\pi}{4}]$

Hence Proved.

==

Ex.15) If $y = \sin^2x \cos^2x$, then show that $y_n = \frac{-4^n}{8} \cos (4x+ \frac{n\pi}{2})$

Proof: Let $y = \sin^2x \cos^2x$

$= \frac{1}{4} (2\sin x \cos x)^2$

$= \frac{1}{4} \sin^2 2x$

$= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)$

$= \frac{1}{8}(1-\cos 4x)$

By taking n^{th} derivative w.r.t. x, we get,

$y_n = \frac{1}{8}\left(0 - 4^n \cos\left(4x+\frac{n\pi}{2}\right)\right)$

$\therefore y_n = \frac{-4^n}{8}\cos\left(4x+\frac{n\pi}{2}\right)$

Hence Proved.

==

Leibnitz's Theorem: Let u and v be any two functions of x, having derivatives of order n, then $(uv)_n = u_n v + n_{C_1} u_{n-1}v_1 + n_{C_2} u_{n-2}v_2 + \ldots\ldots + n_{C_{n-1}} u_1 v_{n-1} + uv_n$

Proof: We prove the theorem by mathematical induction.

By direct differentiation

$(uv)_1 = u_1 v + uv_1$

& $(uv)_2 = u_2 v + u_1 v_1 + u_1 v_1 + uv_2$

i. e. $(uv)_2 = u_2 v + 2_{C_1} u_1 v_1 + uv_2$

Thus theorem is true for n = 1 and 2.

Suppose it is true for n = k

i.e. $(uv)_k = u_k v + k_{C_1} u_{k-1}v_1 + k_{C_2} u_{k-2}v_2 + \ldots\ldots + k_{C_{k-1}} u_1 v_{k-1} + uv_k$ ……..(1)

Differentiating both sides of equation (1), we get,

$(uv)_{k+1} = u_{k+1}v + u_k v_1 + k_{C_1} u_k v_1 + k_{C_1} u_{k-1}v_2 + k_{C_2} u_{k-1}v_2 + k_{C_2} u_{k-2}v_3 + \ldots\ldots\ldots$
$+ k_{C_{k-1}} u_2 v_{k-1} + k_{C_{k-1}} u_1 v_k + u_1 v_k + uv_{k+1}$

i. e. $(uv)_{k+1} = u_{k+1}v + (1 + k_{C_1}) u_k v_1 + (k_{C_1} + k_{C_2}) u_{k-1}v_2 + \ldots\ldots\ldots$
$+ (k_{C_{k-1}} + 1) u_1 v_k + uv_{k+1}$

By using $k_{C_{r-1}} + k_{C_r} = k+1_{C_r}$, we get,

$1 + k_{C_1} = k_{C_0} + k_{C_1} = k+1_{C_1}$, $k_{C_1} + k_{C_2} = k+1_{C_2}$,…….

$k_{C_{k-1}} + 1 = k_{C_{k-1}} + k_{C_k} = k+1_{C_k}$

$\therefore (uv)_{k+1} = u_{k+1}v + k+1_{C_1} u_k v_1 + k+1_{C_2} u_{k-1}v_2 + \ldots + k+1_{C_k} u_1 v_k + uv_{k+1}$

i. e. result is true for n = k $\Longrightarrow$ result is true for n = k+1.

$\therefore$ by mathematical induction, result is true for any natural number n.

i. e. $(uv)_n = u_n v + n_{C_1} u_{n-1}v_1 + n_{C_2} u_{n-2}v_2 + \ldots\ldots + n_{C_{n-1}} u_1 v_{n-1} + uv_n$

Hence proved.

==

Remark: 1) To find n^{th} derivative of y = uv, denote polynomial function by v and other by u.

2) $(y_1v)_n = y_{n+1}v + n\, y_nv_1 + \frac{n(n-1)}{2} y_{n-1}v_2 + \ldots\ldots$

3) $(y_2v)_n = y_{n+2}v + n\, y_{n+1}v_1 + \frac{n(n-1)}{2} y_nv_2 + \ldots\ldots$

==

Ex. Find the n^{th} derivative of x^3e^x.

Solution: Let $y = x^3e^x$

By taking $u = e^x$ and $v = x^3$, we get

$u_n = e^x$, $v_1 = 3x^2$

$u_{n-1} = e^x$, $v_2 = 6x$

$u_{n-2} = e^x$, $v_3 = 6$

$u_{n-3} = e^x$, $v_4 = 0$

and so on.

As y = uv, By using Leibnitz's theorem

$y_n = u_nv + n_{C_1} u_{n-1}v_1 + n_{C_2} u_{n-2}v_2 + n_{C_3} u_{n-3}v_3 + \ldots\ldots + n_{C_{n-1}} u_1v_{n-1} + uv_n$

we get,

$y_n = e^x x^3 + n\, e^x (3x^2) + \frac{n(n-1)}{2} e^x (6x) + \frac{n(n-1)(n-2)}{6} e^x (6) + 0$

$\therefore y_n = e^x [\, x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n-2)]$

==

Ex. If $y = x^2 \sin 3x$, find y_n.

Solution: Let $y = x^2 \sin 3x$

By taking $u = \sin 3x$ and $v = x^2$, we get

$u_n = 3^n \sin(3x+\frac{n\pi}{2})$, $v_1 = 2x$

$u_{n-1} = 3^{n-1} \sin [3x+\frac{(n-1)\pi}{2}]$, $v_2 = 2$

$u_{n-2} = 3^{n-2} \sin [3x+\frac{(n-2)\pi}{2}]$, $v_3 = 0$

and so on.

As y = uv, By using Leibnitz's theorem

$y_n = u_nv + n_{C_1} u_{n-1}v_1 + n_{C_2} u_{n-2}v_2 + \ldots\ldots + n_{C_{n-1}} u_1v_{n-1} + uv_n$ we get,

$y_n = 3^n \sin(3x+\frac{n\pi}{2}) (x^2) + n\, 3^{n-1} \sin [3x+\frac{(n-1)\pi}{2}] (2x)$

$+ \frac{n(n-1)}{2} 3^{n-2} \sin [3x+\frac{(n-2)\pi}{2}] (2) + 0$

$\therefore y_n = 3^n x^2 \sin(3x+\frac{n\pi}{2}) + 2.3^{n-1}nx \sin [3x+\frac{(n-1)\pi}{2}]$

$+ n(n-1)3^{n-2}\sin [3x+\frac{(n-2)\pi}{2}]$

==

Ex. If $y = x^2 \sin(2x+5)$, find y_8.

Solution: Let $y = x^2 \sin(2x+5)$

By taking $u = \sin(2x+5)$ and $v = x^2$, we get

$u_8 = 2^8 \sin(2x+5+\frac{8\pi}{2}) = 256 \sin(2x+5)$, $v_1 = 2x$

$u_7 = 2^7 \sin(2x+5+\frac{7\pi}{2}] = -128 \cos(2x+5)$, $v_2 = 2$

$u_6 = 2^6 \sin(2x+5+\frac{6\pi}{2}] = -64 \sin(2x+5)$, $v_3 = 0$

and so on.

As y = uv, By using Leibnitz's theorem, we get,

$y_8 = u_8 v + 8_{C_1} u_8 v_1 + 8_{C_2} u_6 v_2 + \ldots\ldots + 8_{C_7} u_1 v_7 + uv_8$

$y_8 = 256 \sin(2x+5)(x^2) + 8.[-128 \cos(2x+5)](2x) + 28[-64 \sin(2x+5)](2) + 0$

$\therefore y_8 = 256 [x^2 \sin(2x+5) - 8x \cos(2x+5) - 14 \sin(2x+5)]$

==

Ex. If $y = x^3 \cos x$, find y_n.

Solution: Let $y = x^3 \cos x$

By taking $u = \cos x$ and $v = x^3$, we get

$u_n = \cos(x+\frac{n\pi}{2})$, $v_1 = 3x^2$

$u_{n-1} = \cos[x+\frac{(n-1)\pi}{2}]$, $v_2 = 6x$

$u_{n-2} = \cos[x+\frac{(n-2)\pi}{2}]$, $v_3 = 6$

$u_{n-3} = \cos[x+\frac{(n-3)\pi}{2}]$, $v_4 = 0$

and so on.

As y = uv, By using Leibnitz's theorem

$y_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + n_{C_3} u_{n-3} v_3 + \ldots\ldots + n_{C_{n-1}} u_1 v_{n-1} + uv_n$

we get,

$y_n = \cos(x+\frac{n\pi}{2})(x^3) + n \cos[x+\frac{(n-1)\pi}{2}](3x^2) + \frac{n(n-1)}{2} \cos[x+\frac{(n-2)\pi}{2}](6x)$

$+ \frac{n(n-1)(n-2)}{6} \cos[x+\frac{(n-3)\pi}{2}](6) + 0$

$\therefore y_n = x^3 \cos(x+\frac{n\pi}{2}) + 3n x^2 \cos[x+\frac{(n-1)\pi}{2}] + 3n(n-1)x \cos[x+\frac{(n-2)\pi}{2}]$

$+ n(n-1)(n-2) \cos[x+\frac{(n-3)\pi}{2}]$

==

Ex. If $y = x^2 \log x$, find y_n.

Solution: Let $y = x^2 \log x$

By taking $u = \log x$ and $v = x^2$, we get

$u_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$, $v_1 = 2x$

$u_{n-1} = \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$, $v_2 = 2$

$u_{n-2} = \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}$, $v_3 = 0$

and so on.

As y = uv, By using Leibnitz's theorem

$y_n = u_n v + n_{C_1} u_{n-1}v_1 + n_{C_2} u_{n-2}v_2 + \ldots\ldots + n_{C_{n-1}} u_1 v_{n-1} + uv_n$

we get,

$$y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}(x^2) + n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}(2x) + \frac{n(n-1)}{2}\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}(2) + 0$$

$$\therefore y_n = \frac{(-1)^{n-3}(n-3)!}{x^{n-2}}[(n-1)(n-2) - 2n(n-2) + n(n-1)]$$

===

Ex. If f(x) = tanx, prove that $f^n(0) - n_{C_2} f^{n-2}(0) + n_{C_4} f^{n-4}(0) \ldots\ldots\ldots = \sin\frac{n\pi}{2}$

Proof: Let $f(x) = \tan x = \frac{\sin x}{\cos x}$

$\therefore$ f(x) cosx = sinx

By taking n^{th} derivative w.r.t. x and using Leibnitz's theorem

We get,

$f^n(x)\cos x + n_{C_1} f^{n-1}(x)(-\sin x) + n_{C_2} f^{n-2}(x)(-\cos x) + n_{C_3} f^{n-3}(x)(\sin x)$

$+ n_{C_4} f^{n-4}(x)(\cos x) + \ldots\ldots\ldots = \sin(x + \frac{n\pi}{2})$

Putting x = 0, we get,

$f^n(0) - n_{C_2} f^{n-2}(0) + n_{C_4} f^{n-4}(0) \ldots\ldots\ldots = \sin\frac{n\pi}{2}$

Hence proved.

===

Ex. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n-1)y_n = 0$

Proof: Let $y = (x^2 - 1)^n$

$\therefore y_1 = n(x^2 - 1)^{n-1}(2x)$

$\therefore (x^2 - 1)y_1 = 2nx(x^2 - 1)^n$ by multiplying $(x^2 - 1)$ on both sides.

$\therefore (x^2 - 1)y_1 = 2nxy$

Differentiating again w.r.t. x, we get,

$(x^2 - 1)y_2 + 2xy_1 = 2nxy_1 + 2ny$

i.e. $(x^2 - 1)y_2 - 2(n-1)xy_1 - 2ny = 0$

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}(x^2-1)+ n\, y_{n+1}(2x) + \frac{n(n-1)}{2} y_n(2) + 0 – 2(n-1)[y_{n+1}(x) + ny_n(1) + 0]- 2ny_n = 0$

$\therefore (x^2-1)\, y_{n+2} + 2nx\, y_{n+1} + (n^2-n)\, y_n + 0 – 2(n-1)\, xy_{n+1}(x) -2(n^2-n)\, y_n – 2ny_n = 0$

$\therefore (x^2-1)\, y_{n+2} + 2(n-n+1)\, xy_{n+1} + (n^2-n-2n^2+2n-2n)\, y_n = 0$

$\therefore (x^2-1)\, y_{n+2} + 2xy_{n+1} – n(n+1)\, y_n = 0.$ Hence proved.

===

Ex. If $y = \cos(\log x)$, prove that $x^2 y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)y_n = 0$

Proof: Let $y = \cos(\log x)$ ……..(1)

$\therefore y_1 = -\sin(\log x)\left(\frac{1}{x}\right)$

$\therefore xy_1 = -\sin(\log x)$ by multiplying x on both sides.

Again differentiating w.r.t. x, we get,

$xy_2 + y_1 = -\cos(\log x)\left(\frac{1}{x}\right)$

$\therefore x^2y_2 + xy_1 = -\cos(\log x)$ by multiplying x on both sides.

i.e. $x^2y_2 + xy_1 = -y$ by (1)

i.e. $x^2y_2 + xy_1 + y = 0$

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}(x^2)+ n\, y_{n+1}(2x) + \frac{n(n-1)}{2} y_n(2) + 0 + [\, y_{n+1}(x) + n\, y_n(1) + 0\,] + y_n = 0$

$\therefore x^2y_{n+2} + 2nx\, y_{n+1} + (n^2-n)\, y_n + 0 + xy_{n+1} + ny_n + y_n = 0$

$\therefore x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2-n+n+1)\, y_n = 0$

$\therefore x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)y_n = 0.$ Hence proved.

===

Ex. If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2 y_{n+2}+(2n+1)\, xy_{n+1}+ (n^2+1)y_n = 0$

Proof: Let $y = a\cos(\log x) + b\sin(\log x)$ ……..(1)

$\therefore y_1 = -a\sin(\log x)\left(\frac{1}{x}\right)+ b\cos(\log x)\left(\frac{1}{x}\right)$

$\therefore xy_1 = -a\sin(\log x) + b\cos(\log x)$ by multiplying x on both sides.

Again differentiating w.r.t. x, we get,

$xy_2 + y_1 = -a\cos(\log x)\left(\frac{1}{x}\right) - b\sin(\log x)\left(\frac{1}{x}\right)$

$\therefore x^2y_2 + xy_1 = -[\, a\cos(\log x) + b\sin(\log x)]$ by multiplying x on both sides.

i.e. $x^2y_2 + xy_1 = -y$ by (1)

i.e. $x^2y_2 + xy_1 + y = 0$

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}(x^2)+ n\, y_{n+1}(2x) + \frac{n(n-1)}{2} y_n(2) + 0 + [\, y_{n+1}(x) + n\, y_n(1) + 0\,] + y_n = 0$

$\therefore x^2y_{n+2} + 2nx\, y_{n+1} + (n^2-n)\, y_n + 0 + xy_{n+1} + ny_n + y_n = 0$

$\therefore x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2-n+n+1)\, y_n = 0$

$\therefore x^2y_{n+2} + (2n+1)\, xy_{n+1} + (n^2+1)y_n = 0$. Hence proved.

==

Ex. If $y = \sin^{-1}x$, prove that $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2y_n = 0$

Proof: Let $y = \sin^{-1}x$

$\therefore y_1 = \frac{1}{\sqrt{1-x^2}}$

$\therefore (\sqrt{1-x^2}\,)y_1 = 1$ by multiplying $\sqrt{1-x^2}$ on both sides.

$\therefore (1-x^2)(y_1)^2 = 1$ by squaring both sides.

Again differentiating w.r.t. x, we get,

$(1-x^2)(2y_1)y_2 – 2x(y_1)^2 = 0$

$\therefore (1-x2)y2 - xy_1 = 0$ by dividing $2y_1$ on both sides.

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}\, (1-x^2)+ n\, y_{n+1}(-2x) + \frac{n(n-1)}{2}\, y_n\, (-2) + 0 - [\, y_{n+1}(x) + n\, y_n(1)\, + 0\,] = 0$

$\therefore (1-x^2)y_{n+2} - 2nx\, y_{n+1} - (n^2-n)\, y_n + 0 - xy_{n+1} - ny_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n)\, y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Hence proved.

==

Ex. If $y = \sin(m\sin^{-1}x)$, prove that $(1-x^2)\, y_{n+2} = (2n+1)\, xy_{n+1} + (n^2 – m^2)\, y_n$

Proof: Let $y = \sin(m\sin^{-1}x)$ ………(1)

$\therefore y_1 = \cos(m\sin^{-1}x)\, \frac{m}{\sqrt{1-x^2}}$

$\therefore (\sqrt{1-x^2}\,)y_1 = m\cos(m\sin^{-1}x)$ by multiplying $\sqrt{1-x^2}$ on both sides.

$\therefore (1-x^2)(y_1)^2 = m^2\cos^2(m\sin^{-1}x)$ by squaring both sides.

$\therefore (1-x^2)(y_1)^2 = m^2\,[1- \sin^2(m\sin^{-1}x)]$

$\therefore (1-x^2)(y_1)^2 = m^2\,(1- y^2)$ by (1)

$\therefore (1-x^2)(y_1)^2 + m^2\, y^2 = m^2$

Again differentiating w.r.t. x, we get,

$(1-x^2)(2y_1)y_2 – 2x(y_1)^2 + m^2\,(2y)y_1 = 0$

$\therefore (1-x2)y2 - xy_1 + m^2y = 0$ by dividing $2y_1$ on both sides.

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}\, (1-x^2)+ n\, y_{n+1}(-2x) + \frac{n(n-1)}{2}\, y_n\, (-2) + 0 - [y_{n+1}(x) + n\, y_n(1)+0] + m^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - 2nx\, y_{n+1} - (n^2-n)\, y_n + 0 - xy_{n+1} - ny_n + m^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n- m^2)\, y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 - m^2)y_n = 0$

$\therefore (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$ Hence proved.

==

Ex. If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 + a^2)\, y_n = 0$

Proof: Let $y = e^{a\sin^{-1}x}$ ....(1)

$\therefore y_1 = e^{a\sin^{-1}x}\left(\frac{a}{\sqrt{1-x^2}}\right)$

$\therefore (\sqrt{1-x^2}\,)y_1 = ae^{a\sin^{-1}x}$ by multiplying $\sqrt{1-x^2}$ on both sides.

$\therefore (\sqrt{1-x^2}\,)y_1 = ay$ by (1)

$\therefore (1-x^2)(y_1)^2 = a^2y^2$ by squaring both sides.

$\therefore (1-x^2)(y_1)^2 - a^2y^2 = 0$

Again differentiating w.r.t. x, we get,

$(1-x^2)(2y_1)y_2 - 2x(y_1)^2 - a^2\,(2y)\,y_1 = 0$

$\therefore (1-x2)y_2 - xy_1 - a^2y = 0$ by dividing $2y_1$ on both sides.

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}\,(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2}\,y_n\,(-2) + 0 - [\,y_{n+1}(x) + n\,y_n(1) + 0] - a^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - 2nx\,y_{n+1} - (n^2-n)\,y_n + 0 - xy_{n+1} - ny_n - a^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+a^2)\,y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$ Hence proved.

==

Ex. If $y = e^{m\cos^{-1}x}$, prove that $(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2 + m^2)\, y_n = 0$

Proof: Let $y = e^{m\cos^{-1}x}$ ....(1)

$\therefore y_1 = e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right)$

$\therefore (\sqrt{1-x^2}\,)y_1 = -me^{m\cos^{-1}x}$ by multiplying $\sqrt{1-x^2}$ on both sides.

$\therefore (\sqrt{1-x^2}\,)y_1 = -my$ by (1)

$\therefore (1-x^2)(y_1)^2 = m^2y^2$ by squaring both sides.

$\therefore (1-x^2)(y_1)^2 - m^2y^2 = 0$

Again differentiating w.r.t. x, we get,

$(1-x^2)(2y_1)y_2 - 2x(y_1)^2 - m^2\,(2y)\,y_1 = 0$

$\therefore (1-x2)y_2 - xy_1 - m^2y = 0$ by dividing $2y_1$ on both sides.

By taking n^{th} derivative w.r.t. x of every term and using Leibnitz's theorem, we get,

$y_{n+2}\,(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2}\,y_n\,(-2) + 0 - [\,y_{n+1}(x) + n\,y_n(1) + 0] - m^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - 2nx\,y_{n+1} - (n^2-n)\,y_n + 0 - xy_{n+1} - ny_n - m^2y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+m^2)\, y_n = 0$

$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$ Hence proved.

UNIT-3: SUCCESSIVE DIFFERENTIATION

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1) The process of differentiating the same function again and again is called ……...

A) successive integration B) differentiation

C) successive differentiation D) integration

2) If $y = x^m$ then for $m > n$, y_n = ……….

A) $\frac{m!}{(m-n)!}x^{m-n}$ B) $n!$ C) 0 D) mx^{m-1}

3) If $y = x^n$ then y_n = ……….

A) 0 B) n C) $n!$ D) nx^{n-1}

4) If $y = x^m$ then for $m < n$, y_n = ……….

A) m B) 0 C) n D) n!

5) If $y = (ax+b)^m$ then for $m > n$, y_n = ……….

A) $\frac{m!a^n}{(m-n)!}(ax+b)^{m-n}$ B) $n!\,a^n$ C) 0 D) $m(ax+b)^{m-1}$

6) If $y = (ax+b)^n$ then y_n = ……….

A) n B) $n!\,a^n$ C) 0 D) $n(ax+b)^{n-1}$

7) If $y = (ax+b)^m$ then for $m < n$, y_n = ……….

A) $\frac{m!a^n}{(m-n)!}(ax+b)^{m-n}$ B) $n!\,a^n$ C) 0 D) n

8) If $y = e^{ax}$ then y_n = ………

A) e^{ax} B) ae^{ax} C) $a^n e^{ax}$ D) a^2e^{ax}

9) If $y = e^{5x}$ then y_3 = ………

A) $5e^{5x}$ B) $25e^{5x}$ C) $125e^{5x}$ D) e^{5x}

10) If $y = e^{ax+b}$ then y_n = ………

A) e^{ax+b} B) ae^{ax+b} C) e^{ax} D) a^ne^{ax+b}

11) If $y = \frac{1}{ax+b}$ then y_n = ………

A) $\frac{n!a^n}{(ax+b)^{n+1}}$ B) $\frac{(-1)^n n!a^n}{(ax+b)^n}$ C) $\frac{n!a^n}{(ax+b)^n}$ D) $\frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$

12) If $y = \frac{1}{x+a}$ then y_n = ……..

A) $\frac{(-1)^n n!}{(x+a)^{n+1}}$ B) $\frac{(-1)^n n!}{(x+a)^n}$ C) $\frac{(-1)^n n!}{(x+a)^{n-1}}$ D) None of these

13) If $y = \frac{1}{x-a}$ then y_n = ……..

A) $\frac{(-1)^n.n!}{(x-a)^{n+1}}$ B) $\frac{n!}{(x-a)^{n+1}}$ C) $\frac{1}{(x-a)^{n+1}}$ D) $\frac{-1}{(x-a)^2}$

14) If y = log(ax+b) then y_n = …………

A) $\frac{1}{(ax+b)^2}$ B) $\frac{(-1)^n}{(ax+b)^{n+1}}$ C) $\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$ D) $\frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}$

15) If y = log(x+5) then y_n = …………

A) $\frac{1}{x+5}$ B) $\frac{(-1)^{n-1}}{(x+5)^n}$ C) $\frac{(n-1)!}{(x+5)^n}$ D) $\frac{(-1)^{n-1}(n-1)!}{(x+5)^n}$

16) If y = sin(ax+b) then y_n = ……..

A) $a^n \sin(ax+b+\frac{n\pi}{2})$ B) asin(ax+b) C) $a^n \cos(ax+b+\frac{n\pi}{2})$ D) acos(ax+b)

17) If y = sin(3x) then that y_n = ……..

A) $3^n \sin(3x+\frac{n\pi}{2})$ B) 3sin(3x) C) $3^n \cos(3x+\frac{n\pi}{2})$ D) 3cos(3x)

18) If y = cos(ax+b), then y_n = ……

A) $a^n \cos(ax+b+\frac{n\pi}{2})$ B) asin(ax+b) C) $a^n \sin(ax+b+\frac{n\pi}{2})$ D) acos(ax+b)

19) If y = cos(3x) then y_2 = …………..

A) -3sin3x B) -9cos3x C) –sin3x D) –cos3x

20) If y = e^{ax} sin(bx+c) then y_n = ………..

A) $(\sqrt{a^2+b^2})^n e^{ax} \cos(bx+c)$ B) $(\sqrt{a^2+b^2})^n e^{ax} \cos(bx+c+n\tan^{-1}\frac{b}{a})$

C) $(\sqrt{a^2+b^2})^n e^{ax} \cos(bx+c)$ D) $(\sqrt{a^2+b^2})^n e^{ax} \sin(bx+c+n\tan^{-1}\frac{b}{a})$

21) If y = e^{ax} cos(bx+c) then y_n = ………..

A) $(\sqrt{a^2+b^2})^n e^{ax} \sin(bx+c)$ B) $(\sqrt{a^2+b^2})^n e^{ax} \sin(bx+c+n\tan^{-1}\frac{b}{a})$

C) $(\sqrt{a^2+b^2})^n e^{ax} \sin(bx+c)$ D) $(\sqrt{a^2+b^2})^n e^{ax} \cos(bx+c+n\tan^{-1}\frac{b}{a})$

22) By……..Theorem, if u and v are any two functions of x having derivatives of order n, then $(uv)_n = u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \ldots\ldots + {}^nC_{n-1} u_1 v_{n-1} + u v_n$

A) Leibnit'z B) Lagrange's C) Rolle's D) None of these

23) If U and V are functions of x and D ≡ $\frac{d}{dx}$ then D(UV) = ……….

A) (DU)V B) (DU)V+U(DV) C) U(DV) D) (DU)(DV)

24) n^{th} derivative of xe^x is……

A) e^x+xe^x B) xe^x+1 C) xe^x+ne^x D) e^x

25) n^{th} derivative of xsin5x is……

A) $5^n x\sin(5x+n\frac{\pi}{2})+n.5^{n-1}\sin[5x+(n-1)\frac{\pi}{2}]$ B) $5^n x\sin(5x+n\frac{\pi}{2})$

C) $x\sin(5x+n\frac{\pi}{2})+n\sin[5x+(n-1)\frac{\pi}{2}]$ D) $5^n x\sin5x$

==

UNIT-4: APPLICATION OF CALCULUS

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❖ Taylor's theorem with Lagrange's form of remainder after n terms:

If a function f(x) is defined on [a, a+h], such that

i) f(x), $f'(x)$, $f''(x)$, …………$f^{n-1}(x)$ are continuous in the interval [a, a+h],

ii) $f^{n}(x)$ exists in the interval (a, a+h), then there exists at least one real number θ between 0 to 1 such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

Proof: Consider a function

$$F(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \frac{(a+h-x)^3}{3!} f'''(x) \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{n!} A \quad \ldots\ldots(1)$$

Where A is a constant to be determined such that F(a) = F(a+h).

Now F(a+h) = F(a) gives

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \quad \ldots(2)$$

By given hypothesis i) f(x), $f'(x)$, $f''(x)$, …………$f^{n-1}(x)$ are continuous in the interval [a, a+h], ii) $f^{n}(x)$ exists in the interval (a, a+h)

∴ F(x) is i) continuous in [a, a+h],

ii) derivable in (a, a+h) with

$$F'(x) = f'(x) + (a+h-x) f''(x) - f'(x) + \frac{(a+h-x)^2}{2!} f'''(x) - (a+h-x) f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-2}}{(n-2)!} f^{n-1}(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} [f^n(x) - A]$$

and iii) F(a) = F(a+h)

Thus F(x) satisfies all conditions of Rolle's Theorem.

∴ Rolle's Theorem is applicable. i. e. there exists θ, $0 < \theta < 1$ such that

$F'(a+\theta h) = 0$

$$\therefore \frac{(a+h-a-\theta h)^{n-1}}{(n-1)!} [f^n(a+\theta h) - A] = 0$$

$$\therefore \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} [f^n(a+\theta h) - A] = 0$$

$$\therefore [f^n(a+\theta h) - A] = 0 \qquad \because \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} \neq 0$$

∴ $A = f^n(a+\theta h)$

Putting value of A in (2), we get,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) \ldots\ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

Hence proved.

==

❖ Maclaurin's theorem with Lagrange's form of remainder after n terms:

If a function f(x) is defined on [0, x], such that

i) $f(x), f'(x), f''(x), \ldots\ldots\ldots\ldots f^{n-1}(x)$ are continuous in the interval [0, x],

ii) $f^n(x)$ exists in the interval (0, x), then there exists at least one real number θ between 0 to 1 such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \ldots\ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

==

❖ **REMARK:**

1) $f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots\ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots$

is called Taylor's series expansion of f(x) in powers (x-a) or about point x = a.

2) $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots\ldots$

is called Maclaurin's series expansion of f(x) in powers x or about point x = 0.

3) $R_n = \frac{h^n}{n!} f^n(a+\theta h)$ or $R_n = \frac{x^n}{n!} f^n(\theta x)$ is called Lagrange's form of remainder after n^{th} term of Taylor's or Maclaurin's Theorem respectively.

==

Ex. Find the expansion of e^x in powers of x.

Solution: Maclauri's series expansion of f(x) in powers of x i. e. about point x = 0 is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \ldots\ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots \text{-------(1)}$$

Here $f(x) = e^x$

∴ $f^n(x) = e^x \ \forall\ n \in N$

∴ $f^n(0) = 1 \ \forall\ n \in N$

∴ $f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1, \ldots\ldots\ldots, f^n(0) = 1$

Putting these values in (1), we get,

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$ be the required expansion.

==

Ex. Find the expansion of sinx in powers of x.

Solution: Maclauri's theorem expansion of f(x) in powers of x i. e. about point x = 0 is

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f'''(0) + \frac{x^5}{5!}f'''(0) + \ldots$ ------(1)

Here f(x) = sinx

$\therefore f^n(x) = \sin(x+\frac{n\pi}{2})\ \forall\ n \in N$

$\therefore f^n(0) = \sin(\frac{n\pi}{2})\ \forall\ n \in N$

$\therefore f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{iv}(0) = 0, f^{v}(0) = 1$

Putting these values in (1), we get,

$\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \ldots\ldots\ldots$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots$ be the required expansion.

==

Ex. Find the expansion of cosx in powers of x.

Solution: Maclauri's theorem expansion of f(x) in powers of x i. e. about point x = 0 is

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f'''(0) + \frac{x^5}{5!}f'''(0) + \ldots$ ------(1)

Here f(x) = cosx

$\therefore f^n(x) = \cos(x+\frac{n\pi}{2})\ \forall\ n \in N$

$\therefore f^n(0) = \cos(\frac{n\pi}{2})\ \forall\ n \in N$

$\therefore f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{iv}(0) = 1, f^{v}(0) = 0$ and so on.

Putting these values in (1), we get,

$\cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \ldots\ldots\ldots$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots$ be the required expansion.

==

Ex. Find the expansion of $(1+x)^m$ for $m \in N$

Solution: Maclauri's theorem expansion of f(x) in powers of x i.e. about point x = 0 is

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots\ldots + \frac{x^m}{m!} f^m(0) + \ldots\ldots$ -------(1)

Here $f(x) = (1+x)^m$

$$\therefore f^n(x) = \begin{cases} \frac{m!}{(m-n)!}(1+x)^{m-n} & if\ m > n \\ m! & if\ m = n \\ 0 & if\ m < n \end{cases}$$

$$\therefore f^n(0) = \begin{cases} \frac{m!}{(m-n)!} & if\ m > n \\ m! & if\ m = n \\ 0 & if\ m < n \end{cases}$$

$\therefore$ f (0) = 1, f '(0) = m, f '' (0) = m(m-1), f ''' (0) = m(m-1)(m-2), ………, $f^m(0)$ = m!
and $f^n(0) = 0\ \forall\ n > m$.
Putting these values in (1), we get,

$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \ldots\ldots\ldots + \frac{m!x^m}{m!} + 0$

$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \ldots\ldots\ldots + x^m$

be the required expansion.

==

Ex. Find the expansion of log (1+x)

Solution: Maclauri's theorem expansion of f(x) in powers of x i.e. about point x = 0 is

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots \ldots$ -------(1)

Here f(x) = log (1+x)

$\therefore f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore f^n(0) = (-1)^{n-1}(n-1)!$

$\therefore$ f (0) = 0, f '(0) = 1, f '' (0) = -1, f ''' (0) = 2!, ………, $f^n(0) = (-1)^{n-1}(n-1)!$ and so on.
Putting these values in (1), we get,

$\log(1+x) = 0 + x - \frac{x^2}{2!} + \frac{2!x^3}{3!} + \ldots\ldots\ldots + \frac{(-1)^{n-1}(n-1)!x^n}{n!} + \ldots$

i. e. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots + \frac{(-1)^{n-1}x^n}{n} + \ldots$

be the required expansion.

==

Ex. Use Taylor's theorem to express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of (x-2)

Solution: By Taylor's theorem f(x) is expressed in powers of (x-2) i.e. about point x = 2 is

$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \ldots\ldots$ -------(1)

Here $f(x) = 2x^3 + 7x^2 + x - 6$ $\therefore$ f(2) = 16 + 28 + 2 – 6 = 40
$\therefore f'(x) = 6x^2 + 14x + 1$ $\therefore$ f '(2) = 24 + 28 +1 = 53
$\therefore$ f ''(x) = 12x + 14 $\therefore$ f '' (2) = 24 + 14 = 38
$\therefore$ f '''(x) = 12 $\therefore$ f '''(2) = 12
and all higher order derivatives are 0.
Putting these values in (1), we get,

$2x^3 + 7x^2 + x - 6 = 40 + 53(x-2) + \frac{38(x-2)^2}{2} + \frac{12(x-2)^3}{6} + 0$

$\therefore 2x^3 + 7x^2 + x - 6 = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$ be the required expansion.

==

Ex. Expand $2x^3 + 7x^2 + x - 1$ in powers of (x-2)

Solution: By Taylor's theorem f(x) is expressed in powers of (x-2) i.e. about point x = 2 is

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \dots\dots\text{-------}(1)$$

Here $f(x) = 2x^3 + 7x^2 + x - 1$ ∴ $f(2) = 16 + 28 + 2 - 1 = 45$

∴ $f'(x) = 6x^2 + 14x + 1$ ∴ $f'(2) = 24 + 28 + 1 = 53$

∴ $f''(x) = 12x + 14$ ∴ $f''(2) = 24 + 14 = 38$

∴ $f'''(x) = 12$ ∴ $f'''(2) = 12$

and all higher order derivatives are 0.

Putting these values in (1), we get,

$$2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + \frac{38(x-2)^2}{2} + \frac{12(x-2)^3}{6} + 0$$

∴ $2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$

be the required expansion.

==

Ex. Expand $x^4 - 3x^3 + 2x^2 - x + 1$ in powers of (x-3)

Solution: By Taylor's theorem f(x) is expressed in powers of (x-3) i.e. about point x = 3 is

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \frac{(x-3)^3}{3!}f'''(3) + \frac{(x-3)^4}{4!}f^{iv}(3) + \dots\text{----}(1)$$

Here $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$ ∴ $f(3) = 81 - 81 + 18 - 3 + 1 = 16$

∴ $f'(x) = 4x^3 - 9x^2 + 4x - 1$ ∴ $f'(3) = 108 - 81 + 12 - 1 = 38$

∴ $f''(x) = 12x^2 - 18x + 4$ ∴ $f''(3) = 108 - 54 + 4 = 58$

∴ $f'''(x) = 24x - 18$ ∴ $f'''(3) = 72 - 18 = 54$

∴ $f^{iv}(x) = 24$ ∴ $f^{iv}(3) = 24$

and all higher order derivatives are 0.

Putting these values in (1), we get,

$$x^4 - 3x^3 + 2x^2 - x + 1 = 16 + 38(x-3) + \frac{58(x-3)^2}{2} + \frac{54(x-3)^3}{6} + \frac{24(x-3)^4}{24} + 0$$

∴ $x^4 - 3x^3 + 2x^2 - x + 1 = 16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4$

be the required expansion.

==

Ex. Expand sinx in ascending powers of $(x-\frac{\pi}{2})$

Solution: By Taylor's theorem f(x) is expressed in powers of $(x-\frac{\pi}{2})$ i.e. about point $x = \frac{\pi}{2}$ is

$$f(x) = f(\tfrac{\pi}{2}) + (x-\tfrac{\pi}{2})f'\left(\tfrac{\pi}{2}\right) + \frac{(x-\frac{\pi}{2})^2}{2!}f''(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!}f'''(\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^4}{4!}f^{iv}(\tfrac{\pi}{2}) + \dots\text{----}(1)$$

Here f(x) = sinx ∴ $f(\frac{\pi}{2}) = 1$

∴ f '(x) = cosx ∴ $f'(\frac{\pi}{2}) = 0$

$\therefore$ f ''(x) = - sinx $\therefore$ f '' $(\frac{\pi}{2})$ = -1

$\therefore$ f '''(x) = - cosx $\therefore$ f '''$(\frac{\pi}{2})$ = 0

$\therefore$ f^{iv}(x) = sinx $\therefore$ $f^{iv}(\frac{\pi}{2})$ = 1

and so on.

Putting these values in (1), we get,

$$\sin x = 1 + (x-\frac{\pi}{2})(0) + \frac{(x-\frac{\pi}{2})^2}{2!}(-1) + \frac{(x-\frac{\pi}{2})^3}{3!}(0) + \frac{(x-\frac{\pi}{2})^4}{4!}(1) + \ldots$$

$$\therefore \sin x = 1 - \frac{1}{2!}(x-\frac{\pi}{2})^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4 + \ldots\ldots.$$

be the required expansion.

==

Ex. Prove that $e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \ldots\ldots\ldots$

Proof: Maclauri's theorem expansion of f(x) in powers of x i. e. about point x = 0 is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f'''(0) + \frac{x^5}{5!}f'''(0) + \ldots\ldots\text{------}(1)$$

Here f(x) = e^x cosx

$\therefore f^n(x) = (1^2+1^2)^{n/2} e^x \cos(x+n\tan^{-1}\frac{1}{1})\ \forall\ n \in N$

$\therefore f^n(0) = (2)^{n/2} \cos(\frac{n\pi}{4})\ \forall\ n \in N$

$\therefore$ f (0) = 1, f '(0) = 1, f '' (0) = 0, f ''' (0) = -2, $f^{iv}(0)$ = -4, $f^{v}(0)$ = -4 and so on.

Putting these values in (1), we get,

$$e^x \cos x = 1 + x + \frac{x^2}{2}(0) + \frac{x^3}{6}(-2) + \frac{x^4}{24}(-4) + \frac{x^5}{120}(-4) + \ldots\ldots\ldots$$

$$\therefore e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \ldots\ldots\ldots$$

Hence proved.

==

❖ Reduction Formula for $\int_0^{\pi/2} \sin^n x\, dx$:

1) Show that $\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$

Proof: Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$

$= \int_0^{\pi/2} \sin^{n-1} x.\sin x\, dx$

$= [\sin^{n-1}x(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2}(n-1)\sin^{n-2} x.\cos x(-\cos x)\, dx$ integrating by parts

$= -0 + 0 + (n-1)\int_0^{\pi/2} \sin^{n-2} x.\cos^2 x\, dx$

$= (n-1)\int_0^{\pi/2} \sin^{n-2} x.(1-\sin^2 x)\, dx$

$= (n-1) \int_0^{\pi/2} sin^{n-2} x\,dx - (n-1) \int_0^{\pi/2} sin^{n} x\,dx$

$\therefore I_n = (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n + (n-1) I_n = (n-1) I_{n-2}$

$\therefore nI_n = (n-1) I_{n-2}$

$\therefore I_n = \frac{n-1}{n} I_{n-2}$

i.e. $\int_0^{\pi/2} sin^n x\,dx = \frac{n-1}{n} \int_0^{\pi/2} sin^{n-2} x\,dx$

Hence proved.

==

❖ Reduction Formula for $\int_0^{\pi/2} cos^n x\,dx$:

2) Show that $\int_0^{\pi/2} cos^n x\,dx = \frac{n-1}{n} \int_0^{\pi/2} cos^{n-2} x\,dx$

Proof: Let $I_n = \int_0^{\pi/2} cos^n x\,dx$

$= \int_0^{\pi/2} cos^{n-1} x.\cos x\,dx$

Integrating by parts, we get,

$= [\cos^{n-1} x(\sin x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (n-1) cos^{n-2} x.(-\sin x)(\sin x)\,dx$

$= 0 - 0 + (n-1) \int_0^{\pi/2} cos^{n-2} x.\sin^2 x\,dx$

$= (n-1) \int_0^{\pi/2} cos^{n-2} x.(1-\cos^2 x)\,dx$

$= (n-1) \int_0^{\pi/2} cos^{n-2} x\,dx - (n-1) \int_0^{\pi/2} cos^{n} x\,dx$

$\therefore I_n = (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n + (n-1) I_n = (n-1) I_{n-2}$

$\therefore nI_n = (n-1) I_{n-2}$

$\therefore I_n = \frac{n-1}{n} I_{n-2}$

i.e. $\int_0^{\pi/2} cos^n x\,dx = \frac{n-1}{n} \int_0^{\pi/2} cos^{n-2} x\,dx$

Hence proved.

==

❖ Remark: $\int_0^{\pi/2} sin^n x\,dx$ or $\int_0^{\pi/2} cos^n x\,dx = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4} \dots\dots..\frac{1}{2}.\frac{\pi}{2} & if\ n\ is\ even \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4} \dots\dots..\frac{2}{3}.1 & if\ n\ is\ odd \end{cases}$

==

Ex. Evaluate $\int_0^{\pi/2} cos^7 x\, dx$

Sol. Let I = $\int_0^{\pi/2} cos^7 x\, dx$

$= \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1$ by reduction formula for n = 7 is odd.

$= \frac{16}{35}$

===

Ex. Evaluate $\int_0^{\pi/2} sin^8 x\, dx$

Sol. Let I = $\int_0^{\pi/2} sin^8 x\, dx$

$= \frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ by reduction formula for n = 8 is even.

$= \frac{35\pi}{256}$

===

Ex. Evaluate $\int_0^{\pi/2} sin^9 x\, dx$

Sol. Let I = $\int_0^{\pi/2} sin^9 x\, dx$

$= \frac{8}{9}.\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1$ by reduction formula for n = 9 is odd.

$= \frac{128}{315}$

===

Ex. Evaluate $\int_0^{\pi/2} cos^{10} x\, dx$

Sol. Let I = $\int_0^{\pi/2} cos^{10} x\, dx$

$= \frac{9}{10}.\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$ by reduction formula for n = 10 is even.

$= \frac{63\pi}{512}$

===

Ex. Evaluate $\int_0^{\pi/6} sin^6 3x\, dx$

Sol. Let I = $\int_0^{\pi/6} sin^6 3x\, dx$

Put 3x = t $\therefore$ 3dx = dt i.e. dx = $\frac{dt}{3}$

When x = 0 $\Rightarrow$ t = 0 & x = $\frac{\pi}{6}$ $\Rightarrow$ t = $\frac{\pi}{2}$

$\therefore$ I = $\int_0^{\pi/2} sin^6 t \frac{dt}{3}$

$= \frac{1}{3}[\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}]$ by reduction formula for n = 6 is even.

$= \frac{5\pi}{96}$

Ex. Evaluate $\int_0^{\pi} cos^7 \frac{x}{2}$ dx

Sol. Let I = $\int_0^{\pi} cos^7 \frac{x}{2}$ dx

Put $\frac{x}{2}$ = t $\therefore$ x = 2t i.e. dx = 2dt

When x = 0 $\Rightarrow$ t = 0 & x = π $\Rightarrow$ t = $\frac{\pi}{2}$

$\therefore$ I = $\int_0^{\pi/2} cos^7$ t (2dt)

$= 2\,[\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1]$ by reduction formula for n = 7 is odd.

$= \frac{32}{35}$

==

Ex. Evaluate $\int_0^{\pi/2} cos4x$ dx, using reduction formula for $\int_0^{\pi/2} cos^n x$ dx.

Sol. Let I = $\int_0^{\pi/2} cos\, 4x$ dx

$= \int_0^{\pi/2}[2cos^2 2x - 1]$ dx

$= \int_0^{\pi/2} 2cos^2 2x\, dx - \int_0^{\pi/2} dx$

$= \int_0^{\pi/2} 2(2cos^2 x - 1)^2\, dx - [x]_0^{\frac{\pi}{2}}$

$= \int_0^{\pi/2} 2(4cos^4 x - 4cos^2 x + 1)\, dx - [\frac{\pi}{2} - 0]$

$= 8\int_0^{\pi/2} cos^4 x\, dx - 8\int_0^{\pi/2} cos^2 x\, dx + 2\int_0^{\pi/2} dx - \frac{\pi}{2}$

$= 8[\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}] - 8[\frac{1}{2}.\frac{\pi}{2}] + 2\,[x]_0^{\frac{\pi}{2}} - \frac{\pi}{2}$ by reduction formula for even number.

$= \frac{3\pi}{2} - 2\pi + \pi - \frac{\pi}{2}$

$= 0$

==

Ex. Evaluate $\int_0^{a} \frac{x^6}{\sqrt{a^2-x^2}}$ dx

Sol. Let I = $\int_0^{a} \frac{x^6}{\sqrt{a^2-x^2}}$ dx

Put x = asint $\therefore$ dx = acost dt

When x = 0 $\Rightarrow$ t = 0 & x = a $\Rightarrow$ t = $\frac{\pi}{2}$

$\therefore$ I = $\int_0^{\pi/2} \frac{(asint)^6}{\sqrt{a^2-(asint)^2}}$ (acost dt)

$= a^6 \int_0^{\pi/2} sin^6 t dt$

$= a^6 [\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}]$ by reduction formula.

$= \frac{5\pi}{32} a^6$

Ex. Evaluate $\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$

Sol. Let I = $\int_0^1 \frac{x^6}{\sqrt{1-x^2}} dx$

Put x = sint ∴ dx = cost dt

When x = 0 ⟹ t = 0 & x = 1 ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{(sint)^6}{\sqrt{1-(sint)^2}}$ (costdt)

$= \int_0^{\pi/2} sin^6 t dt$

$= [\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}]$ by reduction formula.

$= \frac{5\pi}{32}$

===

Ex. Evaluate $\int_0^{\infty} \frac{dx}{(a^2+x^2)^3}$

Sol. Let I = $\int_0^{\infty} \frac{dx}{(a^2+x^2)^3}$

Put x = a tant ∴ dx = a sec^2 t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{asec^2 t\, dt}{(a^2+a^2tan^2 t)^3}$

$= \int_0^{\pi/2} \frac{asec^2 t\, dt}{(a^2+a^2tan^2 t)^3}$

$= \int_0^{\pi/2} \frac{asec^2 t\, dt}{a^6 sec^6 t}$

$= \frac{1}{a^5} \int_0^{\pi/2} \frac{dt}{sec^4 t}$

$= \frac{1}{a^5} \int_0^{\pi/2} cos^4 t\, dt$

$= \frac{1}{a^5} [\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}]$ by reduction formula.

$= \frac{3\pi}{16a^5}$

==

Ex. Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^5}$

Sol. Let I = $\int_0^\infty \frac{dx}{(1+x^2)^5}$

Put x = tant ∴ dx = sec² t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{sec^2 t\, dt}{(1+tan^2 t)^5}$

= $\int_0^{\pi/2} \frac{sec^2 t\, dt}{sec^{10} t}$

= $\int_0^{\pi/2} \frac{dt}{sec^8 t}$

= $\int_0^{\pi/2} cos^8$t dt

= $[\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}]$ by reduction formula.

= $\frac{35\pi}{256}$

==

Ex. Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^{7/2}}$

Sol. Let I = $\int_0^\infty \frac{dx}{(1+x^2)^{7/2}}$

Put x = tant ∴ dx = sec² t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{sec^2 t\, dt}{(1+tan^2 t)^{7/2}}$

= $\int_0^{\pi/2} \frac{sec^2 t\, dt}{sec^7 t}$

= $\int_0^{\pi/2} \frac{dt}{sec^5 t}$

= $\int_0^{\pi/2} cos^5$t dt

= $[\frac{4}{5}.\frac{2}{3}.1]$ by reduction formula.

= $\frac{8}{15}$

==

Ex. Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^{9/2}}$

Sol. Let I = $\int_0^{\infty} \frac{dx}{(1+x^2)^{9/2}}$

Put x = tant ∴ dx = $\sec^2$ t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{\sec^2 t\, dt}{(1+\tan^2 t)^{9/2}}$

= $\int_0^{\pi/2} \frac{\sec^2 t\, dt}{\sec^9 t}$

= $\int_0^{\pi/2} \frac{dt}{\sec^7 t}$

= $\int_0^{\pi/2} \cos^7 t\, dt$

= $[\frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1]$ by reduction formula.

= $\frac{16}{35}$

==

❖ Reduction Formula for $\int_0^{\pi/2} \sin^m x.\cos^n x\, dx$:

3) Show that $\int_0^{\pi/2} \sin^m x.\cos^n x\, dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x.\cos^{n-2} x\, dx$

Proof: Let $I_{m,n} = \int_0^{\pi/2} \sin^m x.\cos^n x\, dx$

$= \int_0^{\pi/2} \cos^{n-1} x.(\sin^m x.\cos x)\, dx$

Integrating by parts, we get,

$= [\cos^{n-1}x(\frac{\sin^{m+1}x}{m+1})]_0^{\frac{\pi}{2}} - \int_0^{\pi/2}(n-1)\cos^{n-2}x.(-\sin x)\,(\frac{\sin^{m+1}x}{m+1})\, dx$

$= 0 - 0 + \frac{n-1}{m+1}\int_0^{\pi/2} \sin^{m+2}x.\cos^{n-2}x\, dx$

$= \frac{n-1}{m+1}\int_0^{\pi/2}(1-\cos^2 x).\sin^m x.\cos^{n-2}x\, dx$

$= \frac{n-1}{m+1}\int_0^{\pi/2} \sin^m x.\cos^{n-2}x\, dx - \frac{n-1}{m+1}\int_0^{\pi/2} \sin^m x.\cos^n x\, dx$

$\therefore I_{m,n} = \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$

$\therefore I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{n-1}{m+1} I_{m,n-2}$

$\therefore (\frac{m+1+n-1}{m+1}) I_{m,n} = \frac{n-1}{m+1} I_{m,n-2}$

$\therefore (\frac{m+n}{m+1}) I_{m,n} = \frac{n-1}{m+1} I_{m,n-2}$

$\therefore I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$

i.e. $\int_0^{\pi/2} \sin^m x.\cos^n x\, dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x.\cos^{n-2} x\, dx$

Hence proved.

==

4) Show that $\int_0^{\pi/2} sin^m x.\,cos^n x\,dx = \frac{m-1}{m+n}\int_0^{\pi/2} sin^{m-2} x.\,cos^n x\,dx$

Proof: Let $I_{m,n} = \int_0^{\pi/2} sin^m x.\,cos^n x\,dx$

$= \int_0^{\pi/2} sin^{m-1} x.\,(cos^n x.\sin x)\,dx$

Integrating by parts, we get,

$= [\sin^{m-1}x(\frac{-cos^{n+1}x}{n+1})]_0^{\frac{\pi}{2}} - \int_0^{\pi/2}(m-1)sin^{m-2} x.(\cos x)\,(\frac{-cos^{n+1}x}{n+1})\,dx$

$= -0 + 0 + \frac{m-1}{n+1}\int_0^{\pi/2} sin^{m-2} x.\cos^{n+2}x\,dx$

$= \frac{m-1}{n+1}\int_0^{\pi/2} sin^{m-2} x.\cos^n x.(1-\sin^2 x)\,dx$

$= \frac{m-1}{n+1}\int_0^{\pi/2} sin^{m-2} x.\cos^n x\,dx - \frac{m-1}{n+1}\int_0^{\pi/2} sin^m x.\cos^n x\,dx$

$\therefore I_{m,n} = \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$

$\therefore I_{m,n} + \frac{m-1}{n+1} I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$

$\therefore (\frac{n+1+m-1}{n+1})I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$

$\therefore (\frac{m+n}{n+1})I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$

$\therefore I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$

i.e. $\int_0^{\pi/2} sin^m x.\,cos^n x\,dx = \frac{m-1}{m+n}\int_0^{\pi/2} sin^{m-2} x.\,cos^n x\,dx$

Hence proved.

==

❖ Remark $\int_0^{\pi/2} sin^m x.\,cos^n x\,dx = \frac{[(m-1)(m-3)......either\ 2\ or\ 1]\times[(n-1)(n-3)........either\ 2\ or\ 1]}{(m+n)(m+n-2)(m+n-4)................either\ 2\ or\ 1}$

and multiply by $\frac{\pi}{2}$ to R.H.S. if both m & n are even.

==

Ex. Evaluate $\int_0^{\pi/2} sin^4 x.\,cos^6 x\,dx$

Sol. Let I = $\int_0^{\pi/2} sin^4 x.\,cos^6 x\,dx$

$= \frac{(4-1)(4-3)\times(6-1)(6-3)(6-5)}{(4+6)(4+6-2)(4+6-4)(4+6-6)(4+6-8)}\frac{\pi}{2}$ by reduction formula for both m & n even.

$= \frac{3\times1\times5\times3\times1}{10\times8\times6\times4\times2}\times\frac{\pi}{2}$

$= \frac{3\pi}{512}$

==

Ex. Evaluate $\int_0^{\pi/2} sin^3 x. cos^6 x\, dx$

Sol. Let I = $\int_0^{\pi/2} sin^3 x. cos^6 x\, dx$

$= \frac{(3-1)\times(6-1)(6-3)(6-5)}{(3+6)(3+6-2)(3+6-4)(3+6-6)(3+6-8)}$ by reduction formula.

$= \frac{2\times5\times3\times1}{9\times7\times5\times3\times1}$

$= \frac{2}{63}$

==

Ex. Evaluate $\int_0^{\pi/2} sin^4 x. cos^5 x\, dx$

Sol. Let I = $\int_0^{\pi/2} sin^4 x. cos^5 x\, dx$

$= \frac{(4-1)(4-3)\times(5-1)(5-3)}{(4+5)(4+5-2)(4+5-4)(4+5-6)(4+5-8)}$ by reduction formula.

$= \frac{3\times1\times4\times2}{9\times7\times5\times3\times1}$

$= \frac{8}{315}$

==

Ex. Evaluate $\int_0^{\pi/2} sin^3 x. cos^5 x\, dx$

Sol. Let I = $\int_0^{\pi/2} sin^3 x. cos^5 x\, dx$

$= \frac{(3-1)\times(5-1)(5-3)}{(3+5)(3+5-2)(3+5-4)(3+5-6)}$ by reduction formula.

$= \frac{2\times4\times2}{8\times6\times4\times2}$

$= \frac{1}{24}$

==

Ex. Evaluate $\int_0^{1} x^{7/2} (1-x)^{5/2}\, dx$

Sol. Let I = $\int_0^{1} x^{7/2} (1-x)^{5/2}\, dx$

Put x = $\sin^2 t$ $\therefore$ dx = 2sintcost dt

When x = 0 $\Longrightarrow$ t = 0 & x = 1 $\Longrightarrow$ t = $\frac{\pi}{2}$

$\therefore$ I = $\int_0^{\pi/2} (sin^2 t)^{7/2} (1-\sin^2 t)^{5/2} 2\sin t\cos t\, dt$

$= 2\int_0^{\pi/2} (sin^7 t)(cos^5 t) \sin t\cos t\, dt$

$= 2\int_0^{\pi/2} sin^8 t. cos^6 t\, dt$

$= 2\times \frac{(8-1)(8-3)(8-5)(8-7)\times(6-1)(6-3)(6-5)}{(8+6)(8+6-2)(8+6-4)(8+6-6)(8+6-8)(8+6-10)(8+6-12)} \frac{\pi}{2}$

by reduction formula for both m & n even.

$= 2 \times \frac{7\times5\times3\times1\times5\times3\times1}{14\times12\times10\times8\times6\times4\times2} \times \frac{\pi}{2}$

$= \frac{5\pi}{2048}$

==

Ex. Evaluate $\int_0^1 x^4 \sqrt{1-x^2}\, dx$

Sol. Let I = $\int_0^1 x^4 \sqrt{1-x^2}\, dx$

Put x = sint ∴ dx = cost dt

When x = 0 ⟹ t = 0 & x = 1 ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} sin^4 t \sqrt{1-sin^2 t}$ cost dt

$= \int_0^{\pi/2} sin^4 t . cos^2 t\, dt$

$= \int_0^{\pi/2} sin^4 t . cos^2 t\, dt$

$= \frac{(4-1)(4-3)\times(2-1)}{(4+2)(4+2-2)(4+2-4)} \frac{\pi}{2}$ by reduction formula for both m & n even.

$= \frac{3\times1\times1}{6\times4\times2} \times \frac{\pi}{2}$

$= \frac{\pi}{32}$

==

Ex. Evaluate $\int_0^\infty \frac{x^4}{(1+x^2)^5} dx$

Sol. Let I = $\int_0^\infty \frac{x^4}{(1+x^2)^5} dx$

Put x = tant ∴ dx = sec^2 t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{tan^4 t}{(1+tan^2 t)^5} sec^2 t\, dt$

$= \int_0^{\pi/2} \frac{tan^4 t}{sec^{10} t} sec^2 t\, dt$

$= \int_0^{\pi/2} \frac{tan^4 t}{sec^8 t} dt$

$= \int_0^{\pi/2} \frac{sin^4 t}{cos^4 t} cos^8 t\, dt$

$= \int_0^{\pi/2} sin^4 t . cos^4 t\, dt$

$= \frac{(4-1)(4-3)\times(4-1)(4-3)}{(4+4)(4+4-2)(4+4-4)(4+4-6)} \frac{\pi}{2}$ by reduction formula for both m & n even.

$= \frac{3\times1\times3\times1}{8\times6\times4\times2} \times \frac{\pi}{2}$

$= \frac{3\pi}{256}$

==

Ex. Evaluate $\int_0^\infty \frac{x^4}{(1+x^2)^4}$ dx

Sol. Let I = $\int_0^\infty \frac{x^4}{(1+x^2)^4}$ dx

Put x = tant ∴ dx = sec² t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

$\therefore\ I = \int_0^{\pi/2} \frac{tan^4 t}{(1+tan^2 t)^4} sec^2 t\, dt$

$= \int_0^{\pi/2} \frac{tan^4 t}{sec^8 t} sec^2 t\, dt$

$= \int_0^{\pi/2} \frac{tan^4 t}{sec^6 t} dt$

$= \int_0^{\pi/2} \frac{sin^4 t}{cos^4 t} cos^6 t\, dt$

$= \int_0^{\pi/2} sin^4 t . cos^2 t\, dt$

$= \frac{(4-1)(4-3)\times(2-1)}{(4+2)(4+2-2)(4+2-4)} \frac{\pi}{2}$ by reduction formula for both m & n even.

$= \frac{3\times1\times1}{6\times4\times2} \times \frac{\pi}{2}$

$= \frac{\pi}{32}$

==

Ex. Evaluate $\int_0^\infty \frac{x^2}{(1+x^2)^{9/2}}$ dx

Sol. Let I = $\int_0^\infty \frac{x^2}{(1+x^2)^{9/2}}$ dx

Put x = tant ∴ dx = sec² t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

$\therefore\ I = \int_0^{\pi/2} \frac{tan^2 t}{(1+tan^2 t)^{9/2}} sec^2 t\, dt$

$= \int_0^{\pi/2} \frac{tan^2 t}{sec^9 t} sec^2 t\, dt$

$= \int_0^{\pi/2} \frac{tan^2 t}{sec^7 t} dt$

$= \int_0^{\pi/2} \frac{sin^2 t}{cos^2 t} cos^7 t\, dt$

$= \int_0^{\pi/2} sin^2 t . cos^5 t\, dt$

$= \frac{(2-1)\times(5-1)(5-3)}{(2+5)(2+5-2)(2+5-4)(2+5-6)}$ by reduction formula.

$= \frac{1\times4\times2}{7\times5\times3\times1}$

$= \frac{8}{105}$

==

Ex. Evaluate $\int_0^{\infty} \frac{x^5}{(1+x^2)^{9/2}}$ dx

Sol. Let I = $\int_0^{\infty} \frac{x^5}{(1+x^2)^{9/2}}$ dx

Put x = tant ∴ dx = sec² t dt

When x = 0 ⟹ t = 0 & x = ∞ ⟹ t = $\frac{\pi}{2}$

∴ I = $\int_0^{\pi/2} \frac{tan^5 t}{(1+tan^2 t)^{9/2}}$ sec² t dt

$= \int_0^{\pi/2} \frac{tan^5 t}{sec^9 t}$ sec² t dt

$= \int_0^{\pi/2} \frac{tan^5 t}{sec^7 t}$ dt

$= \int_0^{\pi/2} \frac{sin^5 t}{cos^5 t}\ cos^7 t$ dt

$= \int_0^{\pi/2} sin^5 t . cos^2$ t dt

$= \frac{(5-1)(5-3)\times(2-1)}{(5+2)(5+2-2)(5+2-4)(5+2-6)}$ by reduction formula.

$= \frac{4\times2\times1}{7\times5\times3\times1}$

$= \frac{8}{105}$

==

❖ Reduction formula for $\int \frac{sinnx}{sinx}$ dx (n > 1)

❖ Show that $\int \frac{sinnx}{sinx}$ dx = $\frac{2\sin(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{sinx}$ dx

Proof : As sinC – sinD = 2cos($\frac{C+D}{2}$) sin($\frac{C-D}{2}$)

∴ sinnx – sin(n-2)x = 2cos($\frac{nx+nx-2x}{2}$) sin($\frac{nx-nx+2x}{2}$) = = 2cos(n-1)x sinx

∴ $\frac{sinnx}{sinx}$ - $\frac{\sin(n-2)x}{sinx}$ = 2cos(n-1)x

Integrating both sides, we get,

$\int \frac{sinnx}{sinx}$ dx - $\int \frac{\sin(n-2)x}{sinx}$ dx = $\frac{2\sin(n-1)x}{(n-1)}$

i. e. $\int \frac{sinnx}{sinx}$ dx = $\frac{2\sin(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{sinx}$ dx

Hence proved.

==

Ex. Show that $\int \frac{sin8x}{sinx}$ dx = 2[$\frac{\sin 7x}{7} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3}$ + sinx]

Hence show that $\int_0^{\pi} \frac{sin8x}{sinx}$ dx = 0

Proof : Let $I_n = \int \frac{sinnx}{sinx}$ dx , then by reduction formula

$I_n = \frac{2\sin(n-1)x}{(n-1)} + I_{n-2}$ ....(1)

Putting n = 8, 6, 4, 2 successively in (1), we get,

$I_8 = \frac{2\sin 7x}{7} + I_6$

$I_6 = \frac{2\sin 5x}{5} + I_4$

$I_4 = \frac{2\sin 3x}{3} + I_2$

$I_2 = 2sinx + I_0$, where $I_0 = 0$.

By back substitution, we have

$I_8 = 2[\frac{\sin 7x}{7} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3} + \text{sinx}]$

$\therefore \int_0^{\pi} \frac{sin8x}{sinx}$ dx = 2[$\frac{\sin 7x}{7} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3}$ + sinx$]_0^{\pi}$

= 2[$\frac{\sin 7\pi}{7} + \frac{\sin 5\pi}{5} + \frac{\sin 3\pi}{3} + \sin\pi$ – 0]

= 2 [0-0]

$\therefore \int_0^{\pi} \frac{sin8x}{sinx}$ dx = 0

Hence proved.

==

Ex. Show that $\int \frac{sin10x}{sinx}$ dx = 2[$\frac{\sin 9x}{9} + \frac{\sin 7x}{7} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3}$ + sinx]

Hence show that $\int_0^{\pi} \frac{sin10x}{sinx}$ dx = 0

Proof : Let $I_n = \int \frac{sinnx}{sinx}$ dx , then by reduction formula

$I_n = \frac{2\sin(n-1)x}{(n-1)} + I_{n-2}$ ....(1)

Putting n = 10, 8, 6, 4, 2 successively in (1), we get,

$I_{10} = \frac{2\sin 9x}{9} + I_8$

$I_8 = \frac{2\sin 7x}{7} + I_6$

$I_6 = \frac{2\sin 5x}{5} + I_4$

$I_4 = \frac{2\sin 3x}{3} + I_2$

$I_2 = 2sinx + I_0$, where $I_0 = 0$.

By back substitution, we have

$I_{10} = 2[\frac{\sin 9x}{9} + \frac{\sin 7x}{7} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3} + \sin x]$

$\therefore \int_0^{\pi} \frac{sin10x}{sinx}\,dx = 2[\frac{\sin 9x}{9} + \frac{\sin 7x}{7} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3} + \sin x]_0^{\pi}$

$= 2[\frac{\sin 9\pi}{9} + \frac{\sin 7\pi}{7} + \frac{\sin 5\pi}{5} + \frac{\sin 3\pi}{3} + \sin \pi - 0]$

$= 2\ [0\text{-}0]$

$\therefore \int_0^{\pi} \frac{sin10x}{sinx}\,dx = 0.$ Hence proved.

==

Ex. Show that $\int \frac{sin7x}{sinx}\,dx = 2[\frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \frac{x}{2}]$

Hence show that $\int_0^{\pi} \frac{sin7x}{sinx}\,dx = \pi$

Proof : Let $I_n = \int \frac{sinnx}{sinx}\,dx$, then by reduction formula

$I_n = \frac{2\sin(n-1)x}{(n-1)} + I_{n-2}$ ....(1)

Putting n = 7, 5, 3 successively in (1), we get,

$I_7 = \frac{2\sin 6x}{6} + I_5$

$I_5 = \frac{2\sin 4x}{4} + I_3$

$I_3 = \frac{2\sin 2x}{2} + I_1$, where $I_1 = \int \frac{sinx}{sinx}\,dx = \int 1\,dx = x$

By back substitution, we have

$I_7 = [\frac{2\sin 6x}{6} + \frac{2\sin 4x}{4} + \frac{2\sin 2x}{2} + x]$

$\therefore \int \frac{sin7x}{sinx}\,dx = 2\left[\frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \frac{x}{2}\right]$

$\therefore \int_0^{\pi} \frac{sin7x}{sinx}\,dx = 2[\frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \frac{x}{2}]_0^{\pi}$

$= 2[\frac{\sin 6\pi}{6} + \frac{\sin 4\pi}{4} + \frac{\sin 2\pi}{2} + \frac{\pi}{2} - 0]$

$= 2\ [\frac{\pi}{2}\text{-}0]$

$\therefore \int_0^{\pi} \frac{sin7x}{sinx}\,dx = \pi.$ Hence proved.

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Ex. Show that $\int \frac{sin9x}{sinx}\,dx = 2[\frac{\sin 8x}{8} + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \frac{x}{2}]$

Hence show that $\int_0^{\pi} \frac{sin9x}{sinx}\,dx = \pi$

Proof : Let $I_n = \int \frac{sinnx}{sinx}\,dx$, then by reduction formula

$I_n = \frac{2\sin(n-1)x}{(n-1)} + I_{n-2}$ ....(1)

Putting n = 9, 7, 5, 3 successively in (1), we get,

$I_9 = \frac{2\sin 8x}{8} + I_7$

$I_7 = \frac{2\sin 6x}{6} + I_5$

$I_5 = \frac{2\sin 4x}{4} + I_3$

$I_3 = \frac{2\sin 2x}{2} + I_1$, where $I_1 = \int \frac{sinx}{sinx}$ dx = $\int 1$ dx = x

By back substitution, we have

$I_9 = [\frac{2\sin 8x}{8} + \frac{2\sin 6x}{6} + \frac{2\sin 4x}{4} + \frac{2\sin 2x}{2} + x]$

$\therefore \int \frac{sin9x}{sinx} dx = 2\left[\frac{\sin 8x}{8} + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \frac{x}{2}\right]$

$\therefore \int_0^{\pi} \frac{sin9x}{sinx} dx = 2[\frac{\sin 8x}{8} + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + \frac{x}{2}]_0^{\pi}$

$= 2[\frac{\sin 8\pi}{8} + \frac{\sin 6\pi}{6} + \frac{\sin 4\pi}{4} + \frac{\sin 2\pi}{2} + \frac{\pi}{2} - 0]$

$= 2\ [\frac{\pi}{2}-0]$

$\therefore \int_0^{\pi} \frac{sin9x}{sinx} dx = \pi.$ Hence proved.

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❖ Remark; $\int_0^{\pi} \frac{sinnx}{sinx} dx = \begin{cases} 0 & if\ n\ is\ even \\ \pi & if\ n\ is\ odd \end{cases}$

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UNIT-4: APPLICATION OF CALCULUS

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1) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots$.. is Maclaurin's series expansion of …..

A) sinx B) e^x C) cosx D) $\frac{1}{1-x}$

2) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$….. is Maclaurin's series expansion of …..

A) sinx B) e^x C) cosx D) $\frac{1}{1-x}$

3) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$….. is Maclaurin's series expansion of …..

A) sinx B) e^x C) cosx D) $\frac{1}{1-x}$

4) $1 + x + x^2 + x^3 + \ldots\ldots + x^n + \ldots$.. is Maclaurin's series expansion of …….

A) sinx B) e^x C) cosx D) $\frac{1}{1-x}$

5) $1 - x + x^2 - x^3 + \ldots\ldots + (-1)^n x^n + \ldots$.. is Maclaurin's series expansion of …..

A) $\frac{1}{1+x}$ B) $\frac{1}{1-x}$ C) tanx D) secx

6) $1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \ldots$… …is Maclaurin's series expansion of …..

A) e^x cosx B) e^x sinx C) tanx D) secx

7) By reduction formula $\int_0^{\frac{\pi}{2}} sin^n x$ dx = ……….

A) $\int_0^{\frac{\pi}{2}} sin^{n-2}x$ dx B) $\int_0^{\frac{\pi}{2}} sin^{n-1}x$ dx

C) $\frac{n-1}{n}\int_0^{\frac{\pi}{2}} sin^{n-2}x$ dx D) $\frac{n}{n-1}\int_0^{\frac{\pi}{2}} sin^{n-2}x$ dx

8) $\int_0^{\frac{\pi}{2}} sin^5 x$ dx = ……….

A) 0 B) $\frac{8}{15}$ C) 5π D) $\frac{8\pi}{15}$

9) $\int_0^{\frac{\pi}{2}} sin^6 x$ dx = ……….

A) $\frac{5\pi}{32}$ B) $\frac{8\pi}{15}$ C) $\frac{5}{32}$ D) $\frac{8}{15}$

10) By Reduction Formula for $\int_0^{\pi/2} cos^n$ x dx:

A) $\int_0^{\frac{\pi}{2}} cos^{n-2}x$ dx B) $\int_0^{\frac{\pi}{2}} cos^{n-1}x$ dx

C) $\frac{n-1}{n}\int_0^{\frac{\pi}{2}} cos^{n-2}x$ dx D) $\frac{n}{n-1}\int_0^{\frac{\pi}{2}} cos^{n-2}x$ dx

11) $\int_0^{\pi/2} cos^7$ x dx = …….

A) $\frac{5\pi}{32}$ B) $\frac{8\pi}{15}$ C) $\frac{16}{35}$ D) $\frac{8}{15}$

12) $\int_0^{\pi/2} sin^8$ x dx = ……..

A) $\frac{35\pi}{256}$ B) $\frac{8\pi}{15}$ C) $\frac{16}{35}$ D) $\frac{35}{256}$

13) $\int_0^{\pi/2} sin^9$ x dx

A) $\frac{35\pi}{256}$ B) $\frac{128\pi}{315}$ C) $\frac{128}{315}$ D) $\frac{35}{256}$

14) $\int_0^{\pi/2} cos^{10}$ x dx

A) $\frac{63\pi}{512}$ B) $\frac{128\pi}{315}$ C) $\frac{128}{315}$ D) $\frac{63}{512}$

15) $\int_0^{\pi/6} sin^6$ 3x dx

A) $\frac{5\pi}{32}$ B) $\frac{5\pi}{96}$ C) $\frac{16}{35}$ D) $\frac{8}{15}$

16) $\int_0^{\pi} cos^7\frac{x}{2}$ dx

A) $\frac{5\pi}{32}$ B) $\frac{5\pi}{96}$ C) $\frac{32}{35}$ D) $\frac{8}{15}$

17) By Reduction Formula $\int_0^{\pi/2} sin^m x. cos^n x$ dx = …….

A) $\int_0^{\pi/2} sin^m x. cos^{n-2} x$ dx B) $\frac{m-1}{m+n}\int_0^{\pi/2} sin^{m-2} x. cos^n x$ dx

C) $\frac{m+1}{m+n}\int_0^{\pi/2} sin^{m-2} x. cos^n x$ dx D) $\frac{n+1}{m+n}\int_0^{\pi/2} sin^{m-2} x. cos^n x$ dx

18) $\int_0^{\pi/2} sin^4 x. cos^6 x$ dx = …….

A) $\frac{3\pi}{512}$ B) $\frac{5\pi}{496}$ C) $\frac{32}{512}$ D) $\frac{1}{256}$

19) $\int_0^{\pi/2} sin^3 x. cos^6 x$ dx = …….

A) $\frac{3\pi}{512}$ B) $\frac{5\pi}{496}$ C) $\frac{2}{63}$ D) $\frac{1}{256}$

20) $\int_0^{\pi/2} sin^4 x. cos^5 x$ dx = ……..

A) $\frac{3\pi}{512}$ B) $\frac{8}{315}$ C) $\frac{2}{630}$ D) $\frac{1}{256}$

21) $\int_0^{\pi/2} sin^3 x. cos^5 x$ dx

A) $\frac{3\pi}{512}$ B) $\frac{8}{315}$ C) $\frac{2}{630}$ D) $\frac{1}{24}$

22) By Reduction formula $\int \frac{sinnx}{sinx}$ dx = …….

A) $\frac{2\sin(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{sinx}$ dx B) $\frac{\sin(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{sinx}$ dx

C) $\frac{2\cos(n-1)x}{(n-1)} + \int \frac{\sin(n-2)x}{sinx}$ dx D) $\frac{cos\,x}{(n-1)} + \int \frac{\sin(n-2)x}{sinx}$ dx

23) $\int_0^{\pi} \frac{sin7x}{sinx}$ dx = …….

A) π B) 0 C) $\frac{\pi}{2}$ D) 2π

23) $\int_0^{\pi} \frac{sin8x}{sinx}$ dx = …….

A) π B) 0 C) $\frac{\pi}{2}$ D) 2π

23) $\int_0^{\pi} \frac{sin9x}{sinx}$ dx = …….

A) π B) 0 C) $\frac{\pi}{2}$ D) 2π

23) $\int_0^{\pi} \frac{sin10x}{sinx}$ dx = …….

A) π B) 0 C) $\frac{\pi}{2}$ D) 2π

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."