Pimpalner Education Society's

Karm. A. M. Patil Arts, Commerce and Kai. Annasaheb N. K. Patil Science Senior College Pimpalner, Tal.- Sakri, Dist.- Dhule.

CLASS NOTES

CLASS: F.Y.B.SC SEM.-I

SUBJECT: MTH-101: MATRIX ALGEBRA

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MTH 101: MATRIX ALGEBRA

UNIT-I: Adjoint and Inverse of a matrix. **Marks 15, Hours 08**

a. Elementary operations on matrices
b. Adjoint of a matrix
c. Inverse of a matrix.
c. Existence & uniqueness theorem of inverse of a matrix.
d. Properties of inverse of a matrix,

UNIT-II: Rank of Matrix. **Marks 15, Hours 08**

a. Elementary matrices.
b. Rank and normal form of a matrix.
c. Reduction of a matrix to its normal form.
d. Rank of product of two matrices.

UNIT-III: System of Linear Equations and Eigen Values. **Marks 15, Hours 08**

a. A homogeneous and non-homogeneous system of linear equations.
b. Consistency of system of linear equations.
c. Application of matrices to solve the system of linear equations.
b. Eigen Values and Eigen Vectors of Matrices, Characteristic equation of a matrix.
c. Cayley Hamilton theorem (statement only) and its use to find the inverse of a matrix.

UNIT-IV: Orthogonal Matrices and Quadratic Forms **Marks 15, Hours 08**

a. Orthogonal Matrices.
b. Properties of Orthogonal Matrices.
c. Quadratic forms.
d. Matrix Representation.
e. Elementary congruent transformations, Diagonal form of a quadratic forms, Canonical forms.

REFERENCE BOOKS:

1. Matrix and Linear Algebra, by K. B. Datta, Prentice Hall of India Pvt. Ltd. New Delhi,2000.
2. A Text Book of Matrices, by Shanti Narayan, S. Chand Limited, 2010.
3. Schaum's Outline of Theory and Problems of MATRICES, by Richord Bronson, McGraw-Hill, New York, 1989.

Learning Outcomes:

After successful completion of this course the student will be able to:
a) understand concepts on matrix operations and rank of the matrix.
b) understand use of matrix for solving the system of linear equations.
c) understand basic knowledge of the Eigen values and Eigen vectors.
d) apply Cayley-Hamilton theorem to find the inverse of the matrix.
e) know the matrix transformation and its applications in rotation, reflection, translation.

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UNIT-I: ADJOINT AND INVERSE OF A MATRIX

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INTRODUCTION:

In this section we shall study “Matrix Algebra”. It is an important branch of Mathematics because it finds application in Physics, Statistics, Psychology, Engineering, Computer Sciences etc. The credit of formulation and development of “Matrix Theory” goes to great Mathematicians Hamilton, Cayley and Sylvester.

Matrix: An arrangement of mn numbers in m rows and n columns and enclosed in a square bracket is called a matrix of order mxn (read as m by n).

Note: i) The numbers occurring in a matrix are called elements of matrix.

ii) If a_{ij} denote an element in the i^{th} row and j^{th} column of a matrix A of order mxn then matrix A is written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{mxn}$$

Types of Matrices:

Row Matrix: A matrix containing only one row is called a row matrix.

i.e. A matrix $A = [a_{ij}]_{mxn}$ is called a row matrix if m = 1 and n > 1.

Column Matrix: A matrix containing only one column is called a column matrix.

i.e. A matrix $A = [a_{ij}]_{mxn}$ is called a column matrix if m > 1 and n = 1.

Zero Matrix or Null Matrix: A matrix whose all elements are zero is called a zero matrix or a null matrix.

i.e. A matrix $A = [a_{ij}]_{mxn}$ is called a zero matrix or a null n matrix if $a_{ij} = 0\ \forall$ i and j.

Square Matrix: A matrix containing same number of rows and columns is called a square matrix.

i.e. A matrix $A = [a_{ij}]_{mxn}$ is called a square matrix if m = n.

Diagonal Matrix: A square matrix in which all non-diagonal elements are zero is called a diagonal matrix.

i.e. A square matrix $A = [a_{ij}]_{nxn}$ is called a diagonal matrix if $a_{ij} = 0\ \forall\ i \neq j$.

Scalar Matrix: A diagonal matrix in which all diagonal elements are equal is called a scalar matrix.

i.e. A square matrix $A = [a_{ij}]_{nxn}$ is called a scalar matrix with scalar k if $a_{ij} = 0\ \forall\ i \neq j$ and $a_{ij} = 0\ \forall\ i = j$.

Unit Matrix or Identity Matrix: A diagonal matrix in which all diagonal elements are 1 is called an unit matrix or identity matrix.

i.e. A square matrix $A = [a_{ij}]_{nxn}$ is called an unit matrix or identity matrix if $a_{ij} = 0\ \forall\ i \neq j$ and $a_{ij} = 1\ \forall\ i = j$.

Transpose of a Matrix: A matrix obtained from given matrix A by interchanging its rows and columns is called the transpose of matrix A. Denoted by A'.

i.e. A matrix $A' = [a_{ij}]'_{mxn} = [a_{ji}]_{nxm}$ is called the transpose of matrix A.

Symmetric Matrix: A square matrix $A = [a_{ij}]_{nxn}$ is called a symmetric matrix if $A' = A$ i.e. if $a_{ij} = a_{ji}\ \forall$ i, j.

Skew Symmetric Matrix: A square matrix A is called a skew symmetric matrix if $A' = -A$ i.e. if $a_{ij} = -a_{ji}\ \forall$ i, j.

Upper Triangular Matrix: A square matrix in which all the elements below the diagonal are zero is called an upper triangular matrix.

i.e. A square matrix $A = [a_{ij}]_{nxn}$ is called an upper triangular matrix if $a_{ij} = 0\ \forall\ i > j$.

Lower Triangular Matrix: A square matrix in which all the elements above the diagonal are zero is called a lower triangular matrix.

i.e. A square matrix $A = [a_{ij}]_{nxn}$ is called a lower triangular matrix if $a_{ij} = 0\ \forall\ i < j$.

Triangular Matrix: A square matrix which is either upper triangular matrix or lower triangular matrix is called a triangular matrix.

Singular Matrix: A square matrix A is called a singular matrix if $|A| = 0$

Non-singular Matrix: A square matrix A is called a non-singular matrix if $|A| \neq 0$

Note: i) If k is scalar and A is any matrix then $(kA)' = kA'$

ii) If A and B are matrices of same order then $(A \pm B)' = A' \pm B'$

iii) If A and B are matrices such that product AB is defined then $(AB)' = B'A'$

iv) If A is any square matrix then $|A| = |A'|$

v) If k is any non-zero scalar and A is any square matrix of order n then $|kA| = k^n\ |A|$

vi) If A and B are square matrices of same order then $|AB| = |A||B|$

vii) If A is any square matrix containing zero row or zero column or any two rows or any two columns are identical then $|A| = 0$.

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Ex. Show that a matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ is non-singular matrix.

Proof: Consider $|A| = \begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$

$= 1\,(-2-3) - 3(4-0) + 0$

$= -5-12$

$|A| = -17 \neq 0$

Hence A is a non-singular matrix is proved.

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Ex. If A and B are symmetric matrices then prove that

a) $(AB + BA)$ is a symmetric matrix.

b) $(AB - BA)$ is a skew symmetric matrix.

c) A^m is a symmetric matrix, where m is any positive integer.

Proof: Given A and B are symmetric matrices i.e. $A' = A$ and $B' = B$

a) Consider $(AB + BA)' = (AB)' + (BA)'$

$= B'A' + A'B'$
$= BA + AB$
$= AB + BA$

Hence $(AB + BA)$ is a symmetric matrix is proved.

b) Consider $(AB - BA)' = (AB)' - (BA)'$
$= B'A' - A'B'$
$= BA - AB$
$= -(AB - BA)$

Hence $(AB - BA)$ is a skew symmetric matrix is proved.

c) Consider $(A^m)' = (AA \ldots\ldots A)'$ (m times)
$= A'A' \ldots\ldots\ldots A'$ (m times)
$= AA \ldots\ldots A$ (m times)
$= A^m$

Hence A^m is a symmetric matrix is proved.

Ex. If A is any symmetric matrix then prove that

a) $A + A'$ is a symmetric matrix.
b) $A - A'$ is a skew symmetric matrix.
c) AA' and $A'A$ are both symmetric.

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Minor of an element of a matrix:

Let $A = [a_{ij}]$ be a square matrix of order n, then the determinant obtained by deleting i^{th} row and j^{th} column from $|A|$ is called minor M_{ij} of an element a_{ij}.

Cofactor of an element of a matrix:

Let $A = [a_{ij}]$ be a square matrix of order n, then $A_{ij} = (-1)^{i+j} M_{ij}$ is called cofactor of an element a_{ij}.

Adjoint of a matrix:

Let $A = [a_{ij}]$ be a square matrix of order n, then the transpose of the matrix of cofactors $M = [A_{ij}]_{n\times n}$, is called adjoint of A. It is denoted by adj A.

Thus adj $A = M' = [A_{ij}]'_{n\times n}$.

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Ex. If $A = \begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then find minors and cofactors of a_{11}, a_{21} and a_{22}.

Solution: $M_{11} = |-5| = -5$, $M_{21} = |3| = 3$, $M_{22} = |-2| = -2$,

$\therefore A_{11} = (-1)^{1+1} M_{11} = 1 \times (-5) = -5$

$A_{21} = (-1)^{2+1} M_{21} = (-1) \times (3) = -3$

$A_{22} = (-1)^{2+2} M_{22} = 1 \times (-2) = -2$

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Ex. If $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} = [a_{ij}]_{3\times 3}$ then find minors and cofactors of a_{11}, a_{23} and a_{32}.

Solution: $M_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$,

$M_{23} = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1 - 0 = 1,$

$M_{32} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3,$

$\therefore A_{11} = (-1)^{1+1} M_{11} = 1 \times (-1) = -1$

$A_{23} = (-1)^{2+3} M_{23} = (-1) \times (1) = -1$

$A_{32} = (-1)^{3+2} M_{32} = (-1) \times (3) = -3$

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Ex. Find adjA, where $A = \begin{bmatrix} -2 & -5 \\ 7 & 1 \end{bmatrix}$

Solution: The matrix of cofactors of elements of A is

$M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$= \begin{bmatrix} M_{11} & -M_{12} \\ -M_{21} & M_{22} \end{bmatrix}$

$= \begin{bmatrix} 1 & -7 \\ 5 & -2 \end{bmatrix}$

$\therefore \text{adj}A = M' = \begin{bmatrix} 1 & 5 \\ -7 & -2 \end{bmatrix}$

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Note: If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\text{adj}A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

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Ex. Find adjA, where $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

Solution: The matrix of cofactors of elements of A is

$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$

$= \begin{bmatrix} (5+0) & -(10-0) & (-4-4) \\ -(10-6) & (-5+12) & -(2-8) \\ (0+3) & -(0+6) & (-1-4) \end{bmatrix} = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}$

$\therefore \text{adj}A = M' = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$

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Note: i) The sum of the products of the elements in any row (or column) of a determinant and their corresponding cofactors is equal to the value of the determinant.
ii) The sum of the products of the elements in any row (or column) of a determinant and the cofactors of the corresponding elements in any other row (or column) is zero.

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Theorem: For any square matrix A, A(adjA) = (adjA)A = $|A|$ I

Proof: Let $A = [a_{ij}]_{n\times n}$ is a square matrix of order n.

$\therefore$ adjA = $[A_{ij}]'_{nxn}$ is a square matrix of order n.

$\therefore$ A(adjA) & (adjA)A are square matrices of order n.

Now A(adjA) = $[a_{ij}]_{nxn}[A_{ij}]'_{nxn}$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}$$

$\therefore$ A(adjA) = $|A|$I

Similarly $\therefore$ (adjA)A = $|A|$I

Hence A(adjA) = (adjA)A = $|A|$I is proved.

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Ex: If A is the non-singular matrix of order n, then prove that

i) $|adjA| = |A|^{n-1}$ ii) adjA is non-singular

Proof: Let A is the non-singular matrix of order n i.e. $|A| \neq 0$

i) As A(adjA) = $|A|$I

$\therefore |AadjA| = ||A|I|$

$\therefore |A||adjA| = |A|^n|I|$

$\therefore |adjA| = |A|^{n-1}$ since $|A| \neq 0$ and $|I| = 1$

ii) As $|A| \neq 0 \therefore |adjA| = |A|^{n-1} \neq 0$

$\therefore$ adjA is non-singular matrix is proved.

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Ex: If A is a square matrix then prove that (adjA)' = adjA'

Proof: Let A= $[a_{ij}]_{nxn}$ be a square matrix of order n.

$\therefore$ A' = $[a'_{ij}]_{nxn} = [a_{ji}]_{nxn}$ be a square matrix of order n.

$\therefore$ (adjA)' & adjA' are the square matrices of order n.

As adjA = $[A_{ij}]'_{nxn}$

$\therefore$ $(adjA)' = [A_{ij}]_{nxn} = [A'_{ij}]'_{nxn} = [A_{ji}]'_{nxn}$ = adjA'

Hence proved.

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Ex: If A is a symmetric matrix then prove that adjA is also symmetric matrix.

Proof: Let A be a symmetric matrix.

$\therefore$ A' = A ……(1)

Now $(adjA)'$ = adjA' gives $(adjA)'$ = adjA by (1)

$\therefore$ adjA is also symmetric matrix. Hence proved.

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Inverse of a Matrix:

A square matrix B is said to be inverse of a square matrix A if AB = BA = I

Note: Inverse of a square matrix A is denoted by A^{-1} i.e. $AA^{-1}=A^{-1}A=I$.

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Theorem: The necessary and sufficient condition for a square matrix A to have an inverse is that $|A| \neq 0$ i.e. A square matrix A is invertible if and only if A is non-singular.

Proof: <u>The condition is necessary</u>: Suppose a square matrix A to have an inverse say B.

$\therefore$ AB = I

$\therefore$ |AB| = |I|

$\therefore$ |A||B| = 1

$\therefore$ |A| $\neq$ 0 i. e. A is non-singular matrix.

The condition is sufficient: Suppose a square matrix A is non-singular matrix i.e. |A| $\neq$ 0.

As A(adjA) = (adjA)A = |A|I

$\therefore$ A($\frac{1}{|A|}$ adjA) = ($\frac{1}{|A|}$ adjA)A = I

$\therefore$ AB = BA = I, where B = $\frac{1}{|A|}$ adjA

$\therefore$ A square matrix A is invertible and A^{-1} = B = $\frac{1}{|A|}$ adjA.

Hence proved.

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Theorem: Inverse of a square matrix if it exist, is unique.

Proof: Suppose a square matrix A to have two inverses B and C.

$\therefore$ AB = BA = I and AC = CA = I

Now B = BI

= B (AC) since AC = I

= (BA) C

= IC since BA = I

= C

Thus inverse of a square matrix A is unique.

Hence proved.

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Ex. If A = $\begin{bmatrix} 3 & 5 & 0 \\ -2 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$, Show that A(adjA) is an identity matrix.

Proof: Let A = $\begin{bmatrix} 3 & 5 & 0 \\ -2 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$

$\therefore$ |A| = 3(0+2)-5(-2+3)+0 = 6 - 5 = 1

Now A(adjA) = |A|I gives A(adjA) = I since |A|=1.

Hence A(adjA) is an identity matrix is proved.

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Ex. If A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, Show that A(adjA) is an identity matrix.

Proof: Let A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore$ |A| = 3(-3+4)+3(2-0)+4(-2+0) = 3+6-8 = 1

Now A(adjA) = |A|I gives A(adjA) = I since |A|=1.

Hence A(adjA) is an identity matrix is proved.

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Ex. If A = $\begin{bmatrix} -3 & 1 & 0 \\ 2 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, Show that A(adjA) is a null matrix.

Proof: Let A = $\begin{bmatrix} -3 & 1 & 0 \\ 2 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

∴ |A| = -3(-2+1)-(2+1)+0 = 3-3 = 0

Now $A(\mathrm{adj}A) = |A|I$ gives $A(\mathrm{adj}A) = 0$ since $|A|=0$.

Hence A(adjA) is a null matrix is proved.

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Ex. Let A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ i) Verify that A(adjA) = (adjA)A = |A|I and ii) find A^{-1}

Proof: Let A = $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

∴ |A| = 3(-3+4)+3(2-0)+4(-2-0) = 3+6-8 = 1

∴ |A|I = I……….(1)

The matrix of cofactors of elements of A is

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (-3+4) & -(2-0) & (-2+0) \\ -(-3+4) & (3-0) & -(-3+0) \\ (-12+12) & -(12-8) & (-9+6) \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}$$

$$\therefore \mathrm{adj}A = M' = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\text{Consider } A(\mathrm{adj}A) = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+6-8 & -3-9+12 & 0+12-12 \\ 2+6-8 & -2-9+12 & 0+12-12 \\ 0+2-2 & 0-3+3 & 0+4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ A(adjA) = I ……..(2)

Similarly (adjA)A = I ……..(2)

From equation (1), (2) & (3), we get,

A(adjA) = (adjA)A = |A|I = I,

Hence verified.

ii) Now $A^{-1} = \frac{1}{|A|} adjA = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

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Ex. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ i) Verify that A(adjA) = (adjA)A = |A|I and ii) find A^{-1}

Proof: Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$\therefore |A| = 1(4-0)-2(14-0)+4(7-0) = 4-28+28 = 4$

$\therefore |A|I = 4I$……….(1)

The matrix of cofactors of elements of A is

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (4-0) & -(14-0) & (7-0) \\ -(4-4) & (2-0) & -(1-0) \\ (0-8) & -(0-28) & (2-14) \end{bmatrix} = \begin{bmatrix} 4 & -14 & 7 \\ 0 & 2 & -1 \\ -8 & 28 & -12 \end{bmatrix}$$

$$\therefore adjA = M' = \begin{bmatrix} 4 & 0 & -8 \\ -14 & 2 & 28 \\ 7 & -1 & -12 \end{bmatrix}$$

$$\text{Consider } A(adjA) = \begin{bmatrix} 1 & 2 & 4 \\ 7 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 & -8 \\ -14 & 2 & 28 \\ 7 & -1 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 4-28+28 & 0+4-4 & -8+56-48 \\ 28-28+0 & 0+4+0 & -56+56-0 \\ 0-14+14 & 0+2-2 & 0+28-24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\therefore A(adjA) = 4I$ ……..(2)

Similarly (adjA)A = 4I ……..(3)

From equation (1), (2) & (3), we get,

A(adjA) = (adjA)A = |A|I = 4I,

Hence verified.

ii) Now $A^{-1} = \frac{1}{|A|} adjA = \frac{1}{4}\begin{bmatrix} 4 & 0 & -8 \\ -14 & 2 & 28 \\ 7 & -1 & -12 \end{bmatrix}$

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Ex. Find A^{-1} if it exists, by using adjoint method for $A = \begin{bmatrix} -2 & -1 & 3 \\ 1 & 3 & 7 \\ 4 & 2 & -6 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} -2 & -1 & 3 \\ 1 & 3 & 7 \\ 4 & 2 & -6 \end{bmatrix}$

∴ |A| = -2(-18-14)+(-6-28)+3(2-12) = 64-34-30 = 0

i.e. A is a singular matrix.

∴ A^{-1} does not exists

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Ex. Find A^{-1} if it exists, by using adjoint method for A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

∴ |A| = (16 - 9) – 3 (4 - 3) + 3 (3 - 4) = 7 - 3 - 3 = 1 ≠ 0

i.e. A is a non-singular matrix.

∴ A^{-1} is exists.

The matrix of cofactors of elements of A is

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (16-9) & -(4-3) & (3-4) \\ -(12-9) & (4-3) & -(3-3) \\ (9-12) & -(3-3) & (4-3) \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\therefore \text{adj}A = M' = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

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Ex. Find A^{-1} if it exists, by using adjoint method for A = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

∴ |A| = (3+0) - 2(-1-0) -2(2-0) = 3 + 2 - 4 = 1 ≠ 0

i.e. A is a non-singular matrix.

∴ A^{-1} is exists.

The matrix of cofactors of elements of A is

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (3-0) & -(-1-0) & (2-0) \\ -(2-4) & (1-0) & -(-2-0) \\ (0+6) & -(0-2) & (3+2) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}$$

$$\therefore \text{adj}A = M' = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

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Theorem: If A, B are non-singular square matrices of same order then

a) AB is non-singular (i.e. $(AB)^{-1}$ exists.)
b) $(AB)^{-1} = B^{-1}A^{-1}$(Reversal law for the inverse of a product)
c) adj(AB) = (adjB) (adjA)

Proof: Let A, B are non-singular square matrices of same order.

a) As $|A| \neq 0$ & $|B| \neq 0$
   $\therefore |A||B| \neq 0$
   $\therefore |AB| \neq 0$
   $\therefore$ AB is non-singular i.e. $(AB)^{-1}$ exists.
b) Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$
   & $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$
   $\therefore (AB)^{-1} = B^{-1}A^{-1}$
c) Consider (AB)(adjB)(adjA) = A[B(adjB)](adjA)
   = A[|B|I](adjA)
   = |B|[A(adjA)]
   = |B||A|I
   = |A||B|I
   = |AB|I
   = (AB)(adjAB)
   $\therefore$(AB)(adjAB) = (AB)(adjB)(adjA)
   Premultiplying by $(AB)^{-1}$ on both sides, we get,
   adj(AB) = (adjB)(adjA)
   Hence proved.

==

Theorem: If A is a non-singular matrix and n is any natural number then $(A^n)^{-1} = (A^{-1})^n$

Proof: Let A be a non-singular matrix and n is any natural number then

$(A^n)^{-1} = (AA\ldots\ldots A)^{-1}$ n-times
$= A^{-1} A^{-1} \ldots\ldots A^{-1}$ n-times, by reversal law of inverse.
$\therefore (A^n)^{-1} = (A^{-1})^n$
Hence proved.

==

Theorem: If A is a non-singular matrix then $(A')^{-1} = (A^{-1})'$

Proof: Let A be a non-singular matrix i.e. $|A| \neq 0$

$\therefore A^{-1}$ is exists.

As $AA^{-1} = I = A^{-1}A$

$\therefore (AA^{-1})' = I' = (A^{-1}A)'$

$\therefore (A^{-1})'A' = I = A'(A^{-1})' \quad \because I' = I$

$\therefore (A')^{-1}) = (A^{-1})'$

Hence proved.

==

Ex: If A, B are two square matrices of same order with AB = I, then show that $B = A^{-1}$

Proof: As AB = I……..(1)

$\therefore |AB| = |I|$

$\therefore |A||B| = 1$

$\therefore |A| \neq 0$

$\therefore A^{-1}$ exists.

Pre-multiplying by A^{-1} on both sides of the equation (1), we get,

$A^{-1}(AB) = A^{-1}I$

$\therefore (A^{-1}A)B = A^{-1}$

$\therefore I\,B = A^{-1}$

$\therefore B = A^{-1}$

Hence proved.

==

Theorem: If A is a non-singular matrix of order n and k a non-zero scalar then

a) $(kA)^{-1} = \frac{1}{k}A^{-1}$, b) $|A^{-1}| = \frac{1}{|A|}$, c) $adj(adjA) = |A|^{n-2}A$ and d) $adj(kA) = k^{n-1}(adjA)$

Proof: Let A be a non-singular matrix of order n and k a non-zero scalar.

$\therefore A^{-1}$ is exists and $AA^{-1} = A^{-1}A = I$

a) Consider $(kA)(\frac{1}{k}A^{-1}) = AA^{-1} = I$

$\&\ (\frac{1}{k}A^{-1})(kA) = A^{-1}A = I$

$\therefore (kA)^{-1} = (\frac{1}{k}A^{-1})$

b) As $AA^{-1} = I$

$\therefore |AA^{-1}| = |I|$

$\therefore |A||A^{-1}| = 1$

$\therefore |A^{-1}| = \frac{1}{|A|}$

c) As $A(adjA) = |A|I$

$\therefore |A(adjA)| = \big||A|I\big|$

$\therefore |A||adjA| = |A|^n|I|$

$\therefore |adjA| = |A|^{n-1}$ ……….(1) since $|A| \neq 0$ and $|I| = 1$

$\therefore (adjA)[adj(adjA)] = |adjA|I$

$\therefore (adjA)[adj(adjA)] = |A|^{n-1}I$ by (1)

Premultiplying by A on both sides, we get,

$A(adjA)[adj(adjA)] = |A|^{n-1}AI$

$\therefore |A|I\,[adj(adjA)] = |A|^{n-1}A$

$\therefore adj(adjA) = |A|^{n-2}A$ since $|A| \neq 0$

d) As $A(adjA) = |A|I$

$\therefore (kA)[adj(kA)] = |kA|I$

$\therefore (kA)[adj(kA)] = k^n\,|A|I$ since order of A is n.

$\therefore A[adj(kA)] = k^{n-1}\,A(adjA)$

Premultiplying by A^{-1} on both sides, we get,

$A^{-1}A[adj(kA)] = k^{n-1}\,A^{-1}A(adjA)$

$\therefore I\,[adj(kA)] = k^{n-1}\,I(adjA)$ $\therefore$ $adj(kA) = k^{n-1}(adjA)$

==

Ex. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 7 & 2 \end{bmatrix}$, verify that adj(AB) = (adjB)(adjA)

Solution: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 3 \\ 7 & 2 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 7 & 2 \end{bmatrix}$

$= \begin{bmatrix} -1+14 & 3+4 \\ 0+21 & 0+6 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 13 & 7 \\ 21 & 6 \end{bmatrix}$

$\therefore adj(AB) = \begin{bmatrix} 6 & -7 \\ -21 & 13 \end{bmatrix}$ ……….(1)

Now $adjA = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$ & $adjB = \begin{bmatrix} 2 & -3 \\ -7 & -1 \end{bmatrix}$

$\therefore (adjB)(adjA) = \begin{bmatrix} 2 & -3 \\ -7 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 6+0 & -4-3 \\ -21+0 & 14-1 \end{bmatrix}$

$\therefore (adjB)(adjA) = \begin{bmatrix} 6 & -7 \\ -21 & 13 \end{bmatrix}$ ……….(2)

By equation (1) & (2), adj(AB) = (adjB)(adjA) is verified.

Ex. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 5 & -9 \end{bmatrix}$, verify that adj(AB) = (adjB)(adjA)

Solution: Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & -2 \\ 5 & -9 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 5 & -9 \end{bmatrix}$

$= \begin{bmatrix} 3+25 & -6-45 \\ 1+10 & -2-18 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 28 & -51 \\ 11 & -20 \end{bmatrix}$

$\therefore adj(AB) = \begin{bmatrix} -20 & 51 \\ -11 & 28 \end{bmatrix}$ ……….(1)

Now $adjA = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ & $adjB = \begin{bmatrix} -9 & 2 \\ -5 & 1 \end{bmatrix}$

$\therefore (adjB)(adjA) = \begin{bmatrix} -9 & 2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -18-2 & 45+6 \\ -10-1 & 25+3 \end{bmatrix}$$

$$\therefore \text{(adjB)(adjA)} = \begin{bmatrix} -20 & 51 \\ -11 & 28 \end{bmatrix} \text{..........(2)}$$

By equation (1) & (2), adj(AB) = (adjB)(adjA) is verified.

MULTIPLE CHOICE QUETIONS [MCQ'S]

1) A matrix of order mxn is called a row matrix if ……

a) $m = 1$ & $n > 1$ b) $m \geq 1$ & $n = 1$ c) $m = n$ d) None of these.

2) A matrix of order mxn is called a column matrix if ……

a) $m = 1$ & $n \geq 1$ b) $m > 1$ & $n = 1$ c) $m = n$ d) None of these.

3) A matrix of order mxn is called a square matrix if ……

a) $m = 1$ & $n \geq 1$ b) $m \geq 1$ & $n = 1$ c) $m = n$ d) None of these.

4) A square matrix is called a diagonal matrix if ……

a) every diagonal element is zero b) every non-diagonal element is zero

c) every element is zero d) None of these.

5) A square matrix is called a upper triangular matrix if ……

a) every lower diagonal element is 0, b) every upper diagonal element is 0

c) every diagonal element is 0 d) None of these.

6) A square matrix is called a singular matrix if ……

a) $|A| = 0$ b) $|A| \neq 0$ c) $A = 0$ d) None of these.

7) A square matrix is called a non-singular matrix if ……

a) $|A| = 0$ b) $|A| \neq 0$ c) $A = 0$ d) None of these.

8) A square matrix is called a symmetric matrix if ……

a) $A' = -A$ b) $A' = A$ c) $A' \neq A$ d) $A' = I$

9) A square matrix is called a skew-symmetric matrix if ……

a) $A' = -A$ b) $A' = A$ c) $A' \neq A$ d) None of these.

10) (AB)'= ……

a) AB b) A'B' c) B'A' d) BA

11) If k is any non-zero scalar and A is any square matrix of order n

then $|kA| =$ …..

a) $k|A|$ b) $k^n |A|$ c) kA d) k

12) If A = $\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then cofactor of a_{21} is……

a) -5 b) -3 c) 2 d) 3

13) If A = $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} = [a_{ij}]_{3x3}$ then cofactors of a_{32} is ……

a) -3 b) 3 c) 2 d) 0

14) If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then adjA=……

a) $\begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$, b) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, c) $\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$, d) $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

15) If A = $\begin{bmatrix} -2 & 3 \\ 1 & -5 \end{bmatrix}$ then |A| =……

a) -7 b) 7 c) 10 d) 3

16) For any square matrix A, A(adjA) = (adjA)A = ……

a) |A| I b) |A| c) I d) None of these

17) For any square matrix A of order n, |adjA| = ……

a) |A| b) $|A|^{n-1}$ c) I d) None of these

18) For any square matrix A of order n, adj(adjA) = ……

a) $|A|^{n}A$ b) $|A|^{n-1}A$ c) $|A|^{n-2}A$ d) None of these

19) A square matrix A is invertible if and only if A is……

a) symmetric b) singular c) non-singular d) None of these

20) A(adjA) is an identity matrix iff |A| = ……

a) 1 b) 0 c) -1 d) None of these

21) A(adjA) is a null matrix iff |A| = ……

a) 1 b) 0 c) -1 d) None of these

22) For any non-singular matrix, A^{-1}=……

a) adjA b) |A|adjA c) $\frac{1}{|A|}$ adjA d) None of these

23) If A, B are non-singular square matrices of same order then $(AB)^{-1}$ = ……

a) $B^{-1}A^{-1}$ b) A^{-1} B^{-1} c) BA d) None of these.

24) If A, B are non-singular square matrices of same order then adj(AB) = ……

a) (adjA)(adjB) b) adj(BA) c) (adjB)(adjA) d) None of these.

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UNIT-II: RANK OF MATRIX

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Elementary Transformations:

Following six operations on matrices are called elementary transformations or elementary operations.

i) R_{ij}: Interchange of i^{th} and j^{th} rows.

ii) kR_i or $R_{i(k)}$: Multiplication by k ($\neq 0$) to every element of i^{th} row.

iii) $R_i + kR_j$ or $R_{ij(k)}$: Adding k times an element of j^{th} row to the corresponding element of i^{th} row.

iv) C_{ij}: Interchange of i^{th} and j^{th} columns.

v) kC_i or $C_{i(k)}$: Multiplication by k ($\neq 0$) to every element of i^{th} column.

vi) $C_i + kC_j$ or $C_{ij(k)}$: Adding k times an element of j^{th} column to the corresponding element of i^{th} column.

Equivalent Matrices:

A matrix B is said to be equivalent to a matrix A if B is obtained from A by performing some elementary transformations on A. Written as A~B and read as matrix A is equivalent to matrix B.

Note: i) A~A(Reflexivity), ii) If A~B then B~A(Symmetry) and

iii) If A~B and B~C then A~C (Transitivity).

iv) If σ is an ERT then $\sigma(AB) = (\sigma A)B$ and if σ is an ECT then $\sigma(AB) = A(\sigma B)$.

Elementary Matrices:

A matrix obtained from a unit matrix by performing single elementary transformations on it is called an elementary matrix or E-matrix.

Note: 1) There are six elementary matrices obtained by using six elementary transformations as i) $I \overset{R_{ij}}{\sim} E_{ij}$, ii) $I \overset{R_{i(k)}}{\sim} E_{i(k)}$, iii) $I \overset{R_{ij(k)}}{\sim} E_{ij(k)}$,

iv) $I \overset{C_{ij}}{\sim} E'_{ij}$, v) $I \overset{C_{i(k)}}{\sim} E'_{i(k)}$ and vi) $I \overset{C_{ij(k)}}{\sim} E'_{ij(k)}$.

2) $E_{ij} = E'_{ij}$ and $E_{i(k)} = E'_{i(k)}$ but $E_{ij(k)} \neq E'_{ij(k)}$.

3) $|E_{ij}| = |E'_{ij}| = -1$, $|E_{i(k)}| = |E'_{i(k)}| = k$ and $|E_{ij(k)}| = |E'_{ij(k)}| = 1$

4) If E is an ERM corresponding to an ERT σ then $\sigma(A) = \sigma(IA) = \sigma(I)A = EA$ and if E′ is an ECM corresponding to an ECT σ then $\sigma(A) = \sigma(AI) = A\sigma(I) = AE'$

==

Theorem: The inverse of an elementary matrix is an elementary matrix of the same order.

Proof: Let E_{ij}, $E_{i(k)}$, $E_{ij(k)}$, E'_{ij}, $E'_{i(k)}$ and $E'_{ij(k)}$ are the elementary matrices of order n obtained from the unit matrix I of order n.

i) We have $E_{ij} = I\,E_{ij}$

By performing ERT R_{ij} on both sides, we get,

$I = E_{ij}\,E_{ij}$

$\therefore (E_{ij})^{-1} = E_{ij}$

ii) We have $E_{i(k)} = I\,E_{i(k)}$

By performing ERT $R_i(\frac{1}{k})$ on both sides, we get,

$I = E_i(\frac{1}{k})\, E_{i(k)}$

$\therefore (E_{i(k)})^{-1} = E_i(\frac{1}{k})$

iii) We have $E_{ij(k)} = I\, E_{ij(k)}$

By performing ERT $R_{ij}(-k)$ on both sides, we get,

$I = E_{ij}(-k)\, E_{ij(k)}$

$\therefore (E_{ij(k)})^{-1} = E_{ij}(-k)$

Similarly we can prove $(E'_{ij})^{-1} = E'_{ij}$, $(E'_{i(k)})^{-1} = E'_i(\frac{1}{k})$ and $(E'_{ij(k)})^{-1} = E'_{ij}(-k)$.

Hence the inverse of an elementary matrix is an elementary matrix of the same order is proved.

==

Ex: Let A, B be matrices such that AB is defined. If σ_1 is an ERT and σ_2 is an ECT, then show that $\sigma_1(\sigma_2(AB)) = \sigma_2(\sigma_1(AB))$

Proof: Let σ_1 *is* an ERT and σ_2 is an ECT, then consider

$$\begin{aligned}\sigma_1(\sigma_2(AB)) &= \sigma_1(A\sigma_2(B))\\ &= \sigma_1(A)\sigma_2(B)\\ &= \sigma_2(\sigma_1(A)B)\\ &= \sigma_2(\sigma_1(AB))\end{aligned}$$

Hence proved.

==

Ex: For the elementary matrices of the third order compute $E_{12}[E_{13(-2)}]^{-1}E'_{13}(-1)$

Sol: Let $E_{12}[E_{13(-2)}]^{-1}E'_{13}(-1)$

$= E_{12}E_{13(2)}E'_{13}(-1)$

$$= \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}$$

$$= \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1+0-2&0+0+0&0+0+2\\0+0+0&0+1+0&0+0+0\\0+0-1&0+0+0&0+0+1\end{bmatrix}$$

$$= \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&2\\0&1&0\\-1&0&1\end{bmatrix}$$

$$= \begin{bmatrix}0+0+0&0+1+0&0+0+0\\-1+0+0&0+0+0&2+0+0\\0+0-1&0+0+0&0+0+1\end{bmatrix}$$

$$= \begin{bmatrix}0&1&0\\-1&0&2\\-1&0&1\end{bmatrix}$$

==

Ex: Compute $[E_{3(-2)}]^{-1} E_{13} E'_{23}(-1)$

Sol: Let $[E_{3(-2)}]^{-1} E_{13} E'_{23}(-1)$

$$= E_3(-\tfrac{1}{2}) E_{13} E'_{23}(-1)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0+0+0 & 0+0-1 & 0+0+1 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 1+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & -1+0+0 & 1+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0-\frac{1}{2} & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

==

Ex: Let A be a square matrix of order 3. The ERT's $R_3(-1)$, $R_{12}(-2)$ and $R_{23}(2)$ are applied on A. If the resulting matrix is the matrix B, find a matrix P such that B = PA

Sol: We have $B = E_{23}(2) E_{12}(-2) E_3(-1)A = PA$

$$\therefore P = E_{23}(2) E_{12}(-2) E_3(-1)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+0+0 & 0-2+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & -2+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0-2 \\ 0+0+0 & 0+0+0 & 0+0-1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

==

Submatrix: A matrix obtained from a matrix A, by deleting from it some rows and/or some columns is called a submatrix of the matrix A.

Rank of a Matrix: A non-zero matrix A is said to be a matrix of rank r, if there exist at least one non-zero minor of order r of A and every minor of order (r+1) of A is zero.

Note: i) Rank of null matrix is o i.e.$\rho(0) = 0$

ii) Rank of an unit matrix of order n and every non-singular matrix of order n is n. i.e. $\rho(I) = \rho(A) = n$

iii) If A is non-zero matrix of order mxn, then $1 \le \rho(A) \le \min\{m, n\}$

iv) If every minor of order r of A is zero, then $\rho(A) < r$.

v) If A is any square matrix of order n with $|A| = 0$, then $\rho(A) < n$.

vi) For any matrix $\rho(A) = \rho(A')$.

Normal of a Matrix: Let A be a non-zero matrix of rank r, then it reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by performing an elementary transformations on A is called normal of a matrix A.

Note: i) Normal form of a unit matrix of order n and every non-singular matrix of order n is I_n.

==

Ex. Find rank of a matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore |A| = (-1 + 0) + 2(0 + 0) + 0 = -1 \neq 0$

$\therefore \rho(A) = 3$

==

Ex. Find rank of a matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix}$

$\therefore |A| = (15 - 14) - 0 + (4 - 5) = 1 - 1 = 0$

$\therefore \rho(A) < 3$

Here $\begin{vmatrix} 5 & 7 \\ 2 & 3 \end{vmatrix} = 15 - 14 = 1 \neq 0$

$\therefore \rho(A) = 2$

==

Ex. Find rank of a matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

Sol. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ be a matrix of order 3x4.

$\therefore$ Highest order of minor of A is 3. Consider the minor of order 3 as

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 3 & 0 & -10 \end{vmatrix} = (-10 - 0) - 2(-20 - 9) + 4(0 - 3) = -10 + 58 - 12 = 36 \neq 0$$

$\therefore \rho(A) = 3$

==

Ex. Determine the value of x that will make the matrix $A = \begin{bmatrix} x & x & 1 \\ 1 & x & x \\ x & 1 & x \end{bmatrix}$ of a) rank 3, b) rank 1, c) rank 2.

Sol. Let $A = \begin{bmatrix} x & x & 1 \\ 1 & x & x \\ x & 1 & x \end{bmatrix}$

$\therefore |A| = x(x^2-x)-x(x-x^2)+(1-x^2)$

$= x^2(x-1) + x^2(x-1) - (x^2-1)$

$= 2x^2(x-1) - (x-1)(x+1)$

$= (x-1)(2x^2-x-1)$

$= (x-1)(2x+1)(x-1)$

$= (x-1)^2(2x+1)$

a) $\rho(A) = 3 \Leftrightarrow |A| \neq 0 \Leftrightarrow (x-1)^2(2x+1) \neq 0 \Leftrightarrow x \neq 1 \;\&\; x \neq -\frac{1}{2} \Leftrightarrow x \in R\text{-}\{1, -\frac{1}{2}\}$

$\therefore \rho(A) = 3$ for all $x \in R\text{-}\{1, -\frac{1}{2}\}$.

b) If $x = 1$ then $|A| = 0$ and $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ in which every minor of order 2 is 0.

$\therefore \rho(A) = 1$ for $x = 1$.

c) If $x = -\frac{1}{2}$ then $|A| = 0$ and $A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$

in which $\begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ is a non-minor of order 2.

$\therefore \rho(A) = 2$ for $x = -\frac{1}{2}$.

==

Ex. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ to its normal form. Hence find $\rho(A)$.

Sol. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

By performing R_2-$3R_1$, we get,

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \end{bmatrix}$$

By performing C_2-$2C_1$ & C_3-$3C_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & -7 \end{bmatrix}$$

By performing $(\frac{-1}{5})C_2$ & $(\frac{-1}{7})C_3$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

By performing C_3-C_2, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

i.e. $A \sim [I_2 \quad 0]$ be the normal form of A.

$\therefore \rho(A) = 2$

==

Ex. Reduce the matrix $A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 3 & 6 \end{bmatrix}$ to its normal form. Hence find $\rho(A)$.

Sol. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \\ 3 & 6 \end{bmatrix}$

By performing R_2+2R_1 & R_3-3R_1, we get,

$$A \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By performing C_2-$2C_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. $A \sim \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$ be the normal form of A.

$\therefore \rho(A) = 1$

==

Ex. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ to its normal form. Hence find $\rho(A)$.

Sol. Let $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

By performing R_2-$3R_1$ & R_3+R_1, we get,

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

By performing C_2-$2C_1$, C_3+C_1 & C_4-$3C_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

By performing R_3+R_2, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By performing $\left(\frac{-1}{2}\right)C_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By performing $C_3 - 3C_2$ & $C_4 + 10C_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ be the normal form of A. $\therefore \rho(A) = 2$

==

Ex. Reduce the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ to its normal form. Hence find $\rho(A)$.

Sol. Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

By performing $R_3 - 3R_1$ & $R_4 - R_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

By performing $R_3 - R_2$ & $R_4 - R_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By performing $C_3 - C_1$ & $C_4 - C_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By performing $C_3 + 3C_2$ & $C_4 + C_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $A \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ be the normal form of A. $\therefore \rho(A) = 2$

Ex. Reduce the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ to its normal form. Hence find $\rho(A)$.

Sol. Let $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

By performing R_{12}, we get,

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

By performing R_2-2R_1, R_3-3R_1 & R_4-6R_1, we get,

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

By performing $C_2 + C_1$, $C_3 + 2C_1$ & $C_4 + 4C_1$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

By performing $R_2 - R_3$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

By performing $R_3 - 4R_2$ & $R_4 - 9R_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

By performing $C_3 + 6C_2$ & $C_4 + 3C_2$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

By performing $(\frac{1}{33})C_3$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 2 & 44 \end{bmatrix}$$

By performing $R_4 - 2R_3$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By performing $C_4 - 22C_3$, we get,

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ be the normal form of A.

$\therefore \rho(A) = 3$

==

Theorem: If A is a matrix of rank r then there exists non-singular matrices P and Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is the normal form of A.

Proof: Let A be a matrix of rank r.

$\therefore$ A is reduced to its normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by performing a finite number of elementary transformations on A. Let $E_1, E_2, \ldots\ldots, E_k$ be elementary row matrices corresponding to the elementary row transformations which are applied on A in order and $E'_1, E'_2, \ldots\ldots, E'_s$ be elementary column matrices corresponding to the elementary column transformations which are applied on A in order.

$\therefore (E_kE_{k-1} \ldots\ldots E_2E_1)A(E'_1E'_2 \ldots\ldots E'_s) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

i.e. $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $P = E_kE_{k-1} \ldots\ldots E_2E_1$ & $Q = E'_1E'_2 \ldots\ldots E'_s$ are non-singular matrices. Since every elementary matrix is non-singular.

Hence proved.

==

Theorem: Every non-singular matrix is expressed as a product of a finite number of elementary matrices.

Proof: Let A be a non-singular matrix of order n.

$\therefore \rho(A) = n$ and I_n is the normal form of A. A is reduced to its normal form I_n by performing a finite number of elementary transformations on A. Let $E_1, E_2, \ldots\ldots, E_k$ be elementary row matrices corresponding to the elementary row transformations which are applied on A in order and $E'_1, E'_2, \ldots\ldots, E'_s$ be elementary column matrices corresponding to the elementary column transformations which are applied on A in order.

$\therefore (E_kE_{k-1} \ldots\ldots E_2E_1)A(E'_1E'_2 \ldots\ldots E'_s) = I_n$

$\therefore A = (E_kE_{k-1} \ldots\ldots E_2E_1)^{-1}I_n(E'_1E'_2 \ldots\ldots E'_s)^{-1}$

$\therefore A = E_1^{-1} E_2^{-1} \ldots\ldots E_k^{-1} E'^{-1}_s \ldots\ldots E'^{-1}_2 E'^{-1}_1$

As inverse of an elementary matrix is an elementary matrix of same type. Hence every non-singular matrix is expressed as a product of a finite number of elementary matrices is proved.

==

Ex: If A is a non-singular matrix of order n and P, Q are non-singular matrices such that PAQ is the normal form of A, then prove that $A^{-1} = QP$

Proof: Let A be a non-singular matrix of order n.

$\therefore$ Normal form of A is I_n.

But PAQ is normal form of A.

$\therefore$ $PAQ = I_n$

$\therefore$ $A = P^{-1}I_nQ^{-1}$ Since P and Q are non-singular matrices.

$\therefore$ $A = P^{-1}Q^{-1} = (QP)^{-1}$

$\therefore$ $A^{1} = QP$ Hence proved.

==

Ex: Let A, B be matrices of same order with $\rho(A) = \rho(B)$. Then show that A~B.

Proof: Let A, B be matrices of same order with $\rho(A) = \rho(B) = r$ say.

Then $A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ & $B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \sim B$ by symmetry

$\therefore$ A~B by transitivity

Hence proved.

==

Note: To find non-singular matrices P and Q such that PAQ is the normal form of a matrix A of order mxn. Consider $A = I_mAI_n$ and apply ERT's on LHS A and RHS I_m and apply ECT's on LHS A and RHS I_n upto we get normal form of A in LHS.

Ex: Find non-singular matrices P and Q such that PAQ is the normal form of $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \end{bmatrix}$ be a matrix of order 3x2.

$\therefore$ Consider $A = I_3AI_2$

i.e. $\begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By performing R_{12}, we get,

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By performing $R_2 - 2R_1$ & $R_3 - 3R_1$, we get,

$$\begin{bmatrix}1 & 3\\0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & -2 & 0\\0 & -3 & 1\end{bmatrix} A \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

By performing C_2-$3C_1$, we get,

$$\begin{bmatrix}1 & 0\\0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & -2 & 0\\0 & -3 & 1\end{bmatrix} A \begin{bmatrix}1 & -3\\0 & 1\end{bmatrix}$$

i.e. $\begin{bmatrix}I_1 & 0\\0 & 0\end{bmatrix}$ = PAQ be a normal form of a matrix A.

with P = $\begin{bmatrix}0 & 1 & 0\\1 & -2 & 0\\0 & -3 & 1\end{bmatrix}$ & Q = $\begin{bmatrix}1 & -3\\0 & 1\end{bmatrix}$ are required non-singular matrices.

==

Ex: Find non-matrices P and Q such that PAQ is the normal form of

$$A = \begin{bmatrix}2 & 3 & 1 & 4\\1 & 2 & 2 & 3\\0 & -1 & -3 & -2\end{bmatrix}$$

Sol: Let A = $\begin{bmatrix}2 & 3 & 1 & 4\\1 & 2 & 2 & 3\\0 & -1 & -3 & -2\end{bmatrix}$ be a matrix of order 3x4.

∴ consider $A = I_3AI_4$

i.e. $$\begin{bmatrix}2 & 3 & 1 & 4\\1 & 2 & 2 & 3\\0 & -1 & -3 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

By performing R_{12}, we get,

$$\begin{bmatrix}1 & 2 & 2 & 3\\2 & 3 & 1 & 4\\0 & -1 & -3 & -2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

By performing $R_2 - 2R_1$, we get,

$$\begin{bmatrix}1 & 2 & 2 & 3\\0 & -1 & -3 & -2\\0 & -1 & -3 & -2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & -2 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

By performing C_2-$2C_1$, C_3-$2C_1$ & C_4-$3C_1$, we get,

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & -1 & -3 & -2\\0 & -1 & -3 & -2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & -2 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -2 & -3\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

By performing $(-1)R_2$, we get,

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 3 & 2\\0 & -1 & -3 & -2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\-1 & 2 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -2 & -3\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

By performing R_3+R_2, we get,

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 3 & 2\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 2 & 0\\ -1 & 2 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -2 & -3\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

By performing C_3-3C_2 & C_4-2C_2, we get,

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 2 & 0\\ -1 & 2 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & 4 & 1\\ 0 & 1 & -3 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

i.e. $\begin{bmatrix}I_2 & 0\\ 0 & 0\end{bmatrix}$ = PAQ be a normal form of a matrix A.

with $P = \begin{bmatrix}0 & 1 & 0\\ -1 & 2 & 0\\ -1 & 2 & 1\end{bmatrix}$ & $Q = \begin{bmatrix}1 & -2 & 4 & 1\\ 0 & 1 & -3 & -2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

are required non-singular matrices.

==

Theorem: The rank of product of two matrices can't exceed the rank of either matrix.

i.e. $\rho(AB) \le \min\{\rho(A), \rho(B)\}$

Proof: Let A be a matrix of rank r.

$\therefore$ There exists non-singular matrices P and Q such that

$PAQ = \begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix} = N$ is normal form of A.

$\therefore$ $A = P^{-1}NQ^{-1}$ since P and Q are non-singular matrices. $\therefore$ P^{-1} and Q^{-1} are exists.

$\therefore$ $AB = P^{-1}NQ^{-1}B$.

As N contain r non-zero rows. $\therefore$ $P^{-1}NQ^{-1}B$ contain at most r non-zero rows.

$\therefore$ $\rho(P^{-1}NQ^{-1}B) \le r$ i.e. $\rho(AB) \le \rho(A)$.......(1)

Similarly, we have $\rho(AB) \le \rho(B)$.......(2)

$\therefore$ From equation (1) and (2), we have $\rho(AB) \le \min\{\rho(A), \rho(B)\}$.

Hence proved.

==

Ex. If $A = \begin{bmatrix}1 & 2 & 3\\ 0 & -1 & 1\\ 2 & 7 & 3\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0 & 1\\ -1 & 4 & 2\\ 1 & 3 & 3\end{bmatrix}$, verify that $\rho(AB) \le \min\{\rho(A), \rho(B)\}$

Proof. Let $A = \begin{bmatrix}1 & 2 & 3\\ 0 & -1 & 1\\ 2 & 7 & 3\end{bmatrix}$

$\therefore$ $|A| = (-3-7) - 2(0-2)+3(0+2) = -10 + 4 + 6 = 0$

$\therefore$ $\rho(A) < 3$ but $\begin{vmatrix}-1 & 1\\ 7 & 3\end{vmatrix} = -3-7 = -10 \neq 0$

$\therefore$ $\rho(A) = 2$ ....(1)

& B = $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix}$

$\therefore$ |B| = (12-6) – 0+(-3-4) = 6-7 = -1 ≠ 0

$\therefore$ ρ(B) = 3 ………(2)

Now AB = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 2 & 7 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 4 & 2 \\ 1 & 3 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1-2+3 & 0+8+9 & 1+4+9 \\ 0+1+1 & 0-4+3 & 0-2+3 \\ 2-7+3 & 0+28+9 & 2+14+9 \end{bmatrix}$

$= \begin{bmatrix} 2 & 17 & 14 \\ 2 & -1 & 1 \\ -2 & 37 & 25 \end{bmatrix}$

$\therefore$ |AB| = 2(-25-37) – 17(50+2)+14(74-2) = -124-884+1008 = 0

$\therefore$ ρ(AB) < 3 but $\begin{vmatrix} -1 & 1 \\ 37 & 25 \end{vmatrix}$ = -25-37 = -62 ≠ 0

$\therefore$ ρ(AB) = 2 ………(3)

From equation (1), (2) & (3) ρ(AB) ≤ min{ ρ(A), ρ(B)} is verified.

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MULTIPLE CHOICE QUETIONS [MCQ'S]

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1) Number of elementary row transformation are …….

a) 2 b) 3 c) 4 d) 6

2) Number of elementary column transformation are …….

a) 2 b) 3 c) 4 d) 6

3) Total number of elementary transformation are …….

a) 2 b) 3 c) 4 d) 6

4) An operations R_{ij}, $R_{i(k)}$ & $R_{ij(k)}$ on a matrix are called an …….

a) elementary column transformations b) elementary row transformations

c) elementary matrices d) equivalent matrices

5) An operations C_{ij}, $C_{i(k)}$ & $C_{ij(k)}$ on a matrix are called an …….

a) elementary column transformations b) elementary row transformations

c) elementary matrices d) equivalent matrices

6) An elementary matrix or E-Matrix is obtained from an identity matrix by using ……elementary transformation/s.

a) a single b) two c) three d) six

7) An elementary matrix E_{ij} is obtained from an identity matrix by using an elementary transformation ….

a) R_{ij} b) $R_{i(k)}$ c) $R_{ij(k)}$ d) None of these

8) An elementary matrix $E_{i(k)}$ is obtained from an identity matrix by using an elementary transformation ….

a) R_{ij} b) $R_{i(k)}$ c) $R_{ij(k)}$ d) None of these

9) An elementary matrix $E_{ij(k)}$ is obtained from an identity matrix by using an elementary transformation ….

a) R_{ij} b) $R_{i(k)}$ c) $R_{ij(k)}$ d) None of these

10) An elementary matrix E'_{ij} is obtained from an identity matrix by using an elementary transformation ….

a) C_{ij} b) $C_{i(k)}$ c) $C_{ij(k)}$ d) None of these

11) An elementary matrix $E'_{i(k)}$ is obtained from an identity matrix by using an elementary transformation ….

a) C_{ij} b) $C_{i(k)}$ c) $C_{ij(k)}$ d) None of these

12) An elementary matrix $E'_{ij(k)}$ is obtained from an identity matrix by using an elementary transformation ….

a) C_{ij} b) $C_{i(k)}$ c) $C_{ij(k)}$ d) None of these 8)

13) $(E_{ij})^{-1}$ = ……..

a) E_{ij} b) $E_i(\frac{1}{k})$ c) $E_{i(-k)}$ d) E'_{ij}

14) $(E_{i(k)})^{-1}$ = ……..

a) $E_{i(k)}$ b) $E_i(\frac{1}{k})$ c) $E_{i(-k)}$ d) $E'_i(\frac{1}{k})$

15) $(E_{ij(k)})^{-1}$ = ……..

a) $E_{ij(k)}$ b) $E_{ij}(\frac{1}{k})$ c) $E_{ij(-k)}$ d) $E'_{ij(-k)}$

16) $(E'_{ij})^{-1}$ = ……..

a) E'_{ij} b) $E'_i(\frac{1}{k})$ c) $E'_{i(-k)}$ d) E_{ij}

17) $(E'_{i(k)})^{-1}$ = ……..

a) $E'_{i(k)}$ b) $E'_i(\frac{1}{k})$ c) $E'_{i(-k)}$ d) $E_i(\frac{1}{k})$

18) $(E'_{ij(k)})^{-1}$ = ……..

a) $E'_{ij(k)}$ b) $E'_{ij}(\frac{1}{k})$ c) $E'_{ij(-k)}$ d) $E_{ij(-k)}$

19) Which of the following is not true?

a) $E_{ij} = E'_{ij}$ b) $E_{i(k)} = E'_{i(k)}$ c) $E_{ij(k)} = E'_{ij(k)}$. d) None of these

20) $|E_{ij}| = |E'_{ij}|$ = ….

a) 1 b) 0 c) -1 d) k

21) $|E_{i(k)}| = |E'_{i(k)}|$ = ….

a) 1 b) 0 c) -1 d) k

22) $|E_{ij(k)}| = |E'_{ij(k)}|$ = ….

a) 1 b) 0 c) -1 d) k

23) If E is an ERM corresponding to an ERT σ then $\sigma(A)$ = …

a) EA b) AE c) E'A d) AE'

24) If E' is an ECM corresponding to an ECT σ then $\sigma(A)$ = …

a) E'A b) AE' c) AE d) EA

25) If every minor of order r of matrix A is 0 then …….

a) $\rho(A) < r$ b) $\rho(A) > r$ c) $\rho(A) = 0$ d) $\rho(A) = 1$

26) If A is a non-singular matrix of order n then …….

a) $\rho(A) < n$ b) $\rho(A) > n$ c) $\rho(A) = n$ d) 1

27) If A is a null matrix then …….

a) $\rho(A) < n$ b) $\rho(A) > n$ c) 0 d) 1

28) If A is a unit matrix of order n then …….

a) $\rho(A) < n$ b) $\rho(A) > n$ c) $\rho(A) = n$ d) None of these

29) If A is a non-zero matrix of order mxn then …….

a) $1<\rho(A)\leq\min\{m, n\}$ b) $\rho(A) = m$ c) $\rho(A) = n$ d) $\rho(A) < n$

30) If A and B are non-zero matrices such that AB is exist, then $\rho(AB)$ = …….

a) $\rho(A)$ b) $\rho(B)$ c) $\min\{\rho(A), \rho(B)\}$ d) $\rho(I)$

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UNIT-III-SYSTEM OF LINEAR EQUATIONS AND EIGEN VALUES

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System of Linear Equations:

Let $a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$

be the system m linear equations in n variables.

Written in matrix form as AX = B

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{mxn}$ is called a matrix of coefficients.

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called a matrix of unknowns.

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is called a matrix of constants.

Solution: A set of values of $x_1, x_2, x_3, \ldots\ldots, x_n$ which satisfy all system of linear equations is called solution of system of linear equations.

Consistent: A system of linear equations is said to be consistent if it has solution.

Inconsistent: A system of linear equations is said to be inconsistent if it has no solution.

Augmented Matrix: A matrix [A : B] is called an augmented matrix of a system of linear equations AX = B.

Condition for Consistency: A system of linear equations AX = B is consistent if and only if $\rho(A) = \rho([A : B]$

Homogeneous System of Linear Equations: A system of linear equations AX = B is said to be homogeneous system of linear equations if B = 0.

Non-Homogeneous System of Linear Equations: A system of linear equations AX = B is said to be non-homogeneous system of linear equations if $B \neq 0$.

Trivial Solution: A solution X = 0 is called trivial solution of homogeneous system of linear equations AX = 0.

Non-Trivial Solution: A solution X ≠ 0 is called non-trivial solution of homogeneous system of linear equations AX = 0.

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Remark: i) Homogeneous system of linear equations AX = 0 is always consistent, since it has at least trivial solution X = 0.

ii) If $\rho(A) = \rho([A : B] = m = n$, the number of unknowns then AX = B has a unique solution $X = A^{-1}B$.

iii) If $\rho(A) = \rho([A : B] = r < n$, the number of unknowns then AX = B has infinite number of solutions. In this case we assign n - r variables by n – r arbitrary values.

iv) If $X_1, X_2, \ldots\ldots, X_k$ are solutions of homogeneous system of linear equations AX = 0, then linear combination $\bar{X} = \alpha_1 X_1 + \alpha_2 X_2 + \ldots\ldots + \alpha_k X_k$ is also solution of AX = 0.

==

Ex. Examine for consistency the following system of equations

2x - 3y + 7z = 5, 3x + y - 3z = 13, 2x + 19y - 47z = 32,

Sol.: Let 2x - 3y + 7z = 5, 3x + y - 3z = 13, 2x + 19y - 47z = 32 be the given system of linear equation written in matrix form as

$$\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & -3 & 7 & : & 5 \\ 3 & 1 & -3 & : & 13 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

By R_{12}, we get,

$$\sim \begin{bmatrix} 3 & 1 & -3 & : & 13 \\ 2 & -3 & 7 & : & 5 \\ 2 & 19 & -47 & : & 32 \end{bmatrix}$$

By R_1-R_2 & R_3-R_2, we get,

$$\sim \begin{bmatrix} 1 & 4 & -10 & : & 8 \\ 2 & -3 & 7 & : & 5 \\ 0 & 22 & -54 & : & 27 \end{bmatrix}$$

By R_2-$2R_1$, we get,

$$\sim \begin{bmatrix} 1 & 4 & -10 & : & 8 \\ 0 & -11 & 27 & : & -11 \\ 0 & 22 & -54 & : & 27 \end{bmatrix}$$

By R_3+$2R_2$, we get,

$$\sim \begin{bmatrix} 1 & 4 & -10 & : & 8 \\ 0 & -11 & 27 & : & -11 \\ 0 & 0 & 0 & : & 5 \end{bmatrix}$$

Here $\rho(A) = 2$ and $\rho(A : B) = 3$ i.e. $\rho(A) \neq \rho(A : B)$

$\therefore$ The given system is inconsistent.

==

Ex. Examine for consistency the following system of equations

$x + z = 2$, $-2x + y + 3z = 3$, $-3x + 2y + 7z = 4$, (Mar. 2019)

Sol.: Let $x + z = 2$, $-2x + y + 3z = 3$, $-3x + 2y + 7z = 4$ be the given system of linear equation written in matrix form as

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 3 \\ -3 & 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 0 & 1 & : & 2 \\ -2 & 1 & 3 & : & 3 \\ -3 & 2 & 7 & : & 4 \end{bmatrix}$$

By R_2+$2R_1$ & R_3+$3R_1$, we get

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 2 \\ 0 & 1 & 5 & : & 7 \\ 0 & 2 & 10 & : & 10 \end{bmatrix}$$

By R_3-$2R_2$, we get,

$$\sim \begin{bmatrix} 1 & 0 & 1 & : & 2 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & 0 & : & -4 \end{bmatrix}$$

Here $\rho(A) = 2$ and $\rho(A : B) = 3$ i.e. $\rho(A) \neq \rho(A : B)$

$\therefore$ The given system is inconsistent.

Ex. Examine for consistency the following system of equations

-3x + 5z = 2, 5x + y + 2z = 3, 2x + y + 7z = -2, (Oct.2018)

Sol.: Let -3x + 5z = 2, 5x + y + 2z = 3, 2x + y + 7z = -2 be the given system of linear equation written in matrix form as

$$\begin{bmatrix} -3 & 0 & 5 \\ 5 & 1 & 2 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} -3 & 0 & 5 & : & 2 \\ 5 & 1 & 2 & : & 3 \\ 2 & 1 & 7 & : & -2 \end{bmatrix}$$

By R_1+R_3, we get

$$\sim \begin{bmatrix} -1 & 1 & 12 & : & 0 \\ 5 & 1 & 2 & : & 3 \\ 2 & 1 & 7 & : & -2 \end{bmatrix}$$

By $(-1)R_1$, we get,

$$\sim \begin{bmatrix} 1 & -1 & -12 & : & 0 \\ 5 & 1 & 2 & : & 3 \\ 2 & 1 & 7 & : & -2 \end{bmatrix}$$

By R_2-5R_1 & R_3-2R_1 we get,

$$\sim \begin{bmatrix} 1 & -1 & -12 & : & 0 \\ 0 & 6 & 62 & : & 3 \\ 0 & 3 & 31 & : & -2 \end{bmatrix}$$

By $R_3-\frac{1}{2}R_2$ we get,

$$\sim \begin{bmatrix} 1 & -1 & -12 & : & 0 \\ 0 & 6 & 62 & : & 3 \\ 0 & 0 & 0 & : & -\frac{7}{2} \end{bmatrix}$$

Here $\rho(A) = 2$ and $\rho(A : B) = 3$ i.e. $\rho(A) \neq \rho(A : B)$

$\therefore$ The given system is inconsistent.

==

Ex. Examine the following system of equations for consistency and if consistent then solve them x + y + z = 6, 2x + y + 3z = 13, 5x + 2y + z = 12, 2x - 3y - 2z = -10

Sol.: Let x + y + z = 6, 2x + y + 3z = 13, 5x + 2y + z = 12, 2x - 3y - 2z = -10 be the given system of linear equation written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 5 & 2 & 1 \\ 2 & -3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 12 \\ -10 \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 2 & 1 & 3 & : & 13 \\ 5 & 2 & 1 & : & 12 \\ 2 & -3 & -2 & : & -10 \end{bmatrix}$$

By R_2-$2R_1$, R_3-$5R_1$ & R_4-$2R_1$, we get,

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -1 & 1 & : & 1 \\ 0 & -3 & -4 & : & -18 \\ 0 & -5 & -4 & : & -22 \end{bmatrix}$$

By (-1)R_2, we get,

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & -1 & : & -1 \\ 0 & -3 & -4 & : & -18 \\ 0 & -5 & -4 & : & -22 \end{bmatrix}$$

By R_3+$3R_2$ & R_4+$5R_2$, we get,

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & -7 & : & -21 \\ 0 & 0 & -9 & : & -27 \end{bmatrix}$$

By $(-\frac{1}{7})R_3$, we get,

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & 1 & : & 3 \\ 0 & 0 & -9 & : & -27 \end{bmatrix}$$

By R_4+$9R_3$, we get,

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & -1 & : & -1 \\ 0 & 0 & 1 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Here ρ(A) = 3 and ρ(A : B) = 3 i.e. $\rho(A) = \rho(A : B) = 3$, the number of unknowns.

∴ The given system is consistent and it has a unique solution.

Equivalent system of equation is

x + y + z = 6 …….(1)

y - z = -1 ………..(2)

z = 3

Putting z = 3 in (2), we get, y - 3 = -1 i.e. y = 2.

Again putting z = 3 & y = 2 in (1), we get, x + 2 + 3 = 6 i.e. x = 1.

Hence x = 1, y = 2 & z = 3 be the required solution.

==

Ex. If $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -1 & 3 & -4 \end{bmatrix}$, find A^{-1}. Hence solve the system of linear equations

2x + y - z = -1, x - 2y + 3z = 9, -x + 3y - 4z = -12

Sol.: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -1 & 3 & -4 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -1 & 3 & -4 \end{vmatrix} = 2(8-9) - (-4+3) - (3-2) = -2 + 1 - 1 = -2 \neq 0$

∴ A^{-1} is exists.

Now matrix of cofactors is

$$M = \begin{bmatrix} (8-9) & -(-4+3) & (3-2) \\ -(-4+3) & (-8-1) & -(6+1) \\ (3-2) & -(6+1) & (-4-1) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -9 & -7 \\ 1 & -7 & -5 \end{bmatrix}$$

$$\therefore \operatorname{adj}A = M' = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -9 & -7 \\ 1 & -7 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\operatorname{adj}A = \frac{1}{-2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -9 & -7 \\ 1 & -7 & -5 \end{bmatrix}$$

Given system of linear equation written in matrix form as

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -1 & 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ -12 \end{bmatrix} \text{ i.e. } AX = B$$

As $|A| \neq 0 \Longrightarrow \rho(A) = \rho(A : B) = 3$, the number of unknowns.

∴ The given system is consistent and it has a unique solution $X = A^{-1}B$.

$$\therefore X = \frac{1}{-2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -9 & -7 \\ 1 & -7 & -5 \end{bmatrix}\begin{bmatrix} -1 \\ 9 \\ -12 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1+9-12 \\ -1-81+84 \\ -1-63+60 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Hence x = 1, y = -1 & z = 2 be the required solution.

==

Ex. Solve the following system of equations

2x - y - 5z + 4w = 1, x + 3y + z - 5w = 18, 3x - 2y - 8z + 7w = -1 (Mar. 2019)

Sol.: Let 2x - y - 5z + 4w = 1, x + 3y + z - 5w = 18, 3x - 2y - 8z + 7w = -1

be the given system of linear equation written in matrix form as

$$\begin{bmatrix} 2 & -1 & -5 & 4 \\ 1 & 3 & 1 & -5 \\ 3 & -2 & -8 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \\ -1 \end{bmatrix} \text{ i.e. } AX = B$$

The augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & -1 & -5 & 4 & : & 1 \\ 1 & 3 & 1 & -5 & : & 18 \\ 3 & -2 & -8 & 7 & : & -1 \end{bmatrix}$$

By R_{12}, we get,

$$\sim \begin{bmatrix} 1 & 3 & 1 & -5 & : & 18 \\ 2 & -1 & -5 & 4 & : & 1 \\ 3 & -2 & -8 & 7 & : & -1 \end{bmatrix}$$

By R_2-2R_1 & R_3-3R_1, we get,

$$\sim \begin{bmatrix} 1 & 3 & 1 & -5 & : & 18 \\ 0 & -7 & -7 & 14 & : & -35 \\ 0 & -11 & -11 & 22 & : & -55 \end{bmatrix}$$

By $(-\frac{1}{7})R_2$ & $(-\frac{1}{11})R_3$, we get,

$$\sim \begin{bmatrix} 1 & 3 & 1 & -5 & : & 18 \\ 0 & 1 & 1 & -2 & : & 5 \\ 0 & 1 & 1 & -2 & : & 5 \end{bmatrix}$$

By R_3-R_2, we get,

$$\sim \begin{bmatrix} 1 & 3 & 1 & -5 & : & 18 \\ 0 & 1 & 1 & -2 & : & 5 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Here $\rho(A) = 2$ and $\rho(A : B) = 2$ i.e. $\rho(A) = \rho(A : B) = 2 < 4$, the number of unknowns.

∴ The given system is consistent and it has an infinite number of solutions.

Equivalent system of equation is

x + 3y + z - 5w = 18…….(1)

y + z – 2w = 5 ………..(2)

We assign 4-2=2 variables by arbitrary constants as z = α & w = β

From (2), we get, y + α - 2 β = 5 i. e. y = 5 - α + 2 β

From (1), we get, x + 3(5 - α + 2 β) + α - 5β =18

i.e. x + 15 - 3α + 6 β + α - 5β =18

i.e. x = 3 + 2α - β

Hence x = 3 + 2α - β, y = 5 - α + 2 β, z = α & w = β be the required solution.

==

Ex. Investigate for what values of λ *and* μ the following system of equations

x + 3y + 2z = 2, 2x + 7y - 3z = -11, x + y + λ z = μ

have i) no solution (Mar. 2019),

ii) A unique solution,

iii) An infinite number of solutions.

Sol.: Let x + 3y + 2z = 2, 2x + 7y - 3z = -11, x + y + λ z = μ be the given system of linear equation written in matrix form as

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 7 & -3 \\ 1 & 1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -11 \\ \mu \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 3 & 2 & : & 2 \\ 2 & 7 & -3 & : & -11 \\ 1 & 1 & \lambda & : & \mu \end{bmatrix}$$

By R_2-2R_1 & R_3-R_1, we get

$$\sim \begin{bmatrix} 1 & 3 & 2 & : & 2 \\ 0 & 1 & -7 & : & -15 \\ 0 & -2 & \lambda - 2 & : & \mu - 2 \end{bmatrix}$$

By R_3+2R_2, we get,

$$\sim \begin{bmatrix} 1 & 3 & 2 & : & 2 \\ 0 & 1 & -7 & : & -15 \\ 0 & 0 & \lambda - 16 & : & \mu - 32 \end{bmatrix}$$

have i) no solution if $\rho(A) \neq \rho(A : B)$

i.e. if $\rho(A) = 2$ and $\rho(A : B) = 3$

i.e. if $\lambda - 16 = 0$ and $\mu - 32 \neq 0$

i.e. if $\lambda = 16$ and $\mu \neq 32$

ii) A unique solution if $\rho(A) = \rho(A : B) = 3$

i.e. if $\lambda - 16 \neq 0$

i.e. if $\lambda \neq 16$ and any value of μ.

, iii) An infinite number of solutions if $\rho(A) = \rho(A : B) = 2 < 3$

i.e. if $\lambda - 16 = 0$ and $\mu - 32 = 0$

i.e. if $\lambda = 16$ and $\mu = 32$.

==

Ex. Investigate for what values of λ *and* μ the following system of equations
$2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$

have i) no solution, ii) A unique solution, iii) An infinite number of solutions.

Sol.: Let $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ be the given system of linear equation written in matrix form as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$$

By $R_2 - \frac{7}{2}R_1$ & $R_3 - R_1$, we get

$$\sim \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & : & -\frac{47}{2} \\ 0 & 0 & \lambda - 5 & : & \mu - 9 \end{bmatrix}$$

have i) no solution if $\rho(A) \neq \rho(A : B)$

i.e. if $\rho(A) = 2$ and $\rho(A : B) = 3$

i.e. if $\lambda - 5 = 0$ and $\mu - 9 \neq 0$

i.e. if $\lambda = 5$ and $\mu \neq 9$

ii) A unique solution if $\rho(A) = \rho(A : B) = 3$

i.e. if $\lambda - 5 \neq 0$

i.e. if $\lambda \neq 5$ and any value of μ.

, iii) An infinite number of solutions if $\rho(A) = \rho(A : B) = 2 < 3$

i.e. if $\lambda - 5 = 0$ and $\mu - 9 = 0$

i.e. if $\lambda = 5$ and $\mu = 9$.

===

Ex. For what values of a the equations

$x + y + z = 1$, $2x + 3y + z = a$, $4x + 9y - z = a^2$ have solution.

Sol.: Let $x + y + z = 1$, $2x + 3y + z = a$, $4x + 9y - z = a^2$ be the given system of linear equations written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 4 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$$

i.e. AX = B

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 3 & 1 & : & a \\ 4 & 9 & -1 & : & a^2 \end{bmatrix}$$

By R_2-$2R_1$ & R_3-$4R_1$, we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & -1 & : & a-2 \\ 0 & 5 & -5 & : & a^2-4 \end{bmatrix}$$

By R_3-$5R_2$, we get,

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & -1 & : & a-2 \\ 0 & 0 & 0 & : & a^2-5a+6 \end{bmatrix}$$

have a solution iff $\rho(A) = \rho(A : B)$

i.e. iff $a^2 - 5a + 6 = 0$

i.e. iff $(a-2)(a-3) = 0$

i.e. iff $a = 2$ or $a = 3$

===

Note: Homogeneous system AX = 0 has only trivial solution iff $|A| \neq 0$ and has non trivial solution iff $|A| = 0$.

===

Ex. Examine for non-trivial solution x + y + z = 0, 4x + y = 0, 2x + 2y + 3z = 0.

Sol.: Let x + y + z = 0, 4x + y = 0, 2x + 2y + 3z = 0 be the given homogeneous system of linear equations written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e. } AX = 0$$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 2 & 2 & -3 \end{bmatrix}$

Now $|A|$ = (-3-0) – (-12-0) + (8-2) = -3 + 12 + 6 =15 ≠ 0

∴ Given homogeneous system has trivial solution only.

==

Ex. Show that the following system possesses a non-trivial solution

x + ay +(b+c) z = 0, x + by +(c+a) z = 0, x + cy +(a+b) z = 0. (Mar.2019)

Proof.: Let x + ay +(b+c) z = 0,

x + by +(c+a) z = 0,

x + cy +(a+b) z = 0

be the given homogeneous system of linear equations written in matrix form as

$$\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e. } AX = 0$$

Where $A = \begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}$

Now $|A| = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix}$ by C_3+C_2

$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$

= (a+b+c) (0) ∵ $C_1 = C_3$

∴ $|A| = 0$

∴ Given homogeneous system has non-trivial solution is proved.

Ex. Show that the system of equations

ax + by + cz = 0, bx + cy + az = 0, cx + ay + bz = 0,

has non-trivial solution if and only if a+b+c=0 or a = b = c (Oct.2018)

Sol.: Let ax + by + cz = 0

bx + cy + az = 0

cx + ay + bz = 0

be the given homogeneous system of linear equations written in matrix form as

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e. } AX = 0$$

Where $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Now $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ by } R_1+(R_2+R_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= (a+b+c)[(bc-a^2) - (b^2-ac) + (ba-c^2)]$

$= (a+b+c)(bc - a^2 - b^2 + ac + ba - c^2)$

$= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

$\therefore$ Given homogeneous system has non-trivial solution iff $|A| = 0$.

i.e. iff $(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

i.e. iff $(a+b+c) = o \ or\ [(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

i.e. iff $(a+b+c) = o \ or\ (a-b)^2 = (b-c)^2 = (c-a)^2 = 0$

i.e. iff $(a+b+c) = o \ or\ a = b = c$.

Hence proved.

==

Ex. Show that the following system possesses a nontrivial solution

(a-b)x + (b-c)y +(c-a) z = 0,

(b-c)x + (c-a)y +(a-b) z = 0,

(c-a)x + (a-b)y +(b-c) z = 0. (Oct.2018)

Proof.: Let (a-b)x + (b-c)y +(c-a) z = 0

(b-c)x + (c-a)y +(a-b) z = 0

(c-a)x + (a-b)y +(b-c) z = 0

be the given homogeneous system of linear equations written in matrix form as

$$\begin{bmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e. } AX = 0$$

$$\text{Where } A = \begin{bmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{bmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} \text{ by } R_1+(R_2+R_3)$$

$\therefore |A| = 0$

$\therefore$ Given homogeneous system has non-trivial solution is proved.

==

Ex. Find the value of λ such that following system of homogeneous linear equations Have a non-trivial solutions $3x + y - \lambda z = 0$, $4x - 2y - 3z = 0$, $2\lambda x + 4y + \lambda z = 0$,

Sol.: Let $3x + y - \lambda z = 0$

$4x - 2y - 3z = 0$

$2\lambda x + 4y + \lambda z = 0$

be the given homogeneous system of linear equations written in matrix form as

$$\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e. } AX = 0$$

$$\text{Where } A = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}$$

As given homogeneous system has non-trivial solution

$\therefore |A| = 0$

$$\therefore \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$\therefore 3(-2\lambda+12) - (4\lambda+6\lambda) - \lambda(16+4\lambda) = 0$

$\therefore -6\lambda + 36 - 10\lambda - 16\lambda - 4\lambda^2 = 0$

$\therefore -4\lambda^2 - 32\lambda + 36 = 0$

$\therefore \lambda^2 + 8\lambda - 9 = 0$

$\therefore (\lambda - 1)(\lambda + 9) = 0$

$\therefore \lambda = 1$ or $\lambda = -9$

==

Vectors: A row matrix or a column matrix is called a vector.

e. g. $[2 \quad 3 \quad 7]$, $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ are vectors.

Eigen Values and Eigen Vectors of Matrix: Let A be a given non-zero square matrix of order n. If there exists a scalar λ and a non-zero vector such that AX = λ X then λ is called as eigen value or characteristic value of matrix A and X is called as eigen vector or characteristic vector of A corresponding to an eigen value λ.

Characteristic polynomial: Let A be a non-zero square matrix. Then $\Delta(\lambda) = |A - \lambda I|$ is called characteristic polynomial of matrix A.

Characteristic equation: Let A be a non-zero square matrix. Then $\Delta(\lambda) = |A - \lambda I| = 0$ is called characteristic equation of A.

Note: Let A be a non-zero square matrix of order n. Then

1) The roots of characteristic equation of A are precisely the eigen values A
2) Eigen vectors X of A corresponding to an eigen value λ of A are obtained by solving the homogeneous system $(A - \lambda I)X = 0$.
3) A has at most n distinct eigen values.

Root or Zero: A non-zero square matrix A is said to be root or zero of polynomial

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots\ldots\ldots + a_1 x + a_0$

if $f(A) = a_n A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots\ldots\ldots\ldots + a_1 A + a_0 I = 0$

Cayley Hamilton theorem:

Every non-zero square matrix satisfies its characteristic equation.

==

Ex.: If λ is a non-zero eigen value of a non-singular matrix A, show that $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof: Let λ is a non-zero eigen value of a non-singular matrix A

∴ There exist a non-zero vector X such that

$\therefore (A - \lambda I)\, X = 0$

Pre multiplying by $-\frac{A^{-1}}{\lambda}$, we get,

$\therefore -\frac{A^{-1}}{\lambda}(A - \lambda I)\, X = 0$

$\therefore \left(-\frac{A^{-1}}{\lambda}A + A^{-1}I\right) X = 0$

$\therefore \left(A^{-1} - \frac{1}{\lambda}I\right) X = 0$

Hence $\frac{1}{\lambda}$ is an eigen value of A^{-1} is proved.

==

Ex.: Find characteristic polynomial of $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ (Oct.2018)

Solution: Characteristic polynomial of A is

$\Delta(\lambda) = |A - \lambda I|$

$\therefore \Delta(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 7-\lambda \end{vmatrix}$

$\therefore \Delta(\lambda) = (2-\lambda)(7-\lambda) - 12$

$\therefore \Delta(\lambda) = 14 - 2\lambda - 7\lambda + \lambda^2 - 12$

$\therefore \Delta(\lambda) = \lambda^2 - 9\lambda + 2$

==

Ex.: Find characteristic equation of $A = \begin{bmatrix} 3 & -5 \\ 7 & 8 \end{bmatrix}$ (Mar.2019)

Solution: Characteristic equation of A is

$|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 3-\lambda & -5 \\ 7 & 8-\lambda \end{vmatrix} = 0$

$\therefore (3 - \lambda)(8 - \lambda) + 35 = 0$

$\therefore 24 - 3\lambda - 8\lambda + \lambda^2 + 35 = 0$

$\therefore \lambda^2 - 11\lambda + 59 = 0$

==

Ex.: Find characteristic equation of $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

Solution: Characteristic equation of A is

$|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 3-\lambda & 2 & -1 \\ 1 & 3-\lambda & 0 \\ 2 & -1 & 2-\lambda \end{vmatrix} = 0$

$\therefore (3-\lambda)[(3-\lambda)(2-\lambda)+0] - 2[2-\lambda-0] - [-1-2(3-\lambda)] = 0$

$\therefore (3-\lambda)[6 - 3\lambda - 2\lambda + \lambda^2] -2(2-\lambda) - [-1- 6 + 2\lambda] = 0$

$\therefore (3-\lambda)(\lambda^2 - 5\lambda + 6) - 4 + 2\lambda + 7 - 2\lambda = 0$

$\therefore 3\lambda^2 - 15\lambda + 18 - \lambda^3 + 5\lambda^2 - 6\lambda + 3 = 0$

$\therefore -\lambda^3 + 8\lambda^2 - 21\lambda + 21 = 0$

$\therefore \lambda^3 - 8\lambda^2 + 21\lambda - 21 = 0$

==

Ex.: Find eigen values of $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$ (Oct.2018)

Solution: Characteristic equation of A is

$|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 3 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\therefore (1-\lambda)(3-\lambda) - 6 = 0$

$\therefore 3 - \lambda - 3\lambda + \lambda^2 - 6 = 0$

$\therefore \lambda^2 - 4\lambda - 3 = 0$

$\therefore \lambda = \frac{4 \pm \sqrt{16+12}}{2}$

$\therefore \lambda = \frac{4 \pm \sqrt{28}}{2}$

$\therefore$ Eigen values of A are $2 + \sqrt{7}$ & $2 - \sqrt{7}$

Ex.: Find characteristic equation and eigen values of $A = \begin{bmatrix} 9 & -7 \\ 3 & -1 \end{bmatrix}$

Solution: 1) Characteristic equation of A is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 9-\lambda & -7 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$\therefore (9-\lambda)(-1-\lambda) + 21 = 0$

$\therefore -9 - 9\lambda + \lambda + \lambda^2 + 21 = 0$

$\therefore \lambda^2 - 8\lambda + 12 = 0$

2) We have characteristic equation of A is

$\lambda^2 - 8\lambda + 12 = 0$

$\therefore (\lambda - 2)(\lambda - 6) = 0$

$\therefore \lambda = 2$ or $\lambda = 6$

$\therefore$ Eigen values of A are 2, 6.

==

Ex.: Find characteristic equation and eigen values of $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

Solution: 1) Characteristic equation of A is $|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 2-\lambda & 3 \\ 4 & 7-\lambda \end{vmatrix} = 0$

$\therefore (2-\lambda)(7-\lambda) - 12 = 0$

$\therefore 14 - 2\lambda - 7\lambda + \lambda^2 - 12 = 0$

$\therefore \lambda^2 - 9\lambda + 2 = 0$

2) We have characteristic equation of A is

$\lambda^2 - 9\lambda + 2 = 0$

$\therefore \lambda = \frac{9 \pm \sqrt{81-8}}{2}$

$\therefore \lambda = \frac{9 \pm \sqrt{73}}{2}$

$\therefore$ Eigen values of A are $\frac{9+\sqrt{73}}{2}$ & $\frac{9-\sqrt{73}}{2}$.

==

Ex.: Find eigen values of $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

Solution: Characteristic equation of A is

$|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & -6 & -4 \\ 0 & 4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{vmatrix} = 0$

$\therefore (1-\lambda)[(4-\lambda)(-3-\lambda) + 12] + 6[0 - 0] - 4[0 - 0] = 0$

$\therefore (1-\lambda)[-12 - 4\lambda + 3\lambda + \lambda^2 + 12] + 0 = 0$

$\therefore (1-\lambda)(\lambda^2 - \lambda) = 0$

$\therefore (1-\lambda)\,\lambda\,(\lambda - 1) = 0$

$\therefore$ Eigen values of A are 1, 0 & 1.

==

Ex.: Find eigen values and eigen vectors of $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ (Mar.2019)

Solution: Characteristic equation of A is

$|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\therefore (4-\lambda)(1-\lambda) + 2 = 0$

$\therefore 4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$\therefore \lambda^2 - 5\lambda + 6 = 0$

$\therefore (\lambda-2)(\lambda-3) = 0$

$\therefore$ Eigen values of A are 2 & 3.

i) Eigen vector corresponding to eigen value $\lambda = 2$ is obtained by solving the homogeneous equation $(A - \lambda I)X = 0$

i.e. $(A - 2I)X = 0$

$\therefore \begin{bmatrix} 4-2 & -1 \\ 2 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying $R_2 - R_1$

$\therefore$ The equivalent system of equation is

$2x - y = 0$ with $x = \alpha$ be any arbitrary constant.

$\therefore y = 2\alpha$

$\therefore$ X = $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

In particular $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = 2$.

ii) Eigen vector corresponding to eigen value $\lambda = 3$ is obtained by solving the homogeneous equation (A- λI)X = 0

i.e. (A- 3I) X = 0

$\therefore \begin{bmatrix} 4-3 & -1 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying $R_2 - 2R_1$

$\therefore$ The equivalent system of equation is

x – y = 0 with y = β be any arbitrary constant.

$\therefore$ x = β

$\therefore$ X = $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \beta \\ \beta \end{bmatrix} = \beta\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

In particular $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = 3$.

==

Ex.: Find eigen values and eigen vectors of A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Solution: Characteristic equation of A is

$|A - \lambda I| = 0$

$\therefore \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\therefore (1-\lambda)(3-\lambda) - 8 = 0$

$\therefore 3 - \lambda - 3\lambda + \lambda^2 - 8 = 0$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

$\therefore (\lambda - 5)(\lambda + 1) = 0$

$\therefore$ Eigen values of A are 5 & -1.

i) Eigen vector corresponding to eigen value $\lambda = 5$ is obtained by solving the homogeneous equation (A- λI)X = 0

i.e. (A- 5I) X = 0

$\therefore \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying $(-\frac{1}{4})R_1$

$\therefore \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying $R_2 - 2R_1$

$\therefore$ The equivalent system of equation is

$x - y = 0$ with $y = \alpha$ be any arbitrary constant.

$\therefore x = \alpha$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

In particular $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = 5$.

ii) Eigen vector corresponding to eigen value $\lambda = -1$ is obtained by solving the homogeneous equation $(A - \lambda I) X = 0$

i.e. $(A+I) X = 0$

$\therefore \begin{bmatrix} 1+1 & 4 \\ 2 & 3+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying $(\frac{1}{2})$ R_1

$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by applying $R_2 - 2R_1$

$\therefore$ The equivalent system of equation is

$x + y = 0$ with $x = \beta$ be any arbitrary constant.

$\therefore y = -\beta$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \beta \\ -\beta \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

In particular $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector corresponding to eigen value $\lambda = -1$.

==

Ex.: Verify Cayley Hamilton theorem for A = $\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

Proof: Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)(4-\lambda) + 6 = 0$$

$$\therefore 4 - \lambda - 4\lambda + \lambda^2 + 6 = 0$$

$$\therefore \lambda^2 - 5\lambda + 10 = 0$$

Now consider

$$A^2 - 5A + 10I = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & 2+8 \\ -3-12 & -6+16 \end{bmatrix} + \begin{bmatrix} -5 & -10 \\ 15 & -20 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 10 \\ -15 & 10 \end{bmatrix} + \begin{bmatrix} 5 & -10 \\ 15 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

Hence Cayley Hamilton theorem is verified for A is proved.

==

Ex.: Verify Cayley Hamilton theorem for A = $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ (Oct.2018)

Proof: Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)(1-\lambda) - 6 = 0$$

$$\therefore 1 - \lambda - \lambda + \lambda^2 - 6 = 0$$

$$\therefore \lambda^2 - 2\lambda - 5 = 0$$

Now consider

$$A^2 - 2A - 5I = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + (-2)\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ -6 & -2 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} -7 & -4 \\ -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

Hence Cayley Hamilton theorem is verified for A is proved.

==

Ex.: Verify Cayley Hamilton theorem for A = $\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$. Hence find its inverse.

Proof: Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & -5 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$\therefore$ (1- λ)(2- λ) + 15 = 0

$\therefore$ 2 - λ - 2λ + λ^2 + 15 = 0

$\therefore$ λ^2 - 3λ + 17 = 0

Now consider

$$A^2 - 3A + 17I = \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} + (-3)\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} + 17\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-15 & -5-10 \\ 3+6 & -15+4 \end{bmatrix} + \begin{bmatrix} -3 & 15 \\ -9 & -6 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & -15 \\ 9 & -11 \end{bmatrix} + \begin{bmatrix} 14 & 15 \\ -9 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0$$

Hence Cayley Hamilton theorem is verified for A is proved.

Now $A^2 - 3A + 17I = 0$ gives $17I = 3A - A^2$ …..(1)

Pre-multiplying by A^{-1} to equation (1), we get,

$17 A^{-1} = 3I - A$

$$\therefore 17 A^{-1} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{17}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$$

==

Ex.: Find characteristic equation of A = $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ and using Cayley Hamilton theorem find its inverse.

Proof: Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$\therefore (2 - \lambda)^3 + 0 - (2- \lambda) = 0$

$\therefore 8 - 12\lambda + 6\lambda^2 - \lambda^3 - 2 + \lambda = 0$

$\therefore 6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$

By using Cayley Hamilton theorem, we get,

$6I - 11A + 6A^2 - A^3 = 0$

i.e. $6I = 11A - 6A^2 + A^3$ ……….. (1)

Pre-multiplying by A^{-1} to equation (1), we get,

$6A^{-1} = 11I - 6A + A^2$

$$\therefore 6A^{-1} = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (-6)\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} + \begin{bmatrix} -12 & 0 & 6 \\ 0 & -12 & 0 \\ 6 & 0 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

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UNIT-III-SYSTEM OF LINEAR EQUATIONS AND EIGEN VALUES [MCQ'S]

1) In a system of linear equations AX=B, matrix A is called a matrix of………

a) constants b) coefficients c) unknowns d) None of these.

2) In a system of linear equations AX=B, matrix X is called a matrix of ………

a) constants b) coefficients c) unknowns d) None of these.

3) In a system of linear equations AX=B, matrix B is called a matrix of………

a) constants b) coefficients c) unknowns d) None of these.

4) If a system of linear equations AX=B has a solution, then it said to be ………

a) inconsistent b) consistent c) homogeneous d) None of these.

5) If a system of linear equations AX=B has no solution, then it said to be ………

a) inconsistent b) consistent c) homogeneous d) None of these.

6) If B = 0, then the system of linear equations AX=B is said to be ………

a) inconsistent b) non-homogeneous c) homogeneous d) None of these.

7) If $B \neq 0$, then the system of linear equations AX=B is said to be ………

a) inconsistent b) non-homogeneous c) homogeneous d) None of these.

8) Let AX = B be the system of linear equations, then [A : B] is called …...matrix

a) constants b) coefficients c) augmented d) None of these.

9) The system of linear equations AX = B is consistent iff………..

a) $\rho(\mathrm{A}) = \rho([\mathrm{A}:\mathrm{B}])$ b) $\rho(\mathrm{A}) \neq \rho([\mathrm{A}:\mathrm{B}])$

c) $\rho(\mathrm{A}) > \rho([\mathrm{A}:\mathrm{B}])$ d) None of these.

10) The system of linear equations AX = B is inconsistent iff…

a) $\rho(\mathrm{A}) = \rho([\mathrm{A}:\mathrm{B}])$ b) $\rho(\mathrm{A}) \neq \rho([\mathrm{A}:\mathrm{B}])$

c) $\rho(\mathrm{A}) > \rho([\mathrm{A}:\mathrm{B}])$ d) None of these.

11) A solution X = 0 is called …….solution of homogeneous system AX = 0.

a) non-trivial b) dependent c) trivial d) None of these.

12) A solution $X \neq 0$ is called …….solution of homogeneous system AX=0

a) non-trivial b) independent c) trivial d) None of these.

13) Homogeneous system of linear equations AX = 0 is always ……..

a) inconsistent b) consistent c) None of these.

14) Non-homogeneous system of linear equations AX = B ……..

a) may or may not be consistent b) always consistent

c) always inconsistent d) None of these.

15) If $\rho(A) = \rho([A : B] = r = n$, the number of unknowns then AX = B has …….

a) no solution b) a unique solution c) an infinite number of solutions

16) If $\rho(A) \neq \rho([A : B]$, then the system AX = B has …….

a) no solution b) a unique solution c) an infinite number of solutions

17) If $\rho(A) = \rho([A : B] = r < n$, the number of unknowns then AX = B has …….

a) no solution b) a unique solution c) an infinite number of solutions

18) If $\rho(A) = \rho([A : B] = r < n$ (the number of unknowns), then to find the solution of the system AX = B, we assign ……variables by arbitrary constants.

a) n-r b) r-n c) r d) n

19) Homogeneous system AX = 0 has only trivial solution iff ………

a) $|A| = 0$ b) $|A| \neq 0$ c) $A = 0$ d) $A = I$

20) Homogeneous system AX = 0 has only non-trivial solution iff ……….

a) $|A| = 0$ b) $|A| \neq 0$ c) $A = 0$ d) $A = I$

21) A row matrix or a column matrix is called a …...

a) root b) vector c) zero d) none of these

22) Let A be a given non-zero square matrix of order n. If there exists a scalar λ and a non-zero vector such that AX = λ X then λ & X are called …..&….resp.

a) eigen value & eigen vector b) eigen vector & eigen value c) none of these

23) Characteristic polynomial of a non-zero square matrix A is $\Delta(\lambda) =$ ……

a) $|A - \lambda I|$ b) $|A - \lambda I| = 0$ c) $|A + \lambda I|$ d) $|A + \lambda I| = 0$

24) Characteristic equation of a non-zero square matrix A is……..

a) $|A - \lambda I|$ b) $|A - \lambda I| = 0$ c) $|A + \lambda I|$ d) $|A + \lambda I| = 0$

25) The roots of characteristic equation of A are precisely the ………of A.

a) eigen values b) eigen vectors c) poles d) None of these

26) Eigen vector X of a non zero square matrix A, corresponding to an eigen value λ of A are obtained by solving the system …….

a) $(A - \lambda I) X = 0$ b) $(A + \lambda I) X = 0$ c) $(A - \lambda I) X$ d) $(A + \lambda I) X$

27) A non-zero square matrix of order n has at most ….. distinct eigen values.

a) n-1 b) n c) 1 d) 2

28) A non-zero square matrix of order 3 has at most ……distinct eigen values.

a) 0 b) 1 c) 2 d) 3

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29) If $f(A) = a_nA^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \ldots\ldots\ldots + a_1A + a_0I = 0$, then a non-zero square matrix A is said to be……of polynomial f(x).

a) zero b) pole c) inverse d) None of these

30) By Cayley Hamilton theorem, every non-zero square matrix satisfies its…..

a) characteristic equation b) characteristic polynomial

c) characteristic value d) characteristic vector

31) If λ is a non-zero eigen value of a non-singular matrix A, then an eigen value of A^{-1} is ……

a) λ b) $-\lambda$ c) $\frac{1}{\lambda}$ d) $-\frac{1}{\lambda}$

32) Characteristic polynomial of A = $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ is $\Delta(\lambda)$ =……

a) $\lambda^2 - 9\lambda + 2$ b) $\lambda^2 - 3\lambda + 8 = 0$ c) $\lambda^2 - 3\lambda + 8$ d) $\lambda^2 - 9\lambda + 2 = 0$

33) Characteristic equation of A = $\begin{bmatrix} 3 & -5 \\ 7 & 8 \end{bmatrix}$

a) $\lambda^2 - 3\lambda + 8 = 0$ b) $\lambda^2 - 11\lambda + 59$ c) $\lambda^2 - 11\lambda + 59 = 0$ d) $\lambda^2 - 3\lambda + 8$

34) Characteristic equation of A = $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 3 & 0 \\ 2 & -1 & 2 \end{bmatrix}$

a) $\lambda^3 - 8\lambda^2 + 21\lambda - 21$ b) $\lambda^3 - 8\lambda^2 + 21\lambda - 21 = 0$

c) $\lambda^2 + 21\lambda - 21 = 0$ d) $\lambda^2 + 21\lambda - 21$

35) Eigen values of A = $\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$

a) 1, 3 b) 2, 3 c) 3, 3 d) $2 + \sqrt{7}, 2 - \sqrt{7}$

36) Eigen values of A = $\begin{bmatrix} 9 & -7 \\ 3 & -1 \end{bmatrix}$

a) 2, 6 b) 9, -1 c) 3, -7 d) 3, 9

37) Characteristic equation of A = $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

a) $\lambda^2 - 9\lambda + 2 = 0$ b) $\lambda^2 - 9\lambda + 2$ c) $\lambda^2 - 2\lambda + 7 = 0$ d) $\lambda^2 - 4\lambda + 3$

38) Eigen values of A = $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

a) 4 & 1 b) 2 & 3 c) 2 & -1 d) 2 & 1

39) Eigen values of A = $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

a) 2 & 4 b) 1 & 3 c) 5 & -1 d) 1 & 4

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UNIT-IV: ORTHOGONAL MATRICES AND QUADRATIC FORMS

Orthogonal Matrix: A square matrix A is said to be an orthogonal matrix if AA'= I. Where A' is the transpose of A.

Proper Orthogonal Matrix: A square matrix A is said to be proper orthogonal matrix if AA'= I and $|A| = 1$.

Improper Orthogonal Matrix: A square matrix A is said to be improper orthogonal matrix if AA'= I and $|A| = -1$.

Properties of an Orthogonal Matrices:

1) Determinant of an orthogonal matrix is ± 1 (Mar.2019)

Proof : Let A is an orthogonal matrix.

$\therefore AA' = I$

$\therefore |AA'| = |I|$

$\therefore |A|\,|A'| = 1$

$\therefore |A|^2 = 1 \qquad \because |A| = |A'|\ \&\ |I| = 1$

$\therefore |A| = \pm 1$ is proved.

2) The inverse of an orthogonal matrix is equal to the transpose of that matrix. i.e. $A^{-1} = A'$

Proof : Let A is an orthogonal matrix.

$\therefore AA' = I$ ……..(1)

$\therefore |A| = \pm 1 \neq 0$

$\therefore A^{-1}$ is exists

Pre-multiplying both sides by A^{-1} to equation (1), we get,

$A^{-1}AA' = A^{-1}I$

$\therefore IA' = A^{-1}$

$\therefore A' = A^{-1}$ is proved.

3) Product of two orthogonal matrices of same order is orthogonal.

Proof : Let A and B be any two orthogonal matrices of same order.

$\therefore AA' = I$ ……..(1) and $BB' = I$……..(2)

Now consider

$$\begin{aligned}(AB)(AB)' &= (AB)(B'A') && \because (AB)' = B'A' \\ &= A(BB')A' \\ &= A(I)A' && \text{by (2)} \\ &= AA' \\ &= I && \text{by (1)}\end{aligned}$$

$\therefore AB$ is an orthogonal matrix is proved.

4) Inverse of an orthogonal matrix is orthogonal.

Proof : Let A is an orthogonal matrix.

$\therefore AA' = A'A = I$ and $A^{-1} = A'$

Now consider

$A^{-1}(A^{-1})' = A'(A')' = A'A = I$

$\therefore$ A^{-1} is an orthogonal matrix is proved.

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Ex. Verify that the matrix A = $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is orthogonal.

Proof : Let A = $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\therefore AA' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

$\therefore$ A is an orthogonal matrix is proved.

==

Ex. Show that A = $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is an orthogonal matrix.

Proof : Let A = $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$\therefore AA' = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{4}+\frac{3}{4} & \frac{-\sqrt{3}}{4}+\frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4}+\frac{\sqrt{3}}{4} & \frac{3}{4}+\frac{1}{4} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

$\therefore$ A is an orthogonal matrix is proved.

==

Ex. Show that A = $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ is a proper orthogonal matrix. (Mar.2019)

Proof : Let A = $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$

$\therefore$ AA' = $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

= $\begin{bmatrix} cos^2\theta + sin^2\theta & cos\theta sin\theta - sin\theta cos\theta \\ sin\theta cos\theta - cos\theta sin\theta & sin^2\theta + cos^2\theta \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$= I

& |A| = $\begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix}$ = $cos^2\theta + sin^2\theta$ = 1

$\therefore$ A is a proper orthogonal matrix is proved.

==

Ex. Prove that the matrix A = $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ is a proper orthogonal matrix.

Proof : Let A = $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

$\therefore$ AA' = $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$

= $\begin{bmatrix} \frac{1}{3}+\frac{1}{6}+\frac{1}{2} & \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{1}{6}-\frac{1}{2} \\ \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{4}{6}+0 & \frac{1}{3}-\frac{2}{6}+0 \\ \frac{1}{3}+\frac{1}{6}-\frac{1}{2} & \frac{1}{3}-\frac{2}{6}+0 & \frac{1}{3}+\frac{1}{6}+\frac{1}{2} \end{bmatrix}$

= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= I

& |A| = $\frac{1}{\sqrt{3}}(\frac{2}{\sqrt{12}} - 0) - \frac{1}{\sqrt{6}}(-\frac{1}{\sqrt{6}} - 0) + \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}}) = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$

$\therefore$ A is a proper orthogonal matrix is proved.

==

Ex. Show that A = $\begin{bmatrix} cos\theta & -sin\theta \\ -sin\theta & -cos\theta \end{bmatrix}$ is an improper orthogonal matrix. (Oct.2018)

Proof : Let A = $\begin{bmatrix} cos\theta & -sin\theta \\ -sin\theta & -cos\theta \end{bmatrix}$

$\therefore$ AA' = $\begin{bmatrix} cos\theta & -sin\theta \\ -sin\theta & -cos\theta \end{bmatrix}\begin{bmatrix} cos\theta & -sin\theta \\ -sin\theta & -cos\theta \end{bmatrix}$

= $\begin{bmatrix} cos^2\theta + sin^2\theta & -cos\theta sin\theta + sin\theta cos\theta \\ -sin\theta cos\theta + cos\theta sin\theta & sin^2\theta + cos^2\theta \end{bmatrix}$ = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$= I

& $|A| = \begin{vmatrix} cos\theta & -sin\theta \\ -sin\theta & -cos\theta \end{vmatrix} = -cos^2\theta - sin^2\theta$ = -1

∴ A is an improper orthogonal matrix is proved.

==

Ex. Verify that the matrix $A = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$ is orthogonal.

Proof : Let $A = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$

∴ $AA' = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$

$= \begin{bmatrix} cos^2\theta + 0 + sin^2\theta & 0+0+0 & -cos\theta sin\theta + 0 + sin\theta cos\theta \\ 0+0+0 & 0+1+0 & 0+0+0 \\ -sin\theta cos\theta + 0 + cos\theta sin\theta & 0+0+0 & sin^2\theta + 0 + cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= I

∴ A is an orthogonal matrix is proved.

==

Ex. Prove that the matrix $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ is orthogonal.

Hence find A^{-1}. (Oct.2018)

Proof : Let $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

∴ $AA' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$= \frac{1}{81}\begin{bmatrix} 64+16+1 & -8+16-8 & -32+28+4 \\ -8+16-8 & 1+16+64 & 4+28-32 \\ -32+28+4 & 4+28-32 & 16+49+16 \end{bmatrix}$

$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

= I

∴ A is an orthogonal matrix is proved and $A^{-1} = A' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$.

==

Ex. Verify whether the following matrix is orthogonal.

$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ (Mar.2019)

Solution: Let $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$

$$\therefore AA' = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4+4+1 & 4-2+2 & -2+4+2 \\ 4-2+2 & 4+1+4 & -2-2+4 \\ -2+4+2 & -2-2+4 & 1+4+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 4 & 4 \\ 4 & 9 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

$$\neq I$$

$\therefore$ A is not an orthogonal matrix.

==

Ex. Find the condition that the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal.

Solution: Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal if

$$AA' = I$$

i.e. if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. if $\begin{bmatrix} a^2+b^2 & ac+bd \\ ca+db & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i.e. if $a^2+b^2 = c^2+d^2 = 1$ and $ac+bd = 0$.

==

Quadratic Form : A homogeneous polynomial of second degree in n variables is called a quadratic form in the n variables.

e.g. 1) $ax^2 + 2hxy + by^2$ is a quadratic form in two variables x & y.

2) $x_1^2 + x_2^2 - 3x_3^2 + 4x_1x_2 - x_1x_3 + 5x_2x_3$ is a quadratic form in three variables x_1, x_2 & x_3.

3) $q = \sum_{j=1}^{n} \cdot \sum_{j=1}^{n} a_{ij}\, x_i x_j$ is a general quadratic form in n variables $x_1, x_2, \ldots\ldots. x_n$.

Matrix Notation: A general quadratic form $q = \sum_{j=1}^{n} \cdot \sum_{j=1}^{n} a_{ij}\, x_i x_j$ in n variables $x_1, x_2, \ldots x_n$ is written in matrix form as $q = X'AX$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]_{nxn}$,

a_{ii} = coefficient of x_i^2 & $a_{ij} = a_{ji} = \frac{1}{2}$ coefficient of $x_i x_j$ for $i \neq j$

Rank of Quadratic Form : Let $q = \sum_{j=1}^{n} . \sum_{j=1}^{n} a_{ij} x_i x_j = X'AX$ be a quadratic form in n variables. A rank of a matrix $A = [a_{ij}]_{nxn}$ is called rank of a quadratic form.

Ex. Find the rank of a quadratic form $q = x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$

Solution: Given quadratic form $q = x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$ is written in matrix form as $q = X'AX$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

& $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}$

Now $|A| = (6 - 16) - 2(-6 + 12) + 3(-8 + 6) = -10 - 12 - 6 = -28 \neq 0$

$\therefore \rho(A) = 3$.

Hence rank of given quadratic form q is 3.

Ex. Find the rank of a quadratic form $q = x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_3$

Solution: Given quadratic form $q = x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_3$ is written in matrix form as $q = X'AX$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

& $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Now $|A| = (-2 - 0) - 0 + (0 + 2) = -2 + 2 = 0$ $\therefore \rho(A) < 3$

But $\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = -2 \neq 0$ $\therefore \rho(A) = 2$

Hence rank of given quadratic form q is 2.

Ex. Write the quadratic form corresponding to the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$

Solution: Quadratic form corresponding to the given matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$ is

$q = X'AX = [x_1 \quad x_2] \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 6x_1x_2$

Ex. Write the quadratic form corresponding to the matrix $A = \begin{bmatrix} 5 & 0 & -3 \\ 0 & -2 & 1 \\ -3 & 1 & 1 \end{bmatrix}$

Solution: Quadratic form corresponding to the given matrix $A = \begin{bmatrix} 5 & 0 & -3 \\ 0 & -2 & 1 \\ -3 & 1 & 1 \end{bmatrix}$

is $q = X'AX = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 5 & 0 & -3 \\ 0 & -2 & 1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 5x_1^2 - 2x_2^2 + x_3^2 - 6x_1x_3 + 2x_2x_3$

===

Linear Transformation: The system of linear equations

$x_1 = p_{11}y_1 + p_{12}\, y_2 + \ldots\ldots + p_{1n}\, y_n$

$x_2 = p_{21}y_1 + p_{22}\, y_2 + \ldots\ldots + p_{2n}\, y_n$

… … … … …

… … … … …

… … … … …

$X_n = p_{n1}y_1 + p_{n2}\, y_2 + \ldots\ldots + p_{nn}\, y_n$

i.e. $X = PY$ where $P = [p_{ij}]_{nxn}$ is called linear transformation from $x_1, x_2, \ldots, x_n$ to $y_1, y_2, \ldots, y_n$.

Non-singular Linear Transformation: The linear transformation X = PY is called a non-singular linear transformation if P is non-singular.

Linear Transformation of a Quadratic Form: If X = PY is a non-singular linear transformation, then q = X'AX = (PY)'A(PY) = Y'BY where B = P'AP is called Linear Transformation of a Quadratic Form.

===

Ex. Obtain the linear transformation of the quadratic form $x_1^2 - x_2^2 + x_3^2 - 2x_1x_3 + 4x_2x_3$ under the linear transformations $x_1 = y_1 + y_2 + y_3$, $x_2 = y_2 - y_3$, $x_3 = 2\, y_3$

Solution: The matrix of given quadratic form $x_1^2 - x_2^2 + x_3^2 - 2x_1x_3 + 4x_2x_3$ is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

The matrix of given linear transformations

$x_1 = y_1 + y_2 + y_3$, $x_2 = y_2 - y_3$, $x_3 = 2\, y_3$ is

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

∴ Linear transformations of given quadratic form q = X'AX under linear transformations X = PY is q = Y'BY

Where $B = P'AP = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & -2 & 4 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 6 \\ 2 & 6 & -2 \end{bmatrix}$

$\therefore q = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 6 \\ 2 & 6 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= y_1^2 - 2y_2^2 - 2y_3^2 + 4y_1y_3 + 12y_2y_3$

===

Congruent Matrices: A matrix B is said to be congruent to matrix A if there exist a non-singular matrix P such that B = P'AP

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Properties of Congruent Matrices:

1) Reflexivity: Every square matrix is congruent to itself.

Proof : For any square matrix A there exist a non-singular matrix I such that

A = I'AI

∴ Every square matrix is congruent to itself is proved.

===

2) Symmetry: If A is congruent to B, then B is congruent to A.

Proof : Let A is congruent to B.

∴ there exist a non-singular matrix P such that

A = P'BP

∴ $B = (P')^{-1} AP^{-1}$ since P is non-singular.

∴ $B = (P^{-1})' AP^{-1}$ where P^{-1} is non-singular.

∴ B is congruent to A is proved.

===

3) Transitivity: If A is congruent to B and B is congruent to C, then A is congruent to C.

Proof : Let A is congruent to B and B is congruent to C.

∴ there exists the non-singular matrices P and Q such that

A = P'BP and B = Q'CQ

∴ A = P' Q'CQP

∴ A = (QP)'C(QP) with QP is non-singular.

∴ A is congruent to C is proved.

==

Congruence of quadratic forms: Two quadratic forms are said to be congruent if there corresponding matrices are congruent.

==

Elementary Congruent Transformation:

A pair elementary row and corresponding elementary column transformations is called elementary congruent transformation.

Remark: There are three types of elementary congruent transformations viz. (R_{ij}, C_{ij}), $(R_{i(k)}, C_{i(k)})$ i.e. (kR_i, kC_i) and $(R_{ij(k)}, C_{ij(k)})$ i.e. (R_i+kR_j, C_i+kC_j)

==

Remark: To reduce the symmetric matrix A to congruent diagonal form, consider A = IAI and apply elementary congruent transformations on both sides so that we get diag[$d_1, d_2, \ldots\ldots, d_r, 0, 0, \ldots, 0$] = P'AP

==

Ex. Reduce the symmetric matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & -3 \\ 4 & -3 & 5 \end{bmatrix}$ to its congruent diagonal form.

Solution: To reduce the symmetric matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & -3 \\ 4 & -3 & 5 \end{bmatrix}$ to its congruent diagonal form.

Consider A = IAI

i.e. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & -3 \\ 4 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By applying $(R_2 - 2R_1, C_2 - 2C_1)$ & $(R_3 - 4R_1, C_3 - 4C_1)$ we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & -11 \\ 0 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying $(R_3 - \frac{11}{6}R_2, C_3 - \frac{11}{6}C_2)$ we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -\frac{55}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{1}{3} & -\frac{11}{6} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -\frac{1}{3} \\ 0 & 1 & -\frac{11}{6} \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. D = P'AP where D is the required diagonal matrix congruent to A.

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Ex. Reduce the symmetric matrix A = $\begin{bmatrix} 1 & 2 & 8 \\ 2 & 0 & -3 \\ 8 & -3 & -4 \end{bmatrix}$ to its congruent diagonal form.

Solution: To reduce the symmetric matrix A = $\begin{bmatrix} 1 & 2 & 8 \\ 2 & 0 & -3 \\ 8 & -3 & -4 \end{bmatrix}$ to its congruent diagonal form.

Consider A = IAI

i.e. $\begin{bmatrix} 1 & 2 & 8 \\ 2 & 0 & -3 \\ 8 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By applying ($R_2 - 2R_1$, $C_2 - 2C_1$) & ($R_3 - 8R_1$, $C_3 - 8C_1$) we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -19 \\ 0 & -19 & -68 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying ($R_3 - \frac{19}{4}R_2$, $C_3 - \frac{19}{4}C_2$) we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{89}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{3}{2} & -\frac{19}{4} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{19}{4} \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. D = P'AP where D is the required diagonal matrix congruent to A.

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Canonical Form: Every quadratic form in n variables and of rank r is reduced to sum and difference of the squares of the new variables. This transformed form is called canonical form of given quadratic form.

Index: The number k of positive squares in the canonical form is called the index of a quadratic form.

Signature: The number s = k – (r - k) = 2k – r is called the signature of a quadratic form.

Classification of Quadratic Form: Let q = X'AX be a given quadratic form in n variables of rank r and index k, the q is called

i) positive definite if r = n and k = n

ii) positive semi definite if r < n and k = r

iii) negative definite if r = n and k = 0

iv) negative semi definite if r < n and k = 0

v) indefinite if it is not any of the above forms i.e. k ≠ r and k ≠ 0

==

Ex. Reduce the quadratic form $x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + x_1x_3$ to conical form. Write the linear transformation used. Examine the form for definiteness.

Solution: The matrix of given quadratic form is $A = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix}$

Consider A = IAI

i.e. $\begin{bmatrix} 1 & -1 & \frac{1}{2} \\ -1 & 2 & -1 \\ \frac{1}{2} & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By applying ($R_2 + R_1$, $C_2 + C_1$) & ($R_3 - \frac{1}{2}R_1$, $C_3 - \frac{1}{2}C_1$) we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying ($R_3 + \frac{1}{2}R_2$, $C_3 + \frac{1}{2}C_2$) we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

By applying ($\sqrt{\frac{2}{3}}R_3$, $\sqrt{\frac{2}{3}}C_3$) we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$$

Given quadratic form is congruent to $y_1^2 + y_2^2 + y_3^2$ which is canonical form of given quadratic form with rank r = 3= n and index k = 3 = n.

∴ it is positive definite.

Obtained by using non-singular linear transformation X = PY

i.e. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

i.e. $x_1 = y_1 + y_2$, $x_2 = y_2 + \frac{1}{\sqrt{6}}y_3$, $x_3 = \sqrt{\frac{2}{3}}y_3$

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Ex. Show that the quadratic form $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$ is indefinite.

Solution: The matrix of given quadratic form is $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$

Consider A = IAI

i.e. $\begin{bmatrix}1 & 0 & 3\\0 & -2 & -2\\3 & -2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

By applying ($R_3 - 3R_1$, $C_3 - 3C_1$) we get,

$$\begin{bmatrix}1 & 0 & 0\\0 & -2 & -2\\0 & -2 & -6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & -3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

By applying (R_3 - R_2, C_3 - C_2) we get,

$$\begin{bmatrix}1 & 0 & 0\\0 & -2 & 0\\0 & 0 & -4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3 & -1 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & -3\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$$

By applying ($\frac{1}{\sqrt{2}}R_2$, $\frac{1}{\sqrt{2}}C_2$) & ($\frac{1}{2}R_3$, $\frac{1}{2}C_3$) we get,

$$\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & 0\\-\frac{3}{2} & -\frac{1}{2} & \frac{1}{2}\end{bmatrix} A \begin{bmatrix}1 & 0 & -\frac{3}{2}\\0 & \frac{1}{\sqrt{2}} & -\frac{1}{2}\\0 & 0 & \frac{1}{2}\end{bmatrix}$$

Given quadratic form is congruent to $q = u^2 - v^2 - w^2$ which is canonical form of given quadratic form with rank r = 3= n and index k = 1

∴ it is indefinite.

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Ex. Show that the quadratic form $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz$ is positive semi definite.

Solution: The matrix of given quadratic form is A = $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{bmatrix}$

Consider A = IAI

i.e. $\begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

By applying ($R_2 - 2R_1$, $C_2 - 2C_1$) & ($R_3 - 3R_1$, $C_3 - 3C_1$) we get,

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\-3 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & -2 & -3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Given quadratic form is congruent to $q = u^2$ which is canonical form of given quadratic form with rank r = 1< n and index k = 1 = r

∴ Given quadratic form is positive semi definite is proved.

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UNIT-IV: ORTHOGONAL MATRICES AND QUADRATIC FORMS [MCQ'S]

1) A square matrix A is said to be an orthogonal matrix if …….
   A) AA'≠ I   B) AA'= I   C) A=A'   D) AA'= 0

2) An orthogonal matrix A is said to be proper orthogonal matrix if …...
   A) |A| = 1   B) |A| ≠ 1   C) |A| = -1   D) |A| ≠ -1

3) An orthogonal matrix A is said to be improper orthogonal matrix if …...
   A) |A| = 1   B) |A| ≠ 1   C) |A| = -1   D) |A| ≠ -1

4) Determinant of an orthogonal matrix is …..
   A) 0   B) ±1   C) 2   D) -2

5) The inverse of an orthogonal matrix A is …...
   A) A   B) A'   C) I   D) 0

6) Product of two orthogonal matrices of same order is …..
   A) orthogonal   B) not orthogonal   C) singular   D) symmetric

7) Inverse of an orthogonal matrix is ……..
   A) singular   B) symmetric   C) orthogonal   D) not orthogonal

8) The matrix A = $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ is ……..
   A) singular   B) skew symmetric   C) orthogonal   D) not orthogonal

9) The matrix A = $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is ……. orthogonal matrix.
   A) proper   B) improper   C) None of these

10) The matrix A = $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ is …….. orthogonal matrix.
   A) proper   B) improper   C) None of these

11) The matrix A = $\begin{bmatrix} cos\theta & -sin\theta \\ -sin\theta & -cos\theta \end{bmatrix}$ is ……. orthogonal matrix.
   A) proper   B) improper   C) None of these

12) The matrix A = $\begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$ is ……..orthogonal matrix.
   A) proper   B) improper   C) None of these

13) The inverse of an orthogonal matrix A = $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ is …...

   A) $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ B) $\frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

   C) $\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ D) $\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

14) Whether the matrix A = $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$ is orthogonal?

A) Yes B) No

15) The condition that the matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal is…

A) $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$ B) ad + bc = 0

C) $a^2 + b^2 = c^2 + d^2 = 0$ and $ac + bd = 1$ D) ad + cd = 0

16) A homogeneous polynomial of second degree in n variables is called a …….in the n variables.

A) quadratic form B) canonical form C) congruent form D) None of these

17) $ax^2 + 2hxy + by^2$ is a quadratic form in ….variables x & y.

A) one B) two C) three D) four

18) $x_1^2 + x_2^2 - 3x_3^2 + 4x_1x_2 - x_1x_3 + 5x_2x_3$ is a quadratic form in …… variables

A) one B) two C) three D) four

19) q = $\sum_{j=1}^{n} \cdot \sum_{i=1}^{n} a_{ij}\, x_i x_j$ is a general quadratic form in …. variables

A) 1 B) i C) j D) n

20) The matrix of a quadratic form q = $x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$ is …

A) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -4 \\ 3 & -4 & 3 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 4 & 6 \\ 4 & -2 & -8 \\ 6 & -8 & -3 \end{bmatrix}$ D) $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 2 & -4 \\ 3 & -4 & 3 \end{bmatrix}$

21) The matrix of a quadratic form q = $x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_3$ is ……

A) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ D) $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

22) The rank of a quadratic form q = $x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$ is …

A) 1 B) 2 C) 3 D) 4

23) The rank of a quadratic form q = $x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_3$ is ……

A) 1 B) 2 C) 3 D) 4

24) The quadratic form corresponding to the matrix A = $\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$ is

A) $x_1^2 + 6x_1x_2$ B) $x_1^2 - 3x_2^2$ C) $x_1^2 + 3x_2^2$ D) $x_1^2 + 3x_1x_2$

25) The quadratic form corresponding to the matrix A = $\begin{bmatrix} 5 & 0 & -3 \\ 0 & -2 & 1 \\ -3 & 1 & 1 \end{bmatrix}$ is

A) $5x_1^2 + 2x_2^2 - x_3^2 - 6x_1x_3 + 2x_2x_3$ B) $5x_1^2 - 2x_2^2 + x_3^2 - 6x_1x_3 + 2x_2x_3$

C) $5x_1^2 - 2x_2^2 + x_3^2 - 3x_1x_3 + 1x_2x_3$ D) $x_1^2 - 2x_2^2 + x_3^2 - 3x_1x_3 + x_2x_3$

26) The linear transformation X = PY is called a non-singular linear transformation if P is ……

A) symmetric B) singular C) non-singular D)skew symmetric

27) A matrix B is said to be congruent to matrix A if there exist a non-singular matrix P such that B = ….

A) AP B) PAP' C) PA D) P'AP

28) Statement "Every square matrix is congruent to itself" is ……

A) true B) false

29) Statement "If A is congruent to B, then B is congruent to A" is ……

A) true B) false

30) Statement "If A is congruent to B and B is congruent to C, then A is congruent to C" is ……

A) true B) false

31) Two quadratic forms are said to be congruent if there corresponding matrices are …...

A) not congruent B) congruent C) equal D) not equal

32) A pair elementary row and corresponding elementary column transformations is called …….

A) elementary row transformation B) elementary column transformation

C) elementary congruent transformation D) none of these

33) There are …… elementary congruent transformation.

A) two B) three C) four D) six

34) To reduce the symmetric matrix A to congruent diagonal form, we apply …… on both sides of A = IAI so that we get diag[$d_1, d_2, \ldots, d_r, 0, 0, \ldots, 0$] = P'AP

A) elementary row transformation B) elementary column transformation

C) elementary congruent transformation D) none of these

35) Every quadratic form in n variables and of rank r is reduced to sum and difference of the squares of the new variables. This transformed form is called ……. of given quadratic form.

A) canonical form B) quadratic form

C) congruent form D) linear form

36) A quadratic form q = X'AX in n variables of rank r and index k is said to positive definite if ……

A) r = n and k = r B) r < n and k = r C) r = n and k = 0 D) r < n and k = 0

37) A quadratic form q = X'AX in n variables of rank r and index k is said to positive semi definite if ……

A) r = n and k = r B) r < n and k = r C) r = n and k = 0 D) r < n and k = 0

38) A quadratic form q = X'AX in n variables of rank r and index k is said to negative definite if ……

A) r = n and k = r B) r < n and k = r C) r = n and k = 0 D) r < n and k = 0

39) A quadratic form q = X'AX in n variables of rank r and index k is said to negative semi definite if ……

A) r = n and k = r B) r < n and k = r C) r = n and k = 0 D) r < n and k = 0

40) A quadratic form q = X'AX in n variables of rank r and index k is said to indefinite if ……

A) r = n and k = r B) r < n and k = r C) r = n and k = 0 D) k≠r and k ≠0

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॥ अंतरी पेटवू ज्ञानज्योत ॥

विद्यापीठ गीत

मंत्र असो हा एकच हृदयी 'जीवन म्हणजे ज्ञान'
ज्ञानामधूनी मिळो मुक्ती अन मुक्तीमधूनी ज्ञान ॥धृ ॥
कला, ज्ञान, विज्ञान, संस्कृती साधू पुरूषार्थ
साफल्यास्तव सदा 'अंतरी पेटवू ज्ञानज्योत'
मंगल पावन चराचरातून स्त्रवते अक्षय ज्ञान ॥१ ॥
उत्तम विद्या, परम शक्ति ही आमुची ध्येयासक्ती
शील, एकता, चारित्र्यावर सदैव आमुची भक्ती
सत्य शिवाचे मंदिर सुंदर, विद्यापीठ महान ॥२ ॥
समता, ममता, स्वातंत्र्याचे नांदो जगी नाते,
आत्मबलाने होऊ आम्ही आमुचे भाग्यविधाते,
ज्ञानप्रभुची लाभो करूणा आणि पायसदान ॥३ ॥

— कै.प्रा. राजा महाजन

THE NATIONAL INTERGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation.

I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievance should be settled by peaceful and constitutional means."